9.4 Competing Species

I/ Population dynamics:

Population dynamics of one species is governed by a logistic equation

$$\frac{dx}{dt} = x\left(\varepsilon_1 - \sigma_1 x\right)$$

growth rate. $\frac{\varepsilon_1}{\sigma_1}$ is the saturation point.

The appearance of 2nd species y can decrease the total growth rate (for instance, due to the competition of food supply). For simplicity, we assume the interference of the presence of species y to the growth rate of x is linear. That is,

$$\frac{dx}{dt} = x\left(\varepsilon_1 - \sigma_1 x - \alpha_1 y\right)$$

Similarly, $\frac{dy}{dt} = y\left(\varepsilon_2 - \sigma_2 y - \alpha_2 x\right)$

Let's look at this nonlinear system in detail.

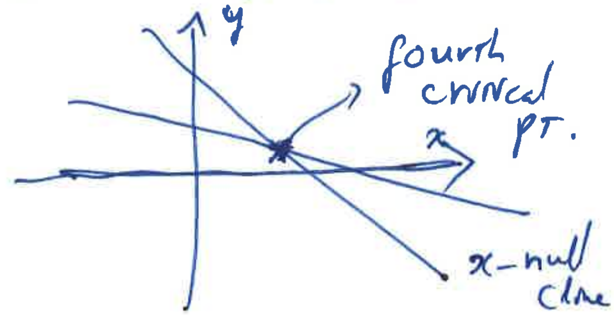
~~example~~: Four critical points:

+ $\begin{cases} x_1 = 0 \\ y_1 = 0 \end{cases}$, $\begin{cases} x_2 = 0 \\ y_2 = \frac{\varepsilon_2}{\sigma_2} \end{cases}$ $\begin{cases} x_3 = \varepsilon_1/\sigma_1 \\ y_3 = 0 \end{cases}$

and $\begin{cases} \sigma_1 x + \alpha_1 y = \varepsilon_1 \\ \alpha_2 x + \sigma_2 y = \varepsilon_2 \end{cases}$ The x/y nullclines.

$$\Leftrightarrow \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \frac{1}{\sigma_1\sigma_2 - \alpha_1\alpha_2} \begin{pmatrix} \sigma_2 & -\alpha_1 \\ -\alpha_2 & \sigma_1 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} x_4 = \dfrac{\varepsilon_1\sigma_2 - \varepsilon_2\alpha_1}{\sigma_1\sigma_2 - \alpha_1\alpha_2} . \\ y_4 = \dfrac{\sigma_1\varepsilon_2 - \varepsilon_1\alpha_2}{\sigma_1\sigma_2 - \alpha_1\alpha_2} . \end{cases}$$



The first three critical points:

Linearization:

$$J_f(x,y) = \begin{pmatrix} \varepsilon_1 - 2\sigma_1 x - \alpha_1 y & -\alpha_1 x \\ -\alpha_2 y & \varepsilon_2 - 2\sigma_2 y - \alpha_2 x \end{pmatrix}$$

At $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$J_f(x_1, y_1) = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}$$

e-values $\lambda_{1,2} = \varepsilon_{1,2} > 0$: unstable node, 3

$\boxed{a_T}$ $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \varepsilon_2/\sigma_2 \end{pmatrix}$

$$J_f(x_2, y_2) = \begin{pmatrix} \varepsilon_1 - \frac{\alpha_1 \varepsilon_2}{\sigma_2} & 0 \\ -\frac{\alpha_2 \varepsilon_2}{\sigma_2} & -\varepsilon_2 \end{pmatrix}$$

e-values: $\begin{cases} \lambda_2 = -\varepsilon_2 & \longrightarrow V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \lambda_1 = \dfrac{\varepsilon_1 \sigma_2 - \alpha_1 \varepsilon_2}{\sigma_2} & \longrightarrow V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$

so stable in V_2-direction $= y$-direction.

if $\varepsilon_1 \sigma_2 < \alpha_1 \varepsilon_2$, it's stable in x-direction as well : stable node.

if $\varepsilon_1 \sigma_2 > \alpha_1 \varepsilon_2$, it's saddle point.

$\boxed{A_T}$ $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} \varepsilon_1/\sigma_1 \\ 0 \end{pmatrix}$

$$J_f(x_3, y_3) = \begin{pmatrix} -\varepsilon_1 & -\varepsilon_1 \alpha_1/\sigma_1 \\ 0 & \varepsilon_2 - \frac{\alpha_2 \varepsilon_1}{\sigma_1} \end{pmatrix}$$

e-values: $\lambda_1 = -\varepsilon_1$, $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = \dfrac{\varepsilon_2 \sigma_1 - \alpha_2 \varepsilon_1}{\sigma_1}$, $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

if $\varepsilon_2 \sigma_1 > \alpha_2 \varepsilon_1$, unstable ; if $\varepsilon_2 \sigma_1 < \alpha_2 \varepsilon_1$: stable node.
$\Rightarrow$ saddle point.

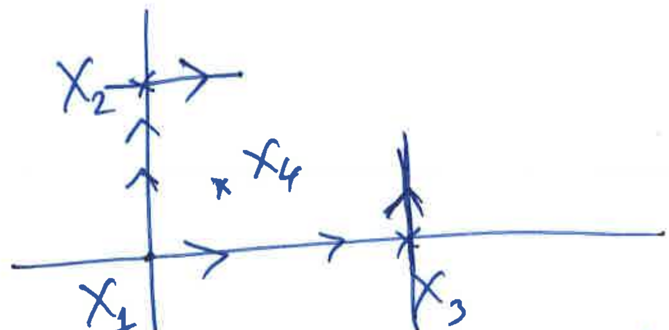
Fourth critical point:

$$x_4 = \frac{\varepsilon_1 \sigma_2 - \varepsilon_2 \alpha_1}{\sigma_1 \sigma_2 - \alpha_1 \alpha_2}, \quad y_4 = \frac{\sigma_1 \varepsilon_2 - \varepsilon_1 \alpha_2}{\sigma_1 \sigma_2 - \alpha_1 \alpha_2}.$$

Only interested in the case $x_4 > 0$, $y_4 > 0$.

