Solution to the Midterm Exam. AM205, Nov 5, 2004.

1. (a) $\cos z$ has no singularity in finite part of z-plane. It is an entire function

(b) $\ln z$ has a branch point at $z=0$, It need a branch cut from $z=0$ to $z=\infty$. It's not entire

(c) $\frac{\ln(1-z)}{z}$. has a branch point $z=1$. It need a branch cut. It's not entire. $z=0$ is not a singularity since $\ln(1-z) = -z + z^2 - \cdots$ $\therefore$ $\frac{\ln(1-z)}{z} = -1$ as $z \to 0$.

(d) $\frac{\sin \pi z}{z^2-1}$, $z=\pm 1$ are not poles, since $\sin \pi z = \pm \pi (z \pm 1)$ It's an entire function

(e) $\frac{e^{iz}}{z}$ has a simple pole at $z=0$.

(f) $\frac{e^{iz}-e^{-iz}}{z} = 2i$ as $z \to 0$, It's an entire function.

2. We have

$$2b \frac{e^{2\phi/\alpha}+1}{e^{2\phi/\alpha}-1} = l$$

$$a = b\sqrt{\left(\frac{e^{2\phi/\alpha}+1}{e^{2\phi/\alpha}-1}\right)^2 - 1}$$

From these, we can solve for b an α ~~as~~ in terms of a and l and ϕ.

$$b = \frac{l}{2}\sqrt{1 - \frac{4a^2}{l^2}}$$

$$\alpha = \frac{2\phi}{\ln \frac{l+2b}{l-2b}}.$$

The solution $T(x,y) = \frac{\alpha}{2} \ln \frac{(x-b)^2+y^2}{(x+b)^2+y^2}$.

now $a=1$.

$l=4$. and $\phi=1$.

We get. $b=\sqrt{3}$.

$$\alpha = \frac{2}{\ln\frac{2+\sqrt{3}}{2-\sqrt{3}}} = 0.7593.$$

So: $T(x,y) = 0.3796 \ln \frac{(x-\sqrt{3})^2+y^2}{(x+\sqrt{3})^2+y^2}$.

3. $I = \int_0^{\infty} \frac{\cos\frac{\pi}{2}x \, dx}{x^2-1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos\frac{\pi}{2}x}{x^2-1} dx.$

At $x=\pm 1$, $\cos\frac{\pi}{2}x = 0$,

One can consider the Taylor series of $\cos\frac{\pi}{2}x$ ~~at x~~ at $x=\pm 1$

denote $x_0=\pm1$,

$$\cos\frac{\pi}{2}x = \underline{\cos\frac{\pi}{2}x_0} + \left(-\frac{\pi}{2}\sin\frac{\pi}{2}x_0\right)(x-x_0) + \left(-\frac{\pi^2}{4}\cos\frac{\pi}{2}x_0\right)(x-x_0)^2 + O((x-x_0)^3)$$

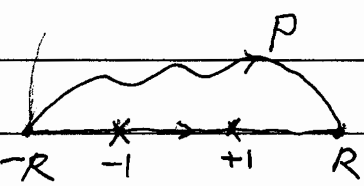
$$= -\frac{\pi}{2}\sin\frac{\pi}{2}x_0\,(x-x_0) + O((x-x_0)^3).$$

$$\frac{\cos\frac{\pi}{2}x}{x-x_0} = -\frac{\pi}{2}\sin\frac{\pi}{2}x_0 = \mp\frac{\pi}{2}$$

So $\frac{\cos\frac{\pi}{2}z}{z^2-1}$ is an entire function

$$\int_{-R}^{R} \frac{\cos\frac{\pi}{2}x}{x^2-1} dx = \int_{-R}^{R} \frac{\cos\frac{\pi}{2}z}{z^2-1} dz$$

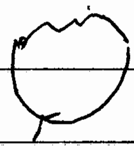
(along the path p, say)



$$= \frac{1}{2}\int_{-R}^{R} \frac{e^{i\frac{\pi}{2}z} + e^{-i\frac{\pi}{2}z}}{z^2-1} dz.$$

For $\int_{-R}^{R} \frac{e^{i\frac{\pi}{2}z}}{z^2-1}\,dz$, $= \oint \frac{e^{i\frac{\pi}{2}z}}{z^2-1}\,dz = 0.$ Jordan's lemma.

$$\int_{-R}^{R} \frac{e^{-i\frac{\pi}{2}z}}{z^2-1}\,dz = \oint_R \frac{e^{-i\frac{\pi}{2}z}}{z^2-1} = (-2\pi i)\left(\frac{e^{-i\frac{\pi}{2}}}{2} - \frac{e^{i\frac{\pi}{2}}}{2}\right) = -2\pi$$



Thus

$$\int_{-\infty}^{\infty} \frac{\cos\frac{\pi}{2}x}{x^2-1}\,dx = \frac{1}{2}\cdot(-2\pi) = -\pi$$

$$I = -\frac{\pi}{2}.$$