20.3. (a) \[ R = \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \lim_{n \to \infty} \frac{\ln(n)}{\ln(n+1)} = 1, \quad a_n = \frac{\ln n}{n}. \]

(b) \[ a_n = \frac{n!}{n^n}, \quad \frac{1}{R} = \lim_{n \to \infty} \left| a_n \right| = \lim_{n \to \infty} \left| \frac{1}{2\pi n n e^{-n}} \right|^\frac{n}{n+1} = e^{-1}. \]

"Different from the answer given in the text."

(c) \[ a_n = n\ln n, \quad \frac{1}{R} = \lim_{n \to \infty} \left| a_n \right| = \lim_{n \to \infty} n \ln n \quad \frac{1}{n} = 0. \]

(d) \[ a_n = \frac{(n+p)^n}{n^n}, \quad \frac{1}{R} = \lim_{n \to \infty} \left| \frac{(n+p)^n}{n^n} \right| = \lim_{n \to \infty} \left( 1 + \frac{p}{n} \right)^n = e^p. \]

To do study at \( \infty \), let \( z = t \) and consider around \( t = 0 \).

(a) \[ \frac{1}{z^{-1}}, \quad \text{not singular at } z = 0. \]

\[ \frac{1}{z^{-2}} = \frac{t}{1-z}, \quad \text{not singular at } t = 0 (z \to \infty) \]

(b) \[ \frac{1+t^3}{z^2}, \quad \text{double pole at } z = 0 \]

\[ \frac{1+t^2}{z^2} = 1 + \frac{1}{z} \quad \text{a simple pole at } t = 0 (z = 0) \]

(c) \[ \sinh \frac{1}{z} = \frac{1}{2} (e^{\frac{1}{z}} - e^{-\frac{1}{z}}) \quad \text{an essential singularity} \]

\[ \sinh \frac{1}{z}, \quad \text{not singular at } t = 0 \]

(d) \[ \frac{z^2}{z^3}, \quad \text{triple pole at } z = 0, \quad \text{essential S at } t = 0 (z = \infty) \]

(e) \[ \frac{z^{1/2}}{(1+z^{1/2})^{1/2}}, \quad \text{branch points } z = 0, \quad (z = \pm i) \text{ also at } t = 0. \]

20.6 (a) \[ \tan z, \quad \text{zeros at } \sin z = 0, \quad \text{simple pole at } \cos z = 0 \]

(b) \[ (z-2) \sin \left( \frac{1}{z} \right), \quad \text{zeros at } z = 2, \quad \text{and } z = 1 - \frac{1}{4\pi}, \]

\[ \text{double pole at } z = 0; \quad \text{essential S at } z = 1, \]

(c) \[ e^{1/z}, \quad \text{zeros at } z = 0, \quad \text{essential S at } z = 0 \]

(d) \[ \tan \frac{1}{z}, \quad \text{zeros at } z = \frac{1}{n\pi}, \quad \text{simple pole at } \frac{1}{2} \approx \frac{\sin(1/2)}{2}, \quad \text{S at } z = 0 \]

(e) \[ z^{3/2}, \quad \text{branch point } z = 0 \text{ at } z = \infty. \]
2. The transformation is \( z = \frac{1 - e^{i\theta}}{1 - e^{-i\theta}} \).

For points on the unit circle in \( z \)-plane, i.e., \( z = e^{i\theta} \), one finds \( z = \frac{\cos \theta}{1 - \cos \theta} \).

Thus the unit circle in \( z \)-plane is mapped into the real axis on \( \zeta \)-plane.

Solution to the Laplace equation in upper half \( \zeta \)-plane with the boundary condition as given is

\[
\phi(\zeta, \eta) = \frac{1}{\pi} \left( \cot \frac{\eta}{\zeta} - \cot \frac{\eta}{\zeta + 1} \right)
\]

or just the imaginary part of the analytic function

\[
W(z) = \frac{1}{\pi} \ln \frac{z - 1}{z + 1}
\]

