

# RESEARCH STATEMENT

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**Overview.** My research centers on the study of steady water waves, particularly those that are periodic, and exhibit a heterogeneous density distribution. These waves are ubiquitous in nature and a classical object of interest. Nonetheless, significant progress in describing them mathematically has only been made quite recently, and a number of fundamental questions remain largely open. My main contribution has been to establish the existence of global continua of solutions with variable density, both with or without surface tension. In what follows I give a description of the problem, summarize my results and mention some future projects.

## 1. BACKGROUND

The basic water wave problem is the following. Consider a large body of water that is bounded from below by an impermeable solid boundary, and above by the atmosphere (which is usually taken to be physical vacuum.) The evolution of the flow is assumed to be governed by the incompressible Euler equations. Physically this is equivalent to requiring that (i) the fluid is inviscid, (ii) the fluid as a whole conserves mass and (iii) each particle in the fluid conserves mass and momentum.

There are two features that make the water wave problem especially difficult. First, it is a free boundary problem. This means that the fluid domain—the set in space inhabited by the water where the governing equations apply—is not known *a priori*, and indeed typically changes with time. Second, we are approaching the Euler equations in their full nonlinearity, i.e. we do not use further length scale arguments to reduce the problem to a model equation, e.g. the Korteweg–de Vries equation or the Camassa–Holm equation.

Steady waves are an important special class of solutions to the water wave problem. A wave is called *steady* or *traveling* provided that there exists a  $c \in \mathbb{R}$  such that, when viewed in a coordinate frame moving with constant speed  $c$ , there is no time dependence. In particular, my research deals with traveling waves that are periodic and two-dimensional, meaning that the motion is identical along any line that runs parallel to a crest.

For definiteness, let us now formulate the problem mathematically. Suppose the  $x$ -axis points in the direction the wave is propagating and that the water lies above a flat bed at  $y = -d$ . We also assume that the interface between the atmosphere and the water is given by the graph of a function,  $\eta = \eta(t, x)$ . Then the fluid domain at each time  $t$  is given by

$$D_\eta := \{(x, y) : x \in \mathbb{R}, -d < y < \eta(t, x)\}.$$

**Problem.** For a fixed wave speed  $c > 0$  and wave length  $L > 0$ , find a smooth (relative) velocity field  $\mathbf{u} = (u - c, v)$ , density function  $\rho$ , pressure  $P$  and free surface  $\eta$  satisfying the steady Euler equations:

$$(W-W1) \quad \nabla \cdot \mathbf{u} = 0, \quad (\mathbf{u} \cdot \nabla)\rho = 0, \quad \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla P + \rho \mathbf{g} = 0, \quad \text{in } D_\eta$$

along with the boundary conditions,

$$(W-W2) \quad v = (u - c)\eta_x \quad \text{and} \quad P = 0, \quad \text{on } y = \eta(x), \quad v = 0, \quad \text{on } y = -d.$$

Here  $\mathbf{g} := (0, g)$  accounts for the force of gravity and all the functions are assumed to be  $L$ -periodic in the  $x$ -direction.

The first set of equations, (W-W1), gives conservation of total mass (incompressibility), conservation of fluid particle mass and conservation of fluid particle momentum, respectively. The lack of any time

dependence is due to having adopted a moving frame of reference via the change of variables  $(x, y) \mapsto (x - ct, y)$ . Note also that the boundary conditions are assuming no surface tension. I address the surface tension case in section 3.

## 2. STRATIFIED STEADY PERIODIC WATER WAVES

The central topic of my research is stratified water waves, i.e., waves where the density is not assumed to be constant throughout the fluid. Stratification is a common feature of ocean waves, where the presence of salinity or temperature gradients can produce a substantial heterogeneity in the fluid. The pronounced effects that may accompany even a moderate density variation have earned stratified flows a great deal of scholarly attention, particularly in the realm of geophysical fluid dynamics.

The vast majority of the literature on stratified water waves considers the case of internal waves, wherein one studies a traveling wave in the interior of the fluid domain whose motion is driven entirely by the heterogeneity of the density. The typical setup is to consider a two-dimension stratified fluid (or several layered homogeneous fluids) confined between two impermeable horizontal boundaries. Some substantial efforts in this genre include [1, 3, 22], among many others.

