Key to problem Set II. AM 205 Oct 19, 2004 1(9)

1. (a) $\frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x} = y$, $\psi = \frac{y^2}{2} + C(x)$. $\frac{\partial\psi}{\partial x} = C'(x)$

$\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y} = -x = C'(x)$ $\therefore$ $C(x) = -\frac{x^2}{2} + \text{Constant}$

Set the constant to be zero. $\psi = (y^2 - x^2)/2$.

(b) $\frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x} = 2x$, $\psi = 2xy + C(x)$, $\frac{\partial\psi}{\partial x} = 2y + C'(x)$

$\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y} = -2y = 2y + C'(x)$. $C'(x) = -4y$.

x^2+y^2 cannot be the real part of any analytic function.

(c) $\frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x} = \frac{1}{2}\left[\frac{2(x-1)}{(x-1)^2+y^2} - \frac{2(x+1)}{(x+1)^2+y^2}\right]$

$\psi(x,y) = \tan^{-1}\frac{y}{x-1} - \tan^{-1}\frac{y}{x+1} + C(x)$.

$\frac{\partial\psi}{\partial x} = -\frac{y}{(x-1)^2+y^2} + \frac{y}{(x+1)^2+y^2} + C'(x)$

$\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y} = \frac{-y}{(x-1)^2+y^2} + \frac{y}{(x+1)^2+y^2}$

$C'(x) = 0$, $C = \text{constant} = 0$.

$\psi(x,y) = \tan^{-1}\frac{y}{x-1} - \tan^{-1}\frac{y}{x+1}$

(d) Use of the C-R conditions as above involves differentiation and integrations. It becomes somewhat complicated in this example. Let's make a guess of the correspond analytic function $f(z)$, (based on the real part as given) as

$$\frac{1}{\sin z} = \frac{1}{\sin x\cosh y + i\cos x\sinh y} = \frac{\sin x\cosh y - i\cos x\sinh y}{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y}$$

2

Use of the identities: $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\sinh^2 y = \frac{1}{2}(\cosh 2y - 1)$$

$$\cosh^2 y = \frac{1}{2}(\cosh 2y + 1)$$

it can be shown that $\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \frac{1}{2}[\cosh(2y) - \cos(2x)]$

Therefor $\frac{1}{2}\frac{1}{\sin \bar{z}} = \dfrac{\sin x \cosh y - i \cos x \sinh y}{\cosh 2y - \cos 2x}$

$$\phi(x,y) = \frac{\sin x \cosh y}{\cosh 2y - \cos 2x}$$

$$\psi(x,y) = -\frac{\cos x \sinh y}{\cosh 2y - \cos 2x}$$

2(a) $f(z) = \ln \dfrac{x-b+iy}{x+b+iy} = \dfrac{1}{2}\ln \dfrac{(x-b)^2+y^2}{(x+b)^2+y^2} + i\left(\tan^{-1}\dfrac{y}{x-b} - \tan^{-1}\dfrac{y}{x+b}\right).$

$$\phi_a(x,y) = \frac{1}{2}\ln\frac{(x-b)^2+y^2}{(x+b)^2+y^2}$$

To get the level curves set $\phi(x,y)$ be a real constant say c

$$\frac{(x-b)^2+y^2}{(x+b)^2+y^2} = e^{2c}$$

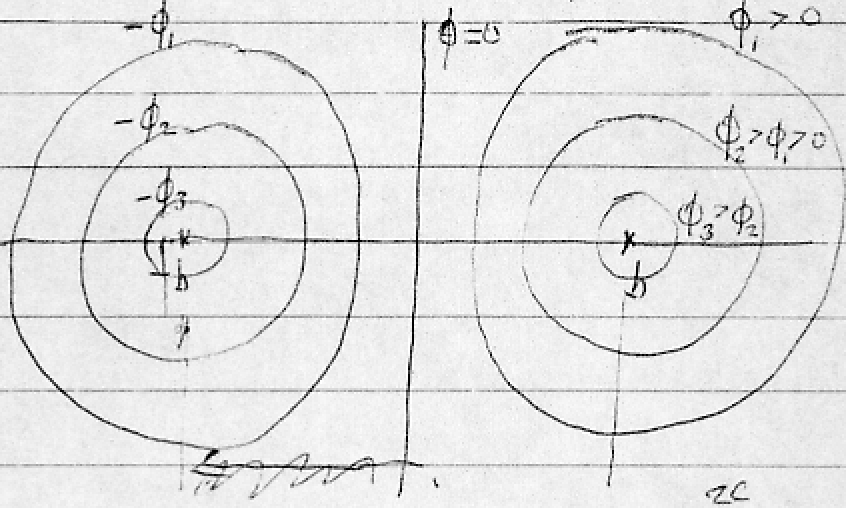
or $\left(x - b\dfrac{e^{2c}+1}{e^{2c}-1}\right)^2 + y^2 = b^2\left[\left(\dfrac{e^{2c}+1}{e^{2c}-1}\right)^2 - 1\right]$

These are circles centered at $b\dfrac{e^{2c}+1}{e^{2c}-1}$ with the radius $= b\sqrt{\left(\dfrac{e^{2c}+1}{e^{2c}-1}\right)^2 - 1}$

