APMA 0360: Methods of Applied Mathematics II

Review for the first midterm, Oct 4th

The first midterm will be tested on the material from Chapter 7 that has been (and will be on this Wed and Friday, Sep 26th & 28th) covered in class. Not only should you know well the following materials, but also be able to solve relevant (similar) problems quickly. Review your homework carefully.

• Reduce high-order differential equations to a system of first-order differential equations (DEs, for short). *Recall:* let

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \cdots$$

and write the differential system (possibly, with some appropriate initial data) for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix}$.

• Compute eigenvalues and eigenvectors of a given matrix. Know how to check whether a set of vectors is linearly independent or dependent.

• Know how to find the general (possibly, real-valued) solution to a given system of *many* first-order differential equations with constant coefficients. Check again Theorem 7.4.2, page 387. In case of complex eigenvalues, remember the fact (you should convince yourself this) that real and imaginary parts of a solution (to a system with real-valued coefficients) are also solutions (are they linearly independent ?). In case of repeated eigenvalues, you should be able to find a generalized eigenvector (only if necessary, of course). *Recall:* generalized eigenvectors V_2 are solutions to

$$(A - \lambda I)V_2 = V_1,$$

where V_1 is an eigenvector associated with eigenvalue λ. How to check that V_1, V_2 are linearly independent ? The second solution is then of the form:

$$X_2(t) = V_1 t e^{\lambda t} + V_2 e^{\lambda t}.$$

• Sketch a phase portrait, determine the type of the origin, draw a few trajectories of solutions to a 2×2 system of DEs. Also, follow the trajectory to discuss the behavior in the large time of the solution that is assumed to begin at a given point. Several cases can happen, below. In all cases, it's always a good idea to check the directional field at a few points on the phase portrait.

(1) Eigenvalues are real, distinct and have the same sign. The origin is (stable or unstable) node. The phase portrait is a bit tricky. Examples:

$$\mathbf{x}'(\mathbf{t}) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} \qquad \text{and} \qquad \mathbf{x}'(\mathbf{t}) = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}.$$

(2) Eigenvalues are real and have opposite sign. The origin is a saddle point. Phase portrait is too easy to sketch (really?). Example:

$$\mathbf{x}'(\mathbf{t}) = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}.$$

(3) Eigenvalues are real and repeated. Tricky. Check if you need to find a generalized eigenvector (see above). If not, the origin is called a proper node (phase plane in this case ?). The other case is improper node (case with generalized eigenvectors). Example:

$$\mathbf{x}'(\mathbf{t}) = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \mathbf{x}.$$

(4) One of the eigenvalues is zero ! Sketch the phase plane. Example:

$$\mathbf{x}'(\mathbf{t}) = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \mathbf{x}.$$

(5) Complex eigenvalues. Origin is a spiral (when real part of eigenvalues are not zero) or a center (real part is zero). Example:

$$\mathbf{x}'(\mathbf{t}) = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \mathbf{x}.$$

Try a few examples with $\alpha > 0$, $\alpha < 0$, and $\alpha = 0$. A possible question is determine the critical value (if any) of α where the qualitative nature of the phase portraits changes.

(6) Eigenvalues are complex and repeated. Can we have this in the case with real-valued coefficients ? Example ?

• A fundamental matrix of a system of first-order DEs

$$X'(t) = A(t)X(t)$$

is defined by

$$\Psi(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}$$

where $X_1(t) = \begin{pmatrix} x_1^{(1)}(t) \\ \vdots \\ x_n^{(1)}(t) \end{pmatrix}, \cdots, X_n(t) = \begin{pmatrix} x_1^{(n)}(t) \\ \vdots \\ x_n^{(n)}(t) \end{pmatrix}$ are solutions and form a fundamental set of solutions to the differential system. The fundamental matrix $\Phi(t)$ is a particular fundamental matrix so that

$$\Phi(0) = I.$$

• The fundamental matrix $\Phi(t)$ can be constructed by $\Phi(t) = \Psi(t)(\Psi(0))^{-1}$ for arbitrary fundamental matrix or, in the case the matrix A is constant, by the exponential matrix

$$\Phi(t) = e^{At}$$

• Know how to solve a system of DEs by using a fundamental matrix. General solution is

$$X(t) = \Psi(t)C$$

for a vector constant C. The unique solution with given initial data is

$$X(t) = \Psi(t)(\Psi(0))^{-1}X(0) \qquad \text{or} \qquad X(t) = \Phi(t)X(0).$$

• Know how to solve inhomogenous system of differential equations:

$$X'(t) = A(t)X(t) + g(t)$$

Possible methods are *undetermined-Coefficients* method and *Variation-of-Parameters* method. For the former method, remember that the general solution is the sum of the general solutions to the homogenous system and the particular solution (which is constructed by matching the coefficients). For the latter method, you should know how to derive to the general formula:

$$X(t) = \Psi(t)C + \Psi(t) \int_0^t \Psi(s)^{-1} g(s)\, ds.$$