The full steady water wave problem—that is, with a free surface, and working from the incompressible Euler equations—has mainly been studied in the context of so-called solitary waves which limit to a constant height at  $x = \pm\infty$ . On the other hand, results concerning stratified and steady periodic waves have been largely restricted to various small amplitude regimes. The first substantial results are due to Dubreil-Jacotin in 1937 [14]. She was able to analyze the fixed boundary problem, i.e., existence of traveling periodic stratified waves between two horizontal plates. Linearizing around these solutions, she gave a system of integral equations governing small amplitude, traveling stratified gravity waves, from which she was able to produce a class of solutions of that type under certain assumptions. Yanowitch [28] later obtained similar results by using a variational argument, again assuming small amplitude. In 1984, Turner [23] succeeded in constructing solutions to the steady stratified water wave problem, both for solitary waves and periodic, by using a variational argument. This means, roughly speaking, that he proved that the solutions of the steady problem can be viewed as minimizers of a certain energy functional, which can then be exploited to prove existence through the direct method of the Calculus of Variations.

One of the main technical obstacles in the stratified water waves setting is that one cannot safely assume that the waves have zero vorticity. The vorticity is defined to be the curl of the velocity field (in the case of two-dimensional waves, the vorticity is defined to be  $z$ -component of the curl.) From (W-W1) we have that  $\nabla \cdot \mathbf{u} = 0$ , thus, if the fluid is also irrotational (i.e. the vorticity is zero throughout), then there exists a pair of conjugate harmonic functions that totally describe the flow. From that perspective, taking vorticity to be nonzero deprives us of some very powerful complex analytic machinery.

Unsurprisingly, therefore, the development of a substantial existence theory for steady rotational waves (those with nonzero vorticity) is quite recent, beginning with the work of Constantin and Strauss (cf. [12]). They were able to construct a large class of rotational, constant density, periodic water waves under very weak assumptions relating the volumetric mass flux and the strength of the vorticity.

In attacking the steady stratified problem, my approach is to reformulate (W-W1)–(W-W2) as a scalar equation and then apply a bifurcation argument to produce global continua of solutions. This is a similar strategy to that employed in [12], but with some major differences in execution. The introduction of stratification brings with it a nonlocal operator term in the scalar equation. Since bifurcation theory relies fundamentally on having a thorough understanding of the linearized problem, as well as strong compactness properties, the presence of a nonlocal operator presents a serious complication.

On the other hand, my method has some substantial benefits. For instance, in the resulting theory there are no mathematical restrictions placed upon the wave speed or wavelength (though clearly there are physical constraints.) Moreover, I need only some inequality between the density variation, specific energy, and volumetric mass flux in order to conclude existence of a global continuum of solutions.

To state my main results, first we need some additional concepts. As the vector space  $\mathbf{u}$  is divergence free (and  $\rho$  is transported), we may introduce a *pseudo-stream function*  $\psi = \psi(x, y)$ , defined uniquely up to a constant by

$$\psi_x = -\sqrt{\rho}v, \quad \psi_y = \sqrt{\rho}(u - c).$$

One can check that  $\psi$  is a stream function in the usual sense, i.e., its gradient is orthogonal to the velocity field in the moving frame at each point in the fluid domain. Moreover, the free surface and flat bed are each level sets of  $\psi$ . For definiteness we choose  $\psi \equiv 0$  on the free boundary, so that  $\psi \equiv -p_0$  on  $y = -d$ , where  $p_0$  is the (relative) pseudo-volumetric mass flux. That is, it gives the rate of fluid flowing through any vertical line with respect to the transformed vector field  $\sqrt{\rho}\mathbf{u}$  and in the moving frame.

Since  $\rho$  is transported, it must be constant on the level sets of the pseudo-stream function, which are called *streamlines*. Hence, we may define the *streamline density function* to be the density viewed as a function of  $-\psi$  (the minus sign is simply for technical reasons.) I will assume that the density is nondecreasing as depth increases, which means that the streamline density function should be nonincreasing. This is a standard assumption for stratified waves. Indeed, hydrodynamic stability requires that the density be monotonically increasing with depth. In the literature this type of density distribution is often referred to as *stable stratification*.

By Bernoulli's theorem, the quantity

$$E := P + \frac{\rho}{2} \left( (u - c)^2 + v^2 \right) + g\rho y$$

is constant along streamlines. In the case of an inviscid fluid,  $E$  represents the energy of the fluid particle at  $(x, y)$ . Under the assumption that  $u < c$  throughout the fluid, then, there exists a function  $\beta$  such that

$$\frac{dE}{d\psi}(x, y) = \beta(\psi(x, y)).$$

For want of a better name, I refer to  $\beta$  as the *Bernoulli function* corresponding to the flow. Physically it describes the variation of specific energy as a function of the streamlines.

