APMA 0360: Midterm I [1]

Some useful facts:

1. Inverse of a 2×2 matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \qquad \text{if} \quad ad-bc \neq 0.$$

2.
$$\int t^n \, dt = \frac{1}{n+1} t^{n+1} + C, \qquad \text{if} \quad n \neq -1.$$

Problem 1 (20 points)
Consider an initial value problem of a third order differential equation:

$$u''' + \sin(t)u'' - 2tu' + u = 0, \qquad u(0) = 1, \quad u'(0) = 0, \quad u''(0) = -2.$$

Transform this problem into an initial value problem of a system of first order equations of the matrix form:

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0.$$

Do NOT solve the resulting system.

Solution: Denote

$$x_1 = u, \quad x_2 = u', \quad x_3 = u''.$$

We easily see that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2t & -\sin t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \text{with initial datum:} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

Problem 2 (20 points)
For each of the following differential equations, determine if the origin is a saddle point, stable node, unstable node, stable improper node, unstable improper node, stable spiral point, or unstable spiral point:

(a) $\mathbf{x}'(t) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x}(t)$ (b) $\mathbf{x}'(t) = \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix}\mathbf{x}(t)$

(c) $\mathbf{x}'(t) = \begin{pmatrix} -2 & 0 \\ 1/2 & -1 \end{pmatrix}\mathbf{x}(t)$ (d) $\mathbf{x}'(t) = \begin{pmatrix} -3 & -2 \\ 1 & -1 \end{pmatrix}\mathbf{x}(t)$

[1] Exam time: 50 minutes. Date: Friday, Feb 24th, 2012

Do NOT solve these systems.

Solution: Only need to calculate the eigenvalues.

(a) It's clear that eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. They are distinct, real, same (positive) sign. The origin is an unstable node.

(b) Eigenvalues are solutions of

$$(\lambda - 2)(\lambda + 2) - 12 = 0$$

so $\lambda^2 = 16$, which yields two eigenvalues -4 and 4. They are real, distinct, different sign. Origin must be a saddle point.

(c) Clearly, the eigenvalues are -1 and -2. Origin is a stable node.

(d) Eigenvalues solve

$$(\lambda + 1)(\lambda + 3) + 2 = 0$$

So $\lambda^2 + 4\lambda + 5 = 0$. Solutions are $\lambda_{1,2} = -2 \pm \sqrt{4 - 5} = -2 \pm i$. Complex eigenvalues with real part equal to -2, negative. The origin is a stable spiral.

Problem 3 (20 points)

Find the general (real-valued) solution to the system of differential equations

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}(t),$$

and sketch the phase portrait.

Solution: First, let's find eigenvalues. They are solutions to

$$(\lambda - 1)(\lambda - 3) + 1 = 0$$

So, eigenvalue is $\lambda = 2$, repeated. Eigenvector associated with the eigenvalue is a solution of the linear algebraic equation:

$$\left[\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Or equivalently,

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

This clearly yields $x_1 + x_2 = 0$ (only one free parameter). An eigenvector is

$$V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We need to find a generalized eigenvector V_2 solving (why so?) $(A - 2I)V_2 = V_1$, or equivalently

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This yields $x_1 + x_2 = 1$. A generalized eigenvector is

$$V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The general solution (why? need to convince that V_1 and V_2 are linearly independent) is then

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \Big[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} \Big]$$

See Figure in the end of this note for the phase portrait. Did you get the "same" ? No, I won't provide you a computer to draw this; you only need to sketch a phase portrait.

Problem 4 (20 points)
Consider an initial value problem of the general nonhomogeneous system:

$$\mathbf{x}'(t) = A\mathbf{x} + g(t), \qquad \mathbf{x}(0) = \mathbf{x}_0,$$

where A is some given constant $n \times n$ matrix. Assume that we can compute the exponential matrix e^{At}.

Question: Present a method for determining the solution to the above initial value problem. Explain clearly all the steps in your method.

Solution 1: Since the matrix A is constant, the exponential matrix e^{At} is the fundamental matrix and so you can present in detail the "variations of parameters" method (since it works for arbitrary fundamental matrix) to obtain the general solution:

$$\mathbf{x}(t) = e^{At}C + e^{At} \int_0^t e^{-As} g(s)\, ds. \tag{1}$$

Now with $\mathbf{x}(0) = \mathbf{x}_0$, we get

$$\mathbf{x}_0 = \mathbf{x}(0) = e^{A0}C + e^{A0} \int_0^0 e^{-As} g(s)\, ds = e^0 C = C$$

That is, $C = \mathbf{x}_0$ and the solution is

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + e^{At} \int_0^t e^{-As} g(s)\, ds. \tag{2}$$

Solution 2: Remembering what we should do if this were a scalar equation, we then compute

$$(e^{-At}\mathbf{x})' = e^{-At}\mathbf{x}' - e^{-At}A\mathbf{x} = e^{-At}(\mathbf{x}' - A\mathbf{x}) = e^{-At}g(t)$$

Here check that $(e{-}At)' = e^{-At}A$ and use $\mathbf{x}' - A\mathbf{x} = g(t)$, which is the given equation. Integrating the above identity yields

$$e^{-At}\mathbf{x} = \mathbf{x}_0 + \int_0^t e^{-As} g(s)\, ds$$

Multiply this by e^{At} we get the formula (2).

