APMA 0360: Methods of Applied Mathematics II
Review for the second midterm, Nov 7th

Below are a few key items that one should master in order to perform well in the second midterm on Wednesday, Nov 7th.

0.1. **Linear approximation.** know how to find (and verify) the linear system that approximates a given nonlinear system near a critical point.

0.2. **Linearization method.** Master the so-called linearization method to understand a given nonlinear system and its phase plane. This consists of 4 steps:

- Find all critical points of the nonlinear system
- Linearize the nonlinear system around each critical point. Determine stability property and type of each critical point of each linearization, using materials from Chapter 7.
- Use the so-called stability table (Table 9.3.2) to draw possible conclusions on the stability property and type of each critical point of the nonlinear system
- Sketch a possible phase portrait of the nonlinear system.

0.3. **Applications.** As an application of the linearization method, study the competing species system and the predator-prey model.

0.4. **The so-called Hamiltonian function approach.** It's an approach to sketch trajectories of a given system of differential equations:

$$x'(t) = F(x, y), \qquad y'(t) = G(x, y) \tag{0.1}$$

Let's recall it a bit here. From the differential equations, deduce

$$G(x, y)dx - F(x, y)dy = 0. \tag{0.2}$$

Solve this scalar differential equation whose solution gives the trajectories of the system (0.1). If the equation (0.2) is exact, that is, if there is a function $H(x, y)$ so that $H_x = G(x, y)$ and $H_y = -F(x, y)$, then the equation (0.2) becomes

$$H_x(x, y)dx + H_y(x, y)dy = 0,$$

or equivalently by chain rule (at least when we can write the variable $y = y(x)$ as a function of x)

$$\frac{d}{dx}H(x, y) = 0.$$

(Hence comes the name "exact"). In this case, the solution (x, y) of (0.2) is given implicitly by the equation

$$H(x, y) = C$$

for arbitrary constant C. Sometimes, other methods might apply to yield the above implicit solutions.

Now the trajectories of the system (0.1) are the level curves of $H(x, y)$. For example, try to sketch a phase plane of the following system:

$$x'(t) = y, \qquad y'(t) = x.$$

0.5. **Lyapunov's second method.** An approach to determine stability property of a critical point, say the origin. To apply this method, one first needs to find a so-called Lyapunov function $V(x,y)$ with $V(0,0)=0$ and check one of the following criteria:

- Stability criterium: If V is positive definite, then $(0,0)$ is asymptotically stable if $\dot{V}$ is negative definite, and $(0,0)$ is stable if $\dot{V}$ is negative semi-definite.
- Instability criterium: If $V(0,0)=0$ and in any neighborhood D of the origin, the function V is positive at least at one point in D and if $\dot{V}$ is positive definite, then the critical point $(0,0)$ is unstable.

A few remarks on the Lyapunov's second method:

- The function $\dot{V}(x,y)$ is computed by
$$\dot{V}(x,y) = x'(t)V_x + y'(t)V_y = F(x,y)V_x(x,y) + G(x,y)V_y(x,y).$$
- The function $V(x,y)$ is often linked with the physical total energy associated with the differential system. Roughly speaking, the stability/instability criteria can verbally be translated as
  - if the total energy is decreasing in time (necessarily towards zero), one should expect stability of the critical point.
  - Otherwise, if there is some point at which the total energy is positive and is increasing in time, the critical point should be unstable.
  - Try two critical points (one stable and the other unstable) in the damped or undamped pendulum model.
- It is often suggested to try with
$$V(x,y) = \frac{1}{2}ax^2 + \frac{1}{2}by^2$$
and search for appropriate constants a and b to match (only) one of the two criteria mentioned above.

0.6. **Stability/instability proof.** Know how to prove stability or instability of critical points of some linear or nonlinear systems. Of course, the above Lyapunov's second method is one nice approach, if applicable. Alternatively, the proof might be given by directly checking the stability definition given in the book, or alternatively by verifying the following stability criterium:

- Stability criterium: Let X_0 be a critical point of the differential system:
$$X'(t) = f(X). \tag{0.3}$$
Assume that one can find two positive constants C_0 and δ_0 so that whenever the initial data $X(0)$ is close to the fixed point X_0 in the sense $\|X(0)-X_0\| \leq \delta_0$, then the solution $X(t)$ starting at the initial data $X(0)$ must satisfy
$$\|X(t)-X_0\| \leq C_0\|X(0)-X_0\|,$$
for all time $t \geq 0$. Then, the critical point is stable. If furthermore one shows that $\|X(t)-X_0\| \to 0$ as $t \to \infty$, the critical point is asymptotically stable.

A few remarks:

- For linear systems, in case of stability (detected from eigenvalues), the constant C_0 can be taken to be the upper bound of the exponential matrix e^{At}, whereas the constant δ_0 could be arbitrarily taken.
- Again for linear systems, in case of instability, the proof is by contradiction. That is, assume the assumption of the stability criterium holds. Then, find a contradiction by specifically taking a solution of the form $cVe^{\lambda t}$. Here λ is positive and V is an associated eigenvector.
- For nonlinear systems, the first step is to linearize around the critical point and determine stability property of the critical point of the linearization. From class, we know of two cases in which nonlinear stability property can be concluded from the linear information:
  - Asymptotically linear stability implies asymptotically nonlinear stability.

– Linear instability implies nonlinear instability.

Check the online class notes for the proof of each of these two cases. Give examples where nonlinear stability property cannot be concluded from linear information.