The solutions I construct are classical, that is, they satisfy the equation in the strong sense. In particular, they are of class  $C^{k+\alpha}(\overline{D_\eta})$ , for some  $\alpha \in (0, 1)$  and various values of  $k \geq 1$ . A complete account of the regularity of the solutions (and the required regularity for the given data) is outlined in [27], but for simplicity I refer to them as smooth here.

With the notation in place, I now summarize the main result which says that there exists a global continuum of smooth, steady, periodic solutions to the stratified water wave problem.

**Theorem 1** ([27]). *Fix any pseudo-volumetric mass flux  $p_0$  and let smooth Bernoulli function  $\beta$  and streamline density function  $\rho$  be given such that a certain size condition holds. Assume the streamline density function  $\rho$  is nonincreasing.*

*Consider traveling periodic solutions to the stratified water wave problem of relative pseudomass flux  $p_0$ , Bernoulli function  $\beta$ , and streamline density function  $\rho$  such that  $u < c$  throughout the fluid. There exists a connected set  $C$  of smooth solutions  $(u, v, \rho, \eta)$  with the following properties.*

- (a)  $C$  contains a laminar flow (with a flat surface  $\eta \equiv 0$  and all streamlines parallel to the bed).
- (b) Either (i)  $C$  contains more than one distinct laminar flow, or, along some sequence  $(u_n, v_n, \rho_n, \eta_n) \in C$ , we have

$$(ii) \max u_n \uparrow c, \quad \text{or} \quad (iii) \min u_n \downarrow -\infty.$$

*Remark.* The size condition above arises in the local theory as a means of guaranteeing the existence of a minimizer to a particular Sturm–Liouville problem and is satisfied, for instance, if  $|p_0|$  and  $\|\rho'\|_{L^\infty}$  are small. In fact, it can be significantly weakened. It should be noted that the nodal properties of the solutions along  $C$  are also known. For instance, I prove that the profile is monotonic with a single crest and trough per period. See Theorem 1.1 of [27] for a more precise statement.

The proof of Theorem 1 inherits its overarching structure from the argument of [12]. There are, however, several significant differences that come with the introduction of stratification. Using  $\psi$  to change coordinates, I prove that there is an equivalent formulation of (W-W1)–(W-W2) in terms of a scalar, nonlinear boundary value problem whose solution describes the height above the flat bed. In the process, the fluid domain is mapped to a fixed, bounded rectangle in  $\mathbb{R}^2$ . The new problem takes the form

$$\mathcal{G}(Q, h) + \mathcal{S}(h) = 0,$$

where  $Q$  is a real-valued parameter,  $\mathcal{G}$  is a quasi-linear elliptic differential operator, and  $\mathcal{S}$  is an integro-differential operator. It is the latter term,  $\mathcal{S}$ — which corresponds to the stratification— that requires the utmost care. Note that  $\mathcal{S}$  is a nonlocal operator, which presents a significant technical challenge.

The continuum  $C$  is constructed by means of a global bifurcation argument, with a 1-parameter family of laminar flows playing the role of the trivial solutions. Analyzing the linearized problem along the curve of laminar flows, I prove that there exists a simple generalized eigenvalue from which bifurcates a local curve of small amplitude solutions.

To continue the local curve globally, I employ a degree theoretic argument in the spirit of Rabinowitz (cf. [21]). However, this requires strong *a priori* estimates in order to guarantee that the operators have the necessary compactness. Towards that end, I introduce the method of “freezing” the stratification term, which is motivated by the following observation. As we are working in spaces of Hölder continuous functions defined on a bounded domain, the integral parts of  $\mathcal{S}$  can be easily controlled. Therefore, if we replace the integral part of  $\mathcal{S}$  with a real constant, the resulting frozen operator is elliptic and, via classical Schauder theory, the desired compactness properties are immediate for the frozen problem. Exploiting the fact that the integral parts of  $\mathcal{S}$  can be estimated, I can then argue back to conclude that the compactness properties hold for the full problem (that is, with the unfrozen operator.) This technique proves extremely useful in the global analysis and eventually leads to Theorem 1.

### 3. STEADY PERIODIC WATER WAVES WITH SURFACE TENSION

Another important class of water waves are those driven by surface tension. These arise, for instance, when wind first begins to blow over a quiescent body of water. If surface tension and gravity are the only restorative forces acting on the water, the wave is called a *capillary-gravity wave*.

Mathematically, the steady water wave problem with surface tension differs from that considered in the previous section in only one respect: on the free surface, the pressure is not constant, but proportional to the mean curvature. That is,

$$P = -\sigma\eta_{xx}(1 + \eta_x^2)^{-3/2}, \quad \text{on } y = \eta(x),$$

where  $\sigma > 0$ , called the coefficient of surface tension, is given.

