Key to Homework #3. AM205

20.3. (a) $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = 1$, $a_n = \frac{1}{\ln n}$

(b) $a_n = \frac{n!}{n^n}$, $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{\sqrt{2\pi n}\, n^n e^{-n}}{n^n}\right|^{1/n} = e^{-1}$

"Different from the answer given in the text"

(c) $a_n = n^{\ln n}$, $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} n^{\frac{1}{n}\ln n} = 1$.

(d) $a_n = \left(\frac{n+p}{n}\right)^{n^2}$

$\frac{1}{R} = \lim_{n\to\infty} \left(\frac{n+p}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{p}{n}\right)^n = e^p$.

20.5 To do study at ∞, let $z = \frac{1}{t}$ and consider around $t=0$.

(a) $\frac{1}{z-2}$ not singular at $z=0$.

$\frac{1}{\frac{1}{t}-2} = \frac{t}{1-2t}$. not singular at $t=0$ $(z\to\infty)$

(b) $\frac{1+z^3}{z^2}$ double ~~pode~~ at $z=0$

$\frac{1+\frac{1}{t^3}}{\frac{1}{t^2}} = \frac{t^3+1}{t}$ a simple pole at $t=0$ $(z=\infty)$

(c) $\sinh\frac{1}{z} = \frac{1}{2}(e^{\frac{1}{z}} - e^{-\frac{1}{z}})$ an essential singularity

$\sinh t$ not singular at $t=0$

(d) $\frac{e^z}{z^3}$ Triple pole at $z=0$. essential S. at $t=0$ $(z=\infty)$

(e) $\frac{z^{1/2}}{(1+z^2)^{1/2}}$. branch points $z=0$, $(z=\pm i)$ also at $t=0$.

20.6 (a) $\tan z$ zeros at $\sin z=0$, simple pole at $\cos z=0$

(b) $\left[\frac{(z-2)}{z^2}\right]\sin\frac{1}{1-z}$ zeros at $z=2$, and $z = 1-\frac{1}{n\pi}$, double pole $z=0$, essential s. at $z=1$,

(c) $e^{\frac{1}{z}}$ zero at $z=\infty$ essential S at $z=0$

(d) $\tan\frac{1}{z}$ zero at $z=\frac{1}{n\pi}$, simple pole at $\frac{1}{z} = \frac{2n+1}{2}\pi$. E.S at $z=0$

(e) $z^{2/3}$, branch point $z=0$ and $z=\infty$

2. The transformation is $\zeta = -i\,\frac{z+i}{z-i}$

For points on the unit circle in z-plane, i.e. $z = e^{i\theta}$, one find $\zeta = \frac{\cos\theta}{1-\sin\theta}$.
Thus the unit circle in z-plane is mapped into the real axis on ζ-plane.

Solution to the Laplace equation in upper half ζ-plane, with the boundary condition as given is

$$\phi(\xi,\eta) = \frac{1}{\pi}\left(\tan^{-1}\frac{\eta}{\xi-1} - \tan^{-1}\frac{\eta}{\xi+1}\right).$$

or just the imaginary part of the analytic function

$$W(\zeta) = \frac{1}{\pi}\ln\frac{\zeta-1}{\zeta+1}$$

Now the boundary value problem in z-plane is

$$W(\zeta(z)) = \tilde{W}(z) = \frac{1}{\pi}\ln\frac{-i\frac{z+i}{z-i} - 1}{-i\frac{z+i}{z-i}+1} = \frac{1}{\pi}\ln\frac{(1-i)(z-1)}{(1+i)(z+1)}$$

$$= \frac{1}{\pi}\ln\frac{1-i}{1+i} + \frac{1}{\pi}\ln\frac{z-1}{z+1}$$

$$= -\frac{i}{2} + \frac{1}{\pi}\ln\frac{z-1}{z+1}.$$

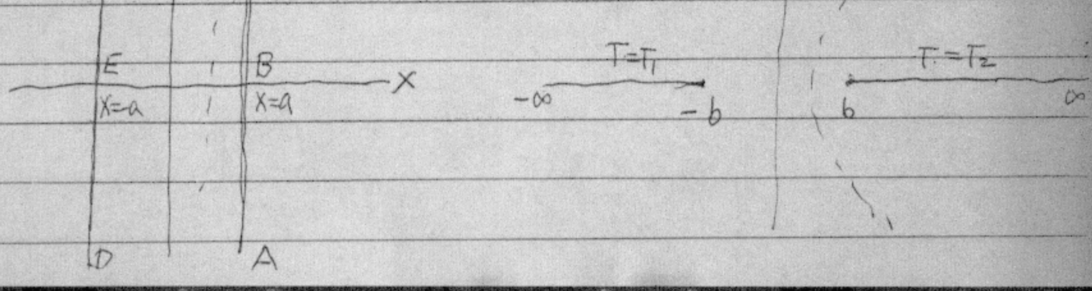
The imaginary part of $\tilde{W}(z)$ is $-\frac{1}{2} + \frac{1}{\pi}(\theta_1 - \theta_2)$.

where $z-1 = r_1 e^{i\theta_1}$, $z+1 = r_2 e^{i\theta_2}$, and $0 < \theta_1 < 2\pi$, $-\pi < \theta_2 < \pi$.