Now the boundary value problem in \( z \)-plane, is

\[
W(\zeta(\theta)) = W(z) = \frac{1}{\pi} \ln \frac{\zeta - 1}{\zeta + 1} = \frac{1}{\pi} \ln \frac{(1 - e^{i\theta})(z - 1)}{(1 + e^{i\theta})(z + 1)}
\]

\[
= \frac{1}{\pi} \ln \frac{1 - e^{i\theta}}{1 + e^{i\theta}} + \frac{1}{\pi} \ln \frac{z - 1}{z + 1}
\]

\[
= -\frac{\bar{z}}{2} + \frac{1}{\pi} \ln \frac{z - 1}{z + 1}
\]

The imaginary part of \( W(\zeta) \) is \( -\frac{1}{\pi} (\theta_1 - \theta_2) \).

where \( \zeta - 1 = r_1 e^{i\theta_1} \), and \( 0 < \theta_1 < 2\pi \), \( -\pi < \theta_2 < \pi \).

3. (a) Real part of the following analytic function solves the problem,

\[
W(z) = \frac{T_2 + T_1}{2z} + \frac{T_2 - T_1}{2z}
\]

(b) \( z \)-plane

---

[Diagram showing \( z \)-plane with points labeled and \( \zeta \)-plane with parameter trajectories]
\[ \frac{z}{3} = \frac{z}{3} + iy, \quad z = x + iy, \]

\[ \frac{z}{3} + iy = b \sin \frac{\pi x}{2a} \cosh \frac{\pi y}{2a} + i b \cos \frac{\pi x}{2a} \sinh \frac{\pi y}{2a}, \]

\[ \frac{z}{3} = b \sin \frac{\pi x}{2a} \cosh \frac{\pi y}{2a}, \]

\[ \eta = b \cos \frac{\pi x}{2a} \sinh \frac{\pi y}{2a}. \]

\[ \frac{x = \text{const}}{\frac{\pi^2}{2a} - \frac{\eta^2}{b} \cosh \frac{\pi y}{2a} = 1}, \quad \text{hyperbolas} \]

\[ \frac{y = \text{const}}{\frac{\pi^2}{2a} + \frac{\eta^2}{b} \cos \frac{\pi x}{2a} = 1}, \quad \text{ellipses} \]

\[ x = -a \Rightarrow \eta = 0, \quad \frac{z}{3} = -b \cosh \frac{\pi y}{2a}, \]

\[ \text{is mapped into} \quad \eta = 0, \quad \frac{z}{3} = \left(-b, -\infty\right). \]

\[ x = a \Rightarrow \eta = 0, \quad \frac{z}{3} = b \cosh \frac{\pi y}{2a}, \]

\[ \text{is mapped into} \quad \eta = 0, \quad \frac{z}{3} = \left(b, +\infty\right). \]

(c) The boundary value problem in \( z \)-plane, is solved by the real part of

\[ W(z) = W(z) = W(z) = \frac{T_2 + T_2}{T_2} + \frac{T_2}{T_2} \cdot \sinh \frac{\pi T_2}{b} \]

One can find the real part of \( \sin \frac{\pi T_2}{b} \) but the expression is rather complicated. However, the solution in \( z \)-plane is more transparent from the mapping in the previous page. We have found each vertical line in \( z \)-plane is mapped into a hyperbola. Since \( T = \text{const.} \) on each vertical line in \( z \)-plane, so \( T = \text{same \& constant} \) on the corresponding hyperbola in \( z \)-plane. For example, \( T = \frac{T_1 + T_2}{2} \) on \( y = 0 \), we have \( T = \frac{T_1 + T_2}{2} \) on \( \eta = 0 \) (the corresponding hyperbola).
4). (a) \[ \frac{x + iy}{x + iy} = \frac{1}{x + iy} \]
\[ = x + iy + \frac{x - iy}{x^2 + y^2} \]
\[ = x \left(1 + \frac{1}{x^2 + y^2}\right) + iy \left(1 - \frac{1}{x^2 + y^2}\right). \]

\[ \frac{y}{x} = x \left(1 + \frac{1}{x^2 + y^2}\right) \]
\[ \eta = y \left(1 - \frac{1}{x^2 + y^2}\right) \]

Need the image of \( x^2 + y^2 = a^2 \), which is
\[ \frac{x^2}{(1 + \frac{1}{a^2})^2} + \frac{\eta^2}{(1 - \frac{1}{a^2})^2} = a^2. \]
\[ a \]
\[ \frac{x^2}{(a + \frac{1}{a})^2} + \frac{\eta^2}{(a - \frac{1}{a})^2} = 1. \]

There are ellipses \( q = 1 \), \( \eta = 0 \), \( \frac{3}{2} = 2x \)

So \( x^2 + y^2 = 1 \) is mapped into a line segment of line from \(-2 \) to \( +2 \), a degenerate ellipse, where \( a > 1 \).

