

MA 0200: Midterm II Solutions

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Question 1. Find the maximal value of the function $f(x, y, z) := x^3 + y^3 + z^2$ on the sphere $x^2 + y^2 + z^2 = a^2$.

Solution. Let $g(x, y, z) := x^2 + y^2 + z^2 - a^2$. We therefore wish to maximize the function f subject to the constraint $g = 0$.

We compute

$$\nabla g(x, y, z) = \langle 2x, 2y, 2z \rangle, \quad \nabla f(x, y, z) = \langle 3x^2, 3y^2, 2z \rangle.$$

The only point where ∇g vanishes is the origin, which does not lie on the sphere so it can be safely ignored.

Now we seek points (x, y, z) and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z).$$

Writing this in components reveals

$$3x^2 = 2\lambda x, \tag{1}$$

$$3y^2 = 2\lambda y, \tag{2}$$

$$2z = 2\lambda z. \tag{3}$$

If $z \neq 0$, then (3) implies $\lambda = 1$. It follows that

$$\frac{2}{3}x = x^2, \quad \frac{2}{3}y = y^2.$$

There are several possibilities. We have that $x = 0$ or $x = 2/3$. Likewise, $y = 0$ or $y = 2/3$. From the fact that $g(x, y, z) = 0$, this shows us that

$$z^2 = a^2 - x^2 - y^2 = \begin{cases} a^2, & \text{if } (x, y) = (0, 0), \\ a^2 - 4/9, & \text{if } (x, y) = (0, 2/3) \text{ or } (2/3, 0), \\ a^2 - 8/9, & \text{if } (x, y) = (2/3, 2/3). \end{cases}$$

Notice that depending on the size of a , some of these solutions may not exist (since $z^2 > 0$, if $a^2 < 4/9$, everything but $z = \pm a$ is ruled out.) Evaluating f at these points yields the following

$$\begin{aligned} f(0, 0, \pm a) &= a^2, \\ f(2/3, 0, \pm \sqrt{a^2 - 4/9}) &= \frac{8}{27} + a^2 - \frac{4}{9}, & (\text{if } a^2 > 4/9), \\ f(0, 2/3, \pm \sqrt{a^2 - 4/9}) &= \frac{8}{27} + a^2 - \frac{4}{9}, & (\text{if } a^2 > 4/9), \\ f(2/3, 2/3, \pm \sqrt{a^2 - 8/9}) &= \frac{16}{27} + a^2 - \frac{8}{9}, & (\text{if } a^2 > 8/9). \end{aligned}$$

If, on the other hand, $z = 0$, then λ need not be equal to 1. Notice that $\lambda \neq 0$, since this would imply $(x, y, z) = 0$, which does not satisfy the constraint. Let's reconsider (1)–(2). We have that either $x = 0$ or $x = 2\lambda/3$. Likewise, either $y = 0$ or $y = 2\lambda/3$. In other words, if x and y are both nonzero, then they are equal (both are $2\lambda/3$.) Recall, though, that we are assuming $z = 0$. By the constraint this implies $x^2 + y^2 = a^2$. The possibilities therefore reduce to

$$(x, y, z) = (0, \pm a, 0), (\pm a, 0, 0), \text{ or } \left(\pm \frac{a}{\sqrt{2}}, \pm \frac{a}{\sqrt{2}}, 0 \right).$$

Evaluating f at these points gives

$$\begin{aligned} f(0, \pm a, 0) &= \pm a^3, \\ f(\pm a, 0, 0) &= \pm a^3, \\ f(a/\sqrt{2}, a/\sqrt{2}, 0) &= \frac{a^3}{\sqrt{2}}, \\ f(-a/\sqrt{2}, -a/\sqrt{2}, 0) &= -\frac{a^3}{\sqrt{2}}, \\ f(-a/\sqrt{2}, a/\sqrt{2}, 0) &= 0, \\ f(a/\sqrt{2}, -a/\sqrt{2}, 0) &= 0. \end{aligned}$$

Bringing all the facts together, we find that the minimum value of f is $-a^3$. The maximum value depends on the size of a : if $a < 1$, then the maximum is a^2 ; if $a \geq 1$, then the maximum is a^3 . \square

Question 2. Compute the integral $\int_0^{\sqrt{\pi}} \int_{x/2}^{\sqrt{\pi}/2} \cos(y^2) dy dx$.

Solution. The trick here is to reverse the order of integration. Notice that we are integrating $dydx$. Thus this is a vertically simple domain where $0 \leq x \leq \sqrt{\pi}$, and $y_1(x) \leq y \leq y_2(x)$, with $y_1(x) = x/2$, $y_2(x) = \sqrt{\pi}/2$. The domain is a triangular region, and is therefore horizontally simple as well; we can describe it as $0 \leq y \leq \sqrt{\pi}/2$, and $0 \leq x \leq 2y$.