3. (a) Real part of the following analytic function solves the problem,

$$W(z) = \frac{T_2+T_1}{2} + \frac{T_2-T_1}{2}\,z.$$

(b)



$\zeta = \xi + i\eta$, $\quad z = x + iy$.

$$\xi + i\eta = b\sin\frac{\pi x}{2a}\cosh\frac{\pi}{2}\frac{y}{a} + ib\cos\frac{\pi}{2}\frac{x}{a}\sinh\frac{\pi}{2}\frac{y}{a}$$

$$\xi = b\sin\frac{\pi}{2}\frac{x}{a}\cosh\frac{\pi}{2}\frac{y}{a}$$

$$\eta = b\cos\frac{\pi}{2}\frac{x}{a}\sinh\frac{\pi}{2}\frac{y}{a}$$

$x = \text{const}$

$$\frac{\xi^2}{b^2\sin^2\frac{\pi}{2}\frac{x}{a}} - \frac{\eta^2}{b^2\cos^2\frac{\pi}{2}\frac{x}{a}} = 1. \qquad \text{hyperbolas}$$

$y = \text{const}$,

$$\frac{\xi^2}{b^2\cosh^2\frac{\pi}{2}\frac{y}{a}} + \frac{\eta^2}{b^2\sinh^2\frac{\pi}{2}\frac{y}{a}} = 1. \qquad \text{ellipses}$$

$x = -a$, $\Rightarrow \eta = 0$, $\xi = -b\cosh\frac{\pi}{2}\frac{y}{a}$

is mapped into $\eta = 0$, $\xi = (-b, -\infty)$.

$x = +a$, $\Rightarrow \eta = 0$, $\xi = b\cosh\frac{\pi}{2}\frac{y}{a}$.

is mapped into $\eta = 0$, $\xi = (b, +\infty)$.

(c) The boundary value problem in ζ-plane, is solved by the real part of

$$W(z) = W(z(\zeta)) = W(\zeta) = \frac{T_2+T_1}{2} + \frac{T_2-T_1}{2}\cdot\frac{2a}{\pi}\sin^{-1}\frac{\zeta}{b}$$

One can find the real part of $\sin^{-1}\frac{\xi+i\eta}{b}$ but the expression is rather complicated. However the solution in ζ-plane is more transparent from the mapping in the previous page. We have found each vertical line in z-plane is mapped into a hyperbola. Since $T = \text{const}$ on each vertical line in z-plane, so $T = \text{same constant}$ on the corresponding hyperbola in ζ-plane. For example, $T = \frac{T_1+T_2}{2}$ on $y = 0$, we have $T = \frac{T_1+T_2}{2}$ on $\eta = 0$ (the corresponding hyperbola).

4).(a) $\xi + i\eta = x + iy + \dfrac{1}{x+iy}$

$$= x + iy + \frac{x-iy}{x^2+y^2}$$

$$= x\left(1+\frac{1}{x^2+y^2}\right) + iy\left(1-\frac{1}{x^2+y^2}\right).$$

$$\begin{cases} \xi = x\left(1+\dfrac{1}{x^2+y^2}\right) \\ \eta = y\left(1-\dfrac{1}{x^2+y^2}\right) \end{cases}$$

Need the image of $\underline{x^2+y^2=a^2}$, which is

$$\frac{\xi^2}{\left(1+\frac{1}{a^2}\right)^2} + \frac{\eta^2}{\left(1-\frac{1}{a^2}\right)^2} = a^2.$$

or

$$\frac{\xi^2}{\left(a+\frac{1}{a}\right)^2} + \frac{\eta^2}{\left(a-\frac{1}{a}\right)^2} = 1.$$

These are ellipses $a=1$ $\eta = 0$, $\xi = 2x$

So $x^2+y^2=1$ is mapped into a ~~line~~ segment of line from -2 to $+2$. a degenerate ellipse.

where $a>1$ where $a<1$

The exterior of (also interior) of the unit circle is mapped into the whole ξ-plane; ~~covered by~~

$a=\frac{1}{2}$, and $a=2$ are mapped into the same ellipse with its majoraxis $= 2+\frac{1}{2} = \frac{5}{2}$

minor axis $= 2-\frac{1}{2} = \frac{3}{2}$.