Briefly, my results for the surface tension case come in five parts, the first two of which relate to the phenomenon of double (local) bifurcation. It is well known in the theory of irrotational capillary-gravity waves that, when the coefficient of surface tension resides in a certain distinguished set, the eigenvalue of the linearized problem at the point of bifurcation will be of multiplicity two and is otherwise simple. Wahlén observed the presence of double bifurcation points in the rotational case and proved the existence of local bifurcation curves in certain regimes (cf. [24]). These results, however, are incomplete in the sense that they do not necessarily describe the entire solution set near the bifurcation point. The first of my contributions is to refine the local bifurcation diagram obtained in [24] at double bifurcation points. In particular, I confirm the existence of curves of so-called mixed solutions conjectured by Wahlén. Secondly, using the bifurcation theory for analytic operators pioneered by Dancer (cf. [13]), I continue the local curves to obtain global curves of solutions. The application of these techniques to steady rotational waves is in and of itself novel and pays some additional dividends. Most notably, it leads to my third result, which is proving that the solution continua are path-connected, a fact that was unknown even for the constant density, gravity wave case originally considered in [12]. Fourthly, for the case of simple local bifurcation, I also use the degree theoretic method to obtain an alternate global theorem. Finally, employing the machinery I outlined in

the previous section, all of these results are obtained for fluids with general (stable) stratification, which is entirely new for both the local bifurcation and global bifurcation theory.

For completeness, let me now give a summary of the theorems. For more precise, detailed statements, see Theorem 1.2, Theorem 1.4 and Theorem 1.6 of [26].

The first theorem states that there exists a global bifurcation curve of solutions when the eigenvalue of the linearized problem is simple. This is known to be the case, for example, when the coefficient of surface tension is sufficiently large.

**Theorem 2** ([26]). *Let the volumetric mass flux  $p_0$  and coefficient of surface tension  $\sigma$  be given. Moreover, let smooth Bernoulli function  $\beta$  and streamline density function  $\rho$  be given such that a certain size condition holds. If  $\sigma$  lies in a certain set,  $\Sigma \subset (0, \infty)$ , then the following statements holds.*

- (a) *There exists a connected set  $C$  of solutions  $(u, v, \rho, \eta)$  of the stratified steady water wave problem with surface tension  $\sigma$ . The set  $C$ , moreover, exhibits properties (a) and (b) of Theorem 1.*
- (b) *Furthermore, there exists a path-connected subset  $\mathcal{K} \subset C$  such that*
  - *$\mathcal{K}$  admits a global, locally injective, continuous parameterization with a locally injective  $C^1$ -reparameterization.*
  - *Either  $\mathcal{K}$  is a closed loop, or it unbounded in the sense that alternative (b)(ii) or (b)(iii) of Theorem 1 must occur along  $\mathcal{K}$ .*

*Remark.* The set  $\Sigma$  contains a half line,  $(\sigma_c, \infty)$ , and its complement is known to be countable in some cases. The size condition is satisfied, in particular, if  $\sigma$  is large, and or if  $|p_0|, \|\rho'\|_{L^\infty}$  are small. In fact, it suffices for  $\rho$  and  $p_0$  to satisfy the hypotheses of Theorem 1.

Theorem 2 corresponds to the case when the local bifurcation is simple, hence its similarity to Theorem 1. The next two results deal with the double bifurcation scenario. The first of these describes the structure of the bifurcation curves near the point of bifurcation.

**Theorem 3** ([26]). *In contrast to Theorem 2, suppose  $\sigma \notin \Sigma$  but that certain non-degeneracy conditions hold. Then there exists a set of four  $C^1$ -curves,  $\{C_{i,\text{loc}}\}_{i=1}^4$ , of small amplitude solutions with the following properties.*

- *Each  $(u, v, \rho, \eta, Q) \in \bigcup_i C_{i,\text{loc}}$  has speed  $c$ , relative mass flux  $p_0$  and satisfies  $u < c$  throughout the fluid; moreover,*
- *each  $(u, v, \rho, \eta, Q) \in C_{1,\text{loc}}$ , has minimal period  $L$  in  $x$ ; each solution in  $(u, v, \rho, \eta, Q) \in C_{2,\text{loc}}$  has minimal period  $L/n$ , for some  $n \geq 3$ .*
- *The curves  $\{C_{i,\text{loc}}\}$  each contain precisely one laminar flow.*

The curves  $C_{\text{loc},1}$  and  $C_{\text{loc},2}$  are pure solutions, while the remaining two curves consist of mixed solutions. The proof of Theorem 3 comes from an application of the classical Lyapunov-Schmidt procedure.