By Fubini–Tonelli,

$$\begin{aligned} \int_0^{\sqrt{\pi}} \int_{x/2}^{\sqrt{\pi}/2} \cos(y^2) dy dx &= \int_0^{\sqrt{\pi}/2} \int_0^{2y} \cos(y^2) dx dy \\ &= \int_0^{\sqrt{\pi}/2} (x \cos(y^2)) \Big|_{x=0}^{x=2y} dy \\ &= \int_0^{\sqrt{\pi}/2} 2y \cos(y^2) dy \\ &= \sin(y^2) \Big|_0^{\sqrt{\pi}/2} = \sin(\pi/4) = \sqrt{2}/2. \quad \square \end{aligned}$$

Question 3. Consider the metal plate corresponding to the region R in the plane give by the inequalities

$$x, y \geq 0, \quad x^2 + y^2 \leq a^2, \quad (a > 0).$$

Assuming that the density of the plate δ is constant, find the coordinates of its centroid.

Solution. First we must compute the mass. The region R is simply the portion of the disc of radius a that lies in the first quadrant. In fact, this domain is a polar rectangle ((r, θ) in $[0, a] \times [0, \pi/2]$), so it's convenient to do everything in polar coordinates. (You can also do it in Cartesian.) Thus

$$\begin{aligned} m &= \iint_R \delta \, dA \\ &= \delta \int_0^{\pi/2} \int_0^a r \, dr \, d\theta \\ &= \frac{\delta}{2} \int_0^{\pi/2} (r^2) \Big|_{r=0}^{r=a} d\theta = \frac{a^2 \delta \pi}{4}. \end{aligned}$$

Now we find the centroid.

$$\begin{aligned} \bar{x} &= \frac{1}{m} \iint_R x \delta \, dA \\ &= \frac{\delta}{m} \int_0^{\pi/2} \int_0^a (r \cos \theta) r \, dr \, d\theta \\ &= \frac{\delta}{3m} \int_0^{\pi/2} (r^3 \cos \theta) \Big|_{r=0}^{r=a} d\theta \\ &= \frac{a^3 \delta}{3m} = \frac{4a}{3\pi}. \end{aligned}$$

Actually, by symmetry we can immediately conclude that $\bar{y} = \bar{x}$. If you don't believe that, you can verify it by a computation along the lines of the one above. We omit it in the interests of brevity. \square

Question 4. Consider the function $f(x, y) := x^3 + y^3 + 3xy + 3$.

(a) Find the directional derivative of f at the point $(0, 1)$ in the direction of the unit vector $\mathbf{u} := \frac{1}{\sqrt{2}}\langle 1, 1 \rangle$.

Solution. By definition $D_{\mathbf{u}}f = \mathbf{u} \cdot \nabla f$. But,

$$\nabla f = \langle 3x^2 + 3y, 3y^2 + 3x \rangle,$$

and thus

$$\mathbf{u} \cdot \nabla f = \frac{1}{\sqrt{2}} (3x^2 + 3y + 3y^2 + 3x).$$

Evaluating the above quantity at $(x, y) = (0, 1)$, we conclude $D_{\mathbf{u}}f(0, 1) = \frac{6}{\sqrt{2}} = 3\sqrt{2}$. \square

(b) Consider the equation $f(x, y) = 4$. Verify that $(0, 1)$ is a solution. Show that near the point $(0, 1)$, one can solve the above equation for y as a function of x and find the value of dy/dx at $x = 0$.

Solution. To see that $(x, y) = (0, 1)$ is a solution, just observe

$$f(0, 1) = 0 + 1 + 0 + 3 = 4.$$

The fact that y can be solved in terms of x is a consequence of the Implicit Function Theorem. All we need to do, therefore, is verify the hypotheses, namely that $\partial f/\partial y \neq 0$ at $(0, 1)$. Since

$$\frac{\partial f}{\partial y} = 3y^2 + 3x,$$

this is obviously the case. We conclude that there exists some function $g = g(x)$ such that $f(x, g(x)) = 4$, for all x sufficiently close to 0. By definition, the value of $\frac{dy}{dx}$ at $x = 0$, is nothing but $\frac{dg}{dx}(0)$. If we simply differentiate the equation $f(x, g(x)) = 4$, we get (via the chain rule) that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dg}{dx} = 0,$$

hence

$$\frac{dg}{dx}(0) = -\frac{\partial f/\partial x}{\partial f/\partial y} \Big|_{(x,y)=(0,1)} = -\left(\frac{3x^2 + 3y}{3y^2 + 3x}\right) \Big|_{(x,y)=(0,1)} = -1.$$

(For full credit, you do not need to rederive this expression. It's OK to simply use the formula for g in the last line.) \square

(c) *Find and classify the critical points of the function f .*

Solution. The critical points are those where ∇f vanishes, or ceases to be defined. Since f is everywhere differentiable, we only care about the former case. Referring back to our work in part (a), we see that $\nabla f(x, y) = 0$ provided that $(x, y) = (0, 0)$ or $x^2 = -y$ and $y^2 = -x$. In the second case, we see that $y^4 = x^2 = -y$. Hence, $y = -1$. But $-x = y^2$, which implies $x = -1$. Altogether then, the critical points are $(0, 0)$ and $(-1, -1)$.

We now compute the second-order partial derivative of f :

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = f_{yx} = 3.$$

The discriminant Δ is thus

$$\Delta = f_{xx}f_{yy} - (f_{xy})^2 = 36xy - 9.$$

At $(0, 0)$, $\Delta = -9 < 0$, so we have a saddle point. At $(-1, -1)$, $\Delta = 36 - 9 > 0$, which means that a (local) maximum or a (local) minimum occurs. To determine which we must consider $f_{xx}(-1)$. Since this quantity is negative, we infer that there is a local maximum. \square