(+) Case 1: $\sigma_1 \sigma_2 > \alpha_1 \alpha_2$. Then $\begin{cases} \varepsilon_1 \sigma_2 - \varepsilon_2 \alpha_1 > 0 \\ \sigma_1 \varepsilon_2 - \varepsilon_1 \alpha_2 > 0 \end{cases}$

for $x_4 > 0$ and $y_4 > 0$. In this case, we know that X_2 is a saddle point, and so is X_3.



we have

$$J_f(x_4, y_4) = \begin{pmatrix} -\sigma_1 x_4 & -\alpha_1 x_4 \\ -\alpha_2 y_4 & -\sigma_2 y_4 \end{pmatrix}$$

evalues: $\lambda^2 + (\sigma_1 x_4 + \sigma_2 y_4)\lambda + (\sigma_1 \sigma_2 - \alpha_1 \alpha_2) x_4 y_4 = 0$

$$\rightarrow \lambda_{1,2} = \frac{-(\sigma_1 x_4 + \sigma_2 y_4) \pm \sqrt{(\sigma_1 x_4 + \sigma_2 y_4)^2 - 4 d\, x_4 y_4}}{2}$$

$d := \sigma_1 \sigma_2 - \alpha_1 \alpha_2 > 0$

$\Rightarrow \lambda_1 \neq \lambda_2$, and λ_1, λ_2 are real and negative or complex with negative real parts.

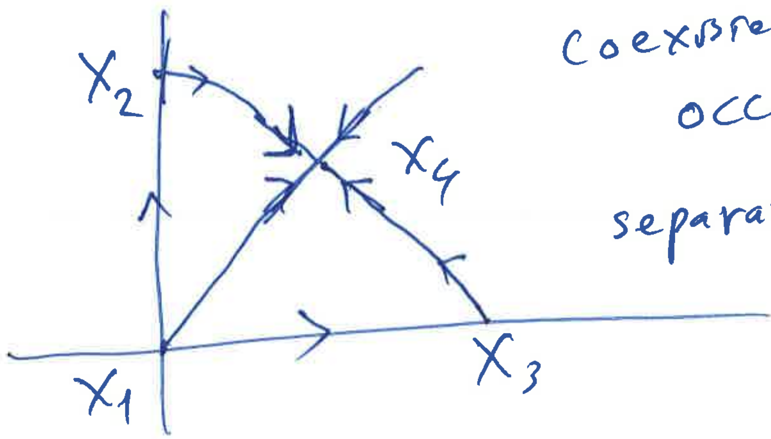
$\to$ asymptotically stable.

Note: $(\sigma_1 x_4 + \sigma_2 y_4)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)x_4 y_4$

$$= \sigma_1^2 x_4^2 + \sigma_2^2 y_4^2 - 2\sigma_1\sigma_2 x_4 y_4 + 4\alpha_1\alpha_2 x_4 y_4$$

$$= (\sigma_1 x_4 - \sigma_2 y_4)^2 + 4\alpha_1\alpha_2 x_4 y_4 > 0$$

$\to$ 2 e-values must be real, negative, distinct.



coexistence occurs!

separatrix.

Case 2: $\sigma_1\sigma_2 < \alpha_1\alpha_2$. In this case, we assume that

$$\begin{cases} \varepsilon_1\sigma_2 - \varepsilon_2\alpha_1 < 0 \\ \sigma_1\varepsilon_2 - \varepsilon_1\alpha_2 < 0 \end{cases}$$

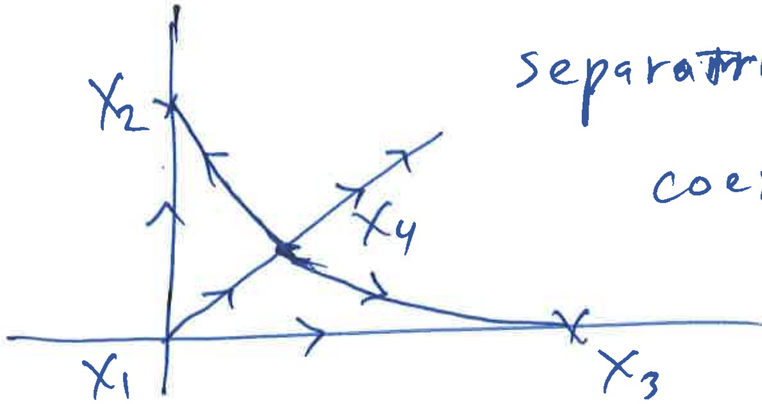
since we're interested in $x_4, y_4 > 0$.

Then X_2 and X_3 are both stable.

at X_4, the e-values are

$$\lambda_{1,2} = \frac{-(\sigma_1 x_4 + \sigma_2 y_4) \pm \sqrt{(\sigma_1 x_4 + \sigma_2 y_4)^2 - 4 d x_4 y_4}}{2}$$

$\Rightarrow \lambda_1 < 0 < \lambda_2 \to$ saddle point.



separatrix

coexistence does not occur!