Using global continuation theory for analytic operators, the curves  $C_{i,\text{loc}}$  are extended globally, yielding the following theorem.

**Theorem 4** ([26]). *Assume the hypotheses of Theorem 3. There exists a global path-connected set of solutions  $\mathcal{K}$  that extends the local curves in the following sense:  $\mathcal{K} = \bigcup_i \mathcal{K}_i$ , where  $C_{i,\text{loc}} \subset \mathcal{K}_i$ , and each  $\mathcal{K}_i$  satisfies (b) of Theorem 2, for  $i = 1, \dots, 4$ .*

#### 4. FUTURE PROJECTS

The natural question to ask, in view of the results of the previous two sections, is whether or not there are other steady wave phenomena that can be approached with the same methodology. One promising problem I plan to consider is that of stratified steady quasi-geostrophic motion on a sphere, which serves as a model for large scale ocean waves. The governing equations in this case can be formulated in terms of an elliptic-type PDE for the stream function, reminiscent of that in section 2 (cf. [20]). This means, roughly speaking, that

the basic components required for the program of [27] are in place. I therefore hope to be able to construct a continuum of solutions along the lines of that in Theorem 1.

Another important problem is that of a discontinuously stratified fluid, as one would find if multiple fluids of different densities are layered vertically. Such waves has been studied extensively in the context of internal waves, but existence results of the type described in sections 2 and 3 have not yet been established. In the ocean, one often finds that the density distribution is essentially constant outside of small regions (called pycnoclines) where it increases rapidly with depth. Understanding discontinuously stratified fluids is thus quite important from the perspective of applications.

Even for the models of stratified steady waves I have already considered, there are many open questions. Much is left to be understood about the qualitative effects of stratification on the flow. It is known, for example, that two-dimensional, constant density, steady, periodic rotational waves are necessarily symmetric if they have a monotonic profile (cf. [11]). Horizontally symmetric here means that there exists a vertical axis of symmetry for each period. I have already provided a partial answer to this question in the stratified case:

**Theorem 5** ([25]). *Under certain size conditions, every stratified steady periodic water wave whose profile is monotonic from crest to trough is symmetric. In particular, this is the case if the wave is sufficiently close to laminar and  $\sqrt{\rho}(c - u)$  is bounded sufficiently away from zero.*

It is not clear, however, if the size assumptions are an artifact of the analysis or a genuinely physical property of stratified waves. A definite answer— in either direction— would be very interesting.

A more long term project is to try to understand the stability (or instability) of stratified steady waves. That is, to determine whether the solutions I have constructed will persist if they are perturbed, and, if not, how precisely does the instability manifest itself. In particular, I plan to approach this problem from a variational angle. Since Turner’s initial work, much progress has been made on understanding the variational characteristics of steady waves, particularly irrotational (constant density) waves. Quite recently, Buffoni, Toland, and Séré developed a general theory for proving the existence of solutions to quasilinear problems that can be expressed in variational terms (cf. [10]). Using some ideas from [23], and a formulation of the steady, irrotational water wave problem due to Babenko (cf. [4, 5]), these authors leveraged the abstract theory to prove existence in a variety of physical regimes (cf. [9, 7, 10]). Moreover, in the surface tension setting, Buffoni has shown that the solutions constructed by these methods are energetically (conditionally) stable (cf. [6, 8]). This was achieved via direct minimization with a concentrated-compactness argument in the vein of Lions (cf. [17, 18]). Similarly, Mielke [19] was able to prove the energetic stability of the set of solitary water waves with surface tension earlier constructed by Amick and Kirchgässner (cf. [2]). In contrast to the approach of Buffoni et al., Mielke established certain spectral properties of the second variation of the functional, which allowed him to apply some machinery of Grillakis, Shatah and Strauss (cf. [15, 16]) to prove his result. Given the substantial progress seen in the past several years, I hope to reexamine the question of (energetic) stability for stratified, steady water waves.

Finally, I hope to apply my methodology to another phenomena in geophysical fluid dynamics: wind waves. It is well-known empirically that the shear force of the wind can create waves in the ocean (the capillary-gravity waves considered in section 3 are one example), but a rigorous mathematical description of the process has so far been elusive. Since the wind is one of the primary reasons for the presence of vorticity in ocean waves, wind waves are a natural object to study in view of Theorem 1–Theorem 4. Under certain physical assumptions, one can describe the behavior of steady wind waves via the homogeneous model, which takes the form of a scalar elliptic equation for the (geostrophic) stream function (cf. [20]). I therefore hope to apply the machinery described in the sections 2 and 3 in order to construct solutions to the wind-driven water wave problem.

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