Week 7: APMA 0360

(+) **Linear systems**: $X' = AX$ with $\det A \neq 0$.

the critical point $X_* = 0$ is

a. stable if all the eigenvalues are either non-positive or complex with non-positive real parts.

b. Asymptotically stable if all the eigenvalues are strictly negative or complex with strictly negative real parts.

c. unstable if one of the eigenvalues is positive or ~~with~~ complex with positive real part.

(+) **Locally linear systems**: $X' = AX + g(X)$ with $\det A \neq 0$. Assume that $g(X)$ is very small in the sense that

$$\frac{\|g(X)\|}{\|X\|} \longrightarrow 0 \quad \text{as} \quad \|X\| \longrightarrow 0 .$$

~~The~~ in this case, the system $X' = AX + g(X)$ is said to be locally linear. Note that $X_* = 0$ is the isolated critical point.

How to study stability property of $X_* = 0$?

$\boxed{\text{Case 1}}$. Assume that all the eigenvalues of A are either strictly negative or complex with negative real parts. We know that we can have

$$\|e^{At}\| \leq C_0 e^{-\alpha t}, \text{ for all } t \geq 0.$$

Here C_0 is some big constant, and $\alpha > 0$.

Now, we shall show that $X_* = 0$ is ~~stab~~ asymptotically stable, that is, we need to show that there is a small number $\delta > 0$ so that whenever $\|X_0\| < \delta$, the solution $X(t)$ must satisfy

$$\|X(t)\| \leq C_0 2 \|X_0\| e^{-\alpha t}, \text{ for all } t \geq 0.$$

Indeed, we assume that the above inequality is not true, that is we can assume that there is some fixed time T so that

$$X(T) = 2\|X_0\| e^{-\alpha T} C_0$$

and

$$X(t) \leq 2\|X_0\| e^{-\alpha t} C_0, \text{ for all time } 0 \leq t \leq T.$$

Now since $X(t)$ is a solution of

$$X'(t) = AX + g(X)$$

where we assume that

$$\|g(X)\| \leq C_1 \|X\|^2 .$$

By the method of Variations-of-Parameters, we know that

$$X(t) = e^{At} X_0 + e^{At} \int_0^t e^{-As} g(X(s))\, ds$$

$$= e^{At} X_0 + \int_0^t e^{A(t-s)} g(X(s))\, ds$$

Therefore,

$$\|X(t)\| \leq \|e^{At}\| \|X_0\| + \int_0^t \|e^{A(t-s)}\| \|g(X(s))\| \, ds$$

$$\leq C_0 e^{-\alpha t} \|X_0\| + \int_0^t C_0 e^{-\alpha(t-s)} C_1 \|X(s)\|^2 \, ds$$

$$\leq C_0 e^{-\alpha t} \|X_0\| + C_0 C_1 \int_0^t e^{-\alpha(t-s)} \left[2\|X_0\| e^{-\alpha s} C_0 \right]^2 ds$$

$$\leq C_0 e^{-\alpha t} \|X_0\| + 4 C_0^3 C_1 \|X_0\|^2 \int_0^t e^{-\alpha t} e^{-\alpha s} \, ds$$

$$\leq C_0 e^{-\alpha t} \|X_0\| + \frac{4 C_0^3 C_1 \|X_0\|^2}{\alpha} e^{-\alpha t}$$

$$= C_0 e^{-\alpha t} \|X_0\| \left[1 + \frac{4 C_0^2 C_1}{\alpha} \|X_0\| \right]$$

for all t: $0 \leq t \leq T$.

Now if we chose δ small enough so that

$$\frac{4C_0^2 C_1}{\alpha}\,\delta < \frac{1}{2},$$

then we just showed that

$$\begin{aligned}
\|X(t)\| &\le C_0 e^{-\alpha t}\|X_0\|\left[1 + \frac{4C_0^2 C_1}{\alpha}\|X_0\|\right] \\
&\le C_0 e^{-\alpha t}\|X_0\|\left[1 + \frac{4C_0^2 C_1}{\alpha}\,\delta\right] \\
&\le C_0 e^{-\alpha t}\|X_0\|\left(1 + \frac{1}{2}\right) \\
&\le \frac{3C_0}{2} e^{-\alpha t}\|X_0\|
\end{aligned}$$

but at $t = T$, we assumed that

$$X(T) = 2C_0 e^{-\alpha t}\|X_0\| < \frac{3}{2} C_0 e^{-\alpha t}\|X_0\|$$

which is a contradiction since $2 > 3/2$.

This proves that for all time $t \ge 0$, we always have

$$\|X(t)\| \le 2C_0\|X_0\| e^{-\alpha t}$$

Hence, $X_* = 0$ is asymptotically stable.

Case 3: If one of the eigenvalues of A is either positive or complex with positive real part, we show that the critical point $X_0 = 0$ is unstable. ~~The~~

(we'll prove this next time).