The exterior of (also interior) of the unit circle is mapped into the whole \( z \)-plane.

\( a = \frac{1}{2} \) and \( a = 2 \) are mapped into the same ellipse with its major axis \( = 2 + \frac{1}{2} = \frac{5}{2} \)
\[ \text{Minor axis } = 2 - \frac{1}{2} = \frac{3}{2}. \]
(b) \( z = 2 + \frac{1}{z} \)

\[
\frac{d}{dz} z = 1 - \frac{1}{z^2}, \quad \frac{d}{dz} \frac{1}{z} = 0, \quad \text{at } z = \pm 1.
\]

Conformal except at \( z = \pm 1 \).

(c) At \( z = \pm 1 \), the angle between any two line elements in \( z \)-plane will be double between the two corresponding line elements in \( \bar{z} \)-plane. Any line going through \( z = \pm 1 \), will be folded (a cusp) in \( \bar{z} \)-plane.

The circle \((x + \alpha)^2 + y^2 = (1 + x)^2\) passes through \( z = 1 \). The image of this circle in \( \bar{z} \)-plane will look like

![Wing-like profile](image)

5) First let's make a change of variable \( z = \pm 1 + t \), then the Legendre equation \( (1 - z^2)w'' - 2zw' + \lambda w = 0 \) (prime = \( \frac{d}{dz} \)) become,

\[
t w'' + p(t) w' + q(t) w = 0, \quad \text{(prime = \( \frac{d}{dt} \))}
\]

and \( p(t) = \frac{t}{t + 1}, \quad q(t) = \frac{\lambda}{t + 2}, \quad \text{upper sign for } z = 1 \).

Both \( p(t) \) and \( q(t) \) are analytic at \( t = 0 \), in fact their Taylor Series are

\[
p(t) = 2 \frac{1 + t}{2} \frac{1 + t}{1 + t} = (1 + t)(1 - \frac{5}{2}) + O(t^3) = 1 + \frac{t}{2} + O(t^3)
\]

\[
q(t) = -\frac{\lambda}{2} \frac{1}{1 + t} = -\frac{\lambda}{2} \frac{1}{1 + t} = \frac{\lambda}{2} \frac{1}{1 + t}
\]
Put the equation in the form we have in the class.

\[ t^2 W'' + t P(t) W' + t Q(t) W = 0 \]

\[ t Q(t) = Q_0 = 0. \quad \text{Start with } Q_1 \]

\[ Q_0 = 0. \]

\[ t Q(t) = \sum_{n=0}^{\infty} Q_n t^n \]

\[ I(x) = \alpha (x-1) + P_0 x + Q_0 \]

\[ = x^2, \quad \alpha_1 = \alpha_2 = 0, \quad \text{double root} \]

\[ I(x+n) = (x+n)^2 \neq 0, \]

\[ P_0 = 1 \]

One can always find an analytic solution with \( \alpha = 0 \).

Cannot get another by using \( \alpha_2 \), (double roots). But use the

Wronskian method.

\[ w_1(t) = W(t) = c e^{\int \frac{P_0(t)}{t} dt} = c e^{\int \frac{\sum P_n t^n}{t} dt} \]

\[ = c e^{\sum \frac{P_n t^n}{n}} \]

\[ P_0 = 1 \]

\[ = c e^{\frac{P_0 t^n}{n}} \]

\[ W_2(t) = W_1(t) \int \frac{W(t) dt}{-w_1^2(t)} = c W_1(t) \int \frac{e^{-\frac{P_0 t^n}{n}}}{W^2} dt \]

\[ R \text{ can be represented by a power series} \]

\[ = \ln(t) \quad \text{[power series in } t] \]