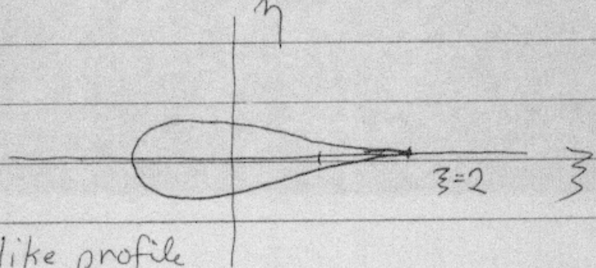
(b) $\zeta = z + \frac{1}{z}$

$\frac{d\zeta}{dz} = 1 - \frac{1}{z^2}$, $\quad \frac{d\zeta}{dz} = 0$, at $z = \pm 1$.

Conformal except at $z = \pm 1$.

At $z = \pm 1$, the angle between any two line elements in z-plane will be double ~~between~~ the two corresponding line elements in ζ-plane. Any line going through $z = \pm 1$, will be folded ~~chang~~ in ζ-plane. (a cusp)

(c) The circle $(x+\alpha)^2 + y^2 = (1+\alpha)^2$ passes through $z = 1$. The image of this circle in ζ-plane will look like



Wing-like profile

5) First let's make a change of variable $z = \pm 1 + t$, then the Legendre equation: $(1-z^2)w'' - 2zw' + \lambda w = 0$ (prime $\frac{d}{dz}$) become

$$t w'' + P(t) w' + Q_R w = 0, \quad (\text{prime} = \tfrac{d}{dt})$$

and $P(t) = 2\frac{t \pm 1}{t \pm 2}$, $\quad Q_R(t) = -\frac{\lambda}{t \pm 2}$, $\quad$ upper sign for $z = 1$, lower " " $z = -1$

Both $P(t)$ and $Q(t)$ are analytic at $t = 0$, in fact their Taylor series are

$$P(t) = 2\frac{1 \pm t}{2 \pm t} = \frac{1 \pm t}{1 \pm t/2} = (1 \pm t)(1 \mp \tfrac{t}{2}) + O(t^2) = 1 \pm \frac{t}{2} + O(t^2)$$

$$Q_R(t) = \mp \frac{\lambda}{2} \frac{1}{1 \pm t/2} = \frac{\lambda}{\mp 2}(1 \mp \tfrac{t}{2}) = \mp \frac{\lambda}{2}(1 \mp \tfrac{t}{2})$$

Put the equation in the form we have in the class.

$$t^2 W'' + t\,p(t)\,W' + t\,Q_R W = 0$$

$$I(\alpha) \equiv \alpha(\alpha-1) + P_0\alpha + \overset{0}{\overset{\|}{Q_0}} \qquad \underline{t\,Q_R(t) = Q}$$

$$\equiv \alpha^2, \qquad \alpha_1 = \alpha_2 = 0, \text{ double root},$$

$$I(\alpha+n) \equiv (\alpha+n)^2 \neq 0,$$

$Q_0 = 0$. if start with Q_1

$$t\,Q_R(t) = \sum_{n=0}^{\infty} Q_n t^n$$

$\underline{\underline{P_0 = 1}}$

One can alway find an analytic solution with $\alpha = 0$.

Cannot get another by using α_2, (double roots). but use the Wronskian method.

$$w_1^2 \frac{d}{dt}\left(\frac{w_2}{w_1}\right) = W(t) = c\, e^{-\int^t p(t_1)\,dt_1} = c\, e^{-\int^t \frac{\sum P_n t^n}{t} dt}$$

$$= c\, e^{-P_0 \ln t - \sum_{n=1} \frac{P_n}{n} t^n} \qquad P_0 = 1$$

$$= \frac{c}{t}\, e^{-\sum \frac{P_n}{n} t^n}$$

$$W_2(t) = w_1(t) \int \frac{W(t)\,dt}{w_1^2(t)} = c\,w_1(t) \cdot \int \frac{dt}{t} \left[\frac{e^{-\sum \frac{P_n}{n} t^n}}{w_1^2} \right]$$

can be represented by a power series

$$= c\,W_1(t) \cdot \left[\ln t + \text{power series in } t \right]$$

$$= \ln(t) \cdot \left[\text{power series in } t \right]$$