linear instability $\Rightarrow$ nonlinear instability.

I. Linear instability:

Consider the linear system:

$$X' = AX \qquad (1)$$

For simplicity, take $A \in M_{2\times 2}$, constant matrix.
Assume that A has 2 distinct eigenvalues $\lambda_1 < \lambda_2$ with λ_2 positive. Let V_1, V_2 be the associated eigenvectors.

$\boxed{\text{Claim 1}}$ $X_0 = 0$ is unstable critical point of (1).

__Proof:__ Assume that $X_0 = 0$ is stable. Then we can find 2 positive constants C_0, δ_0 so that for all time $t \geq 0$,

$$\|X(t)\| \leq C_0 \|X(0)\| , \qquad (*)$$

whenever $\|X(0)\| < \delta_0$.

Now if we take $X(0) = \dfrac{\delta_0}{2\|V_2\|} V_2$, then it is clear that $\|X(0)\| = \dfrac{\delta_0}{2\|V_2\|}\|V_2\| = \dfrac{\delta_0}{2} < \delta_0$ and the solution with this initial value is

$$X(t) = \frac{\delta_0}{2\|V_2\|} V_2 e^{\lambda_2 t}$$

(solution stays on the line of V_2).

So, $\|X(t)\| = \dfrac{\delta_0}{2} e^{\lambda_2 t}$.

Now by (*) (The stability assumption)

we then have

$$\|X(t)\| = \frac{\delta_0}{2} e^{\lambda_2 t} \leq C_0 \|X_0\| = \frac{C_0 \delta_0}{2}$$

which implies that

$$e^{\lambda_2 t} \leq C_0 , \quad \text{for all } t \geq 0.$$

Of course, this is a contradiction since $e^{\lambda_2 t} \to +\infty$ as $t \to +\infty$. For instance, take

$$t = \frac{1}{\lambda_2} \log(C_0 + 1)$$

we have $e^{\lambda_2 t} = e^{\lambda_2 \frac{1}{\lambda_2} \log(C_0+1)} = C_0 + 1 > C_0$.

So (*) is false. That is, $X_0 = 0$ is unstable. ▨

II. Nonlinear instability:

Consider locally linear system (2)

$$X' = AX + g(X)$$

with $\|g(X)\| \leq C_1 \|X\|^2$, for all X.

Assume that A has 2 eigenvalues $\lambda_1 < \lambda_2$ with $\lambda_2 > 0$. Let V_1, V_2 be the associated e-vector.

$\boxed{\text{claim 2}}$ $X_0 = 0$ is unstable.

Remark: with the assumption on A, we just saw that $X_0 = 0$ is unstable critical point of (1), the

Linear system. The claim 2 is to say that linear instability also implies nonlinear instability.

<u>The intuition</u> to verify claim 2 is that we can again try to follow the "unstable direction" of V_2, with $\lambda_2 > 0$. like in the linear case. <u>However</u>, we'll need to be sure how the nonlinear part affects the analysis. that is we need to control the nonlinear part. To do so, we need <u>further</u> information on the linear solution (the fundamental matrix).

$\boxed{\text{claim 3}}$: There is a constant C_2 so that

$$\|e^{At}\| \le C_2 e^{\lambda_2 t}, \quad \text{for all } t \geq 0.$$

<u>Proof</u>: it's straightforward as we did several times in class.

<u>Proof of claim 2</u>: We prove it by contradiction. Indeed, assume that $X_0 = 0$ is stable, that is (*) on page ① valid.

We then take as before the initial value

$$X(0) = \frac{\delta_0}{2\|V_2\|} V_2 = X_0 \quad (\text{notation})$$

we ~~expect~~ the solution $X(t)$ to be near the linear solution $X_{lin}(t) = \frac{\delta_0}{2\|V_2\|} V_2 e^{\lambda_2 t}$.

To make it rigorous, we consider the difference:

$$Y(t) = X(t) - X_{lin}(t)$$

Then of course $Y(t)$ solves

$$\begin{aligned} Y'(t) &= X'(t) - X_{lin}'(t) \\ &= AX + g(X) - AX_{lin} \\ &= A(X - X_{lin}) + g(X) \\ &= AY + g(X). \end{aligned}$$

This is a non-homogenous system for $Y(t)$. We then have by "Variations of Parameters" method

$$Y(t) = e^{At} Y(0) + \int_0^t e^{A(t-s)} g(X(s))\, ds.$$

Clearly, $Y(0) = 0$ by definition of the difference.

So

$$\|Y(t)\| \le \int_0^t \underbrace{\|e^{A(t-s)}\|}_{\text{use claim 3}} \, \|\underbrace{g(X(s))}_{\text{use assumption on } g(x)\text{, below } ②}\| \, ds$$

it yields

$$\|Y(t)\| \leq \int_0^t C_2 e^{\lambda_2(t-s)} C_1 \underbrace{\|X(s)\|^2}_{\text{now use stability assumpt. } \circledast} ds$$

$$\leq C_1 C_2 \int_0^t e^{\lambda_2(t-s)} C_0^2 \|X(0)\|^2 ds$$

$$\leq C_0^2 C_1 C_2 \|X(0)\|^2 \int_0^t e^{\lambda_2(t-s)} ds$$

$$\Rightarrow \boxed{\|Y(t)\| \leq \frac{C_0^2 C_1 C_2}{\lambda_2} \|X(0)\|^2 e^{\lambda_2 t}} \quad , \text{ for all } t \geq 0.$$

Note that we could take $\|X(0)\|$ as small as we want. So $Y(t)$ is indeed small in the following sense. We have

$$X(t) = X_{lin}(t) + \underbrace{Y(t)}_{\text{the difference.}}$$

so

$$\|X(t)\| \geq \|X_{lin}(t)\| - \|Y(t)\| \quad \text{(triangle inequality)}$$

$$\geq \|X_0\| e^{\lambda_2 t} - \frac{C_0^2 C_1 C_2}{\lambda_2} \|X_0\|^2 e^{\lambda_2 t}$$

$$= \|X_0\| e^{\lambda_2 t} \left[1 - \underbrace{\frac{C_0^2 C_1 C_2}{\lambda_2} \|X_0\|}_{\text{small as we want.}} \right]$$

Take $\|X_0\|$ be very really small. Precisely, take X_0 small so that

$$\frac{C_0^2 C_1 C_2}{\lambda_2}\|X_0\| \leq \frac{1}{2}, \text{ but } X_0 \neq 0.$$

Then
$$\|X(t)\| \geq \|X_0\| e^{\lambda_2 t}\left[1 - \frac{1}{2}\right]$$
$$= \frac{1}{2}\|X_0\| e^{\lambda_2 t}$$

Now again, $t \to +\infty$, then $e^{\lambda_2 t} \to +\infty$ and thus contradicts with our stability assumpt. that
$$\|X(t)\| \leq C_0 \|X_0\|.$$

The contradiction ~~proves~~ shows that the claim 2 must be true. $\boxed{///}$

$\boxed{\text{Homework1}}$ Revise the above proof by replacing the assumption $\|g(X)\| \leq C_1\|X\|^2$ by the "smallness" condition

$$\lim_{X \to 0} \frac{\|g(X)\|}{\|X\|} = 0.$$

$\boxed{\text{Homework 2:}}$ Revise the proof when the matrix A has at least one complex eigenvalue with positive real part.

$\boxed{\text{Homework 3}}$ Prove "linear instability $\to$ nonlinear instability" in the case A is an $n \times n$ matrix.