Problem 5 (20 points)
Consider the nonhomogeneous system:

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 0 \\ t^5 e^t \end{pmatrix}$$

We know (that is, you do NOT have to find it) that the exponential matrix of $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} e^t$$

Question: Find the general solution to the above system by ONE of any methods that you know. Hint: use the variation-of-parameters method (or Problem 4).

Solution: Note that

$$e^{-At} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} e^{-t}$$

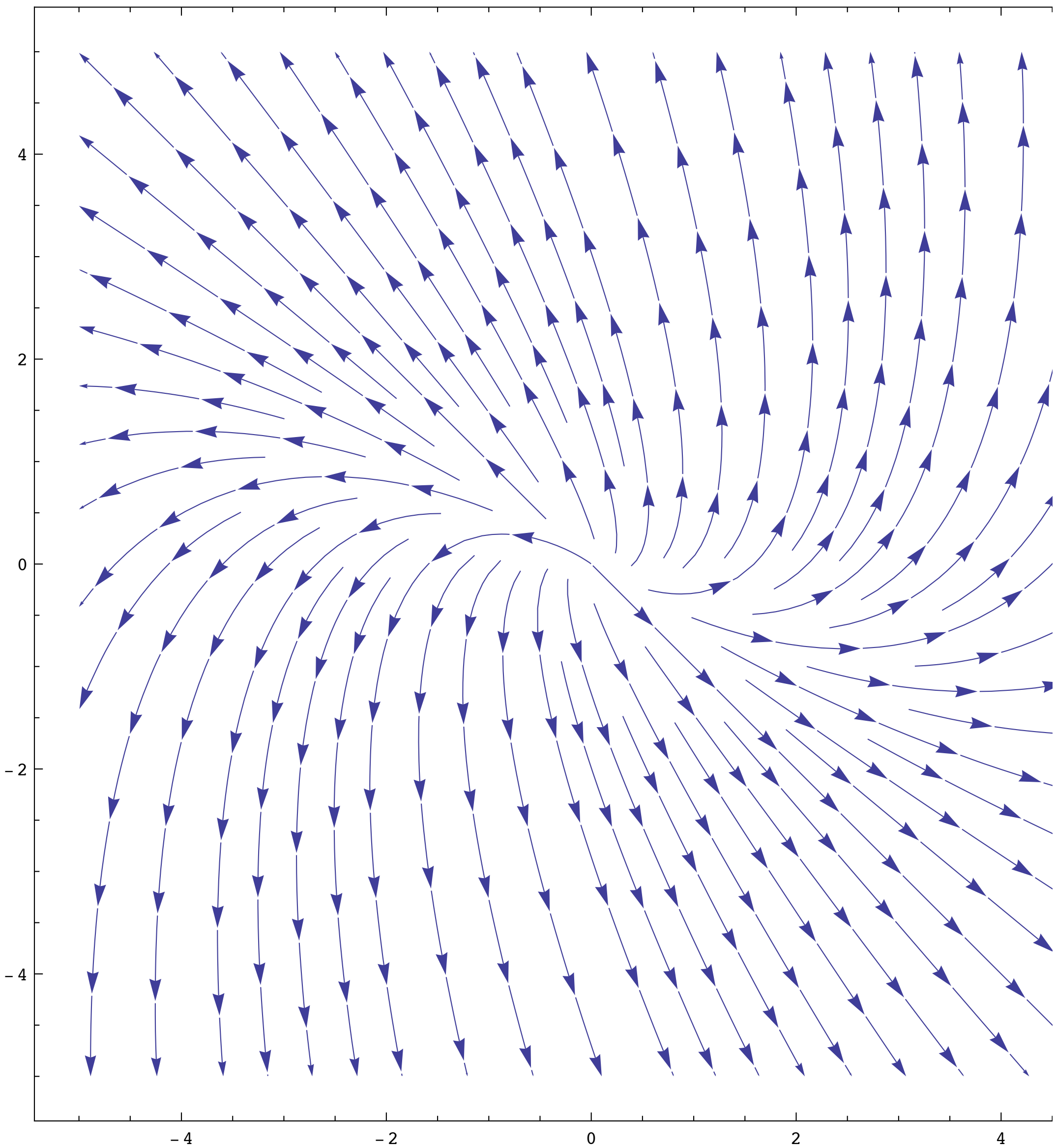
Now using the formula (1) in Problem 4, we get the general solution

$$\begin{aligned}
\mathbf{x}(t) &= e^{At}C + e^{At} \int_0^t e^{-As} g(s)\, ds \\
&= e^{At}C - e^{At} \int_0^t \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ s^5 \end{pmatrix} ds \\
&= e^{At}C - e^{At} \int_0^t \begin{pmatrix} -s^6 \\ s^5 \end{pmatrix} ds \\
&= e^{At}C - e^{At} \begin{pmatrix} -\frac{1}{7}t^7 \\ \frac{1}{6}s^6 \end{pmatrix} \\
&= e^{At}C - \begin{pmatrix} \frac{1}{42}t^7 \\ \frac{1}{6}s^6 \end{pmatrix} e^t
\end{aligned}$$

(well, it wasn't so bad, was it ?) Note in this last problem if you used the so-called "undetermined-coefficients", it would be a bit lengthy since you will have to begin with a particular solution of a form:

$$\mathbf{x}(t) = \begin{pmatrix} a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 \\ b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 + b_6 t^6 + b_7 t^7 \end{pmatrix} e^t$$

and solve for these constants a_j and b_j.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