① $c=0$, the circle become straight line $x=0$; (where $\phi=0$)

② $c>0$, (or $\phi>0$) The center of the circles on the right side of b with the point b alway inside the circle

(3) ϕ negative curves are just the mirror images of ~~these~~ those with $\phi > 0$. as shown in the following diagram



(b). Distance between two centers: $2b \dfrac{e^{2c}+1}{e^{2c}-1} = l$.

radius of the cylinders : $a = b\sqrt{\left(\dfrac{e^{2c}+1}{e^{2c}-1}\right)^2 - 1}$

Eliminating c from these we get $b = \dfrac{l}{2}\sqrt{1 - \dfrac{4a^2}{l^2}} \equiv A \cdot \dfrac{l}{2}$

substituting this b into the first expression, we have.

$$c = \frac{1}{2} \ln \frac{1+A}{1-A}$$

The two cylinders are located at $\pm l$ with the same radius a. The one on the right has a temperature c (as given above in term of $A = \sqrt{1 - \frac{4a^2}{l^2}}$) while the left one has the temperature $-c$.

To satisfy the given boundary condition (T_1 on the right and T_2 on the left), Consider the same geometry but with both cylinders having the same temperature T. The solution to the Laplace equation of this problem is T = constant ~~in~~ everywhere.

Combining these two problem (linear combination) we set

$$\begin{cases} T_0 + c = T_1 \\ T_0 - c = T_2 \end{cases} \quad \text{So} \quad T_0 = \frac{1}{2}(T_1 + T_2)$$

The final solution

$$T(x,y) = \frac{1}{2}\left[T_1 + T_2 + \ln\frac{(x-b)^2+y^2}{(x+b)^2+y^2}\right] \quad \text{with } b = \frac{l}{2}\sqrt{1-\frac{4a^2}{l^2}}.$$

3. This problem is intended for you to go over the problem we did in the class with $\cos m\theta$. Hower the integral can be easily shown to be zero by breaking it up into two parts, one integrating from 0 to π and the other from π to 2π. Two parts cancel each other.

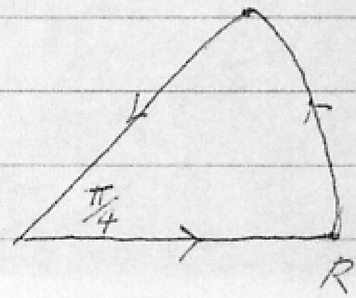
4. (20.13), $e^{iaz^2} = e^{iar^2(\cos 2\theta + i\sin 2\theta)} = e^{iar^2\cos 2\theta} \cdot e^{-ar^2\sin 2\theta}$

$$|e^{iaz^2}| = e^{-ar^2\sin 2\theta} \to 0 \quad \text{as } r\to\infty \quad \text{provided } \sin 2\theta > 0$$

or $0 < \theta < \frac{\pi}{2}$

Consider

$$\oint_C dz\, e^{iaz^2}$$



with C as indicated

$$0 = \oint_C dz\, e^{iaz^2} = \int_0^R dx\, e^{iax^2} + \int dz\, e^{iaz^2} + \int_R^0 dr\, e^{i\frac{\pi}{4}} e^{ia(r^2 i)}$$

by Cauchy theorem

Over the arc

This is zero as $R\to\infty$

$$\int_0^\infty dx\, e^{iax^2} = \frac{1}{\sqrt{2}}(1+i)\int_0^\infty dx\, e^{-ax^2} = \frac{1}{\sqrt{2}}(1+i)\cdot\frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

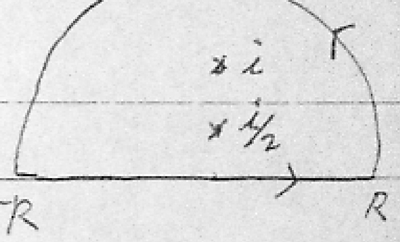
Therefore

$$\int_0^\infty dx\cos ax^2 = \int_0^\infty dx \sin ax^2 = \sqrt{\frac{\pi}{8a}}.$$

(20.21) $\displaystyle\int_0^\infty \frac{\cos mx\, dx}{(4x^2+1)(x^2+1)} = \frac{1}{8}\int_{-\infty}^{\infty}\frac{\cos mx\, dx}{(x^2+\frac{1}{4})(x^2+1)}$

Consider

$$\oint_C \frac{e^{imz}\,dz}{(z^2+\frac{1}{4})(z^2+1)} = \int_{-R}^{R} \frac{e^{imx}\,dx}{(x^2+\frac{1}{4})(x^2+1)} + \int_{\substack{\text{half upper}\\ \text{circle}}}$$



(This is zero by Jordan's lemma)

Therefore

$$\int_{-\infty}^{\infty} \frac{e^{imx}\,dx}{(x^2+\frac{1}{4})(x^2+1)} = 2\pi i \text{ Residue in } C$$

$$= 2\pi i \left[\frac{e^{im(i/2)}}{i(1-\frac{1}{4})} + \frac{e^{im(i)}}{2i(-1+\frac{1}{4})} \right]$$

$$= 2\pi \left(\frac{4}{3} e^{-m/2} - \frac{2}{3} e^{-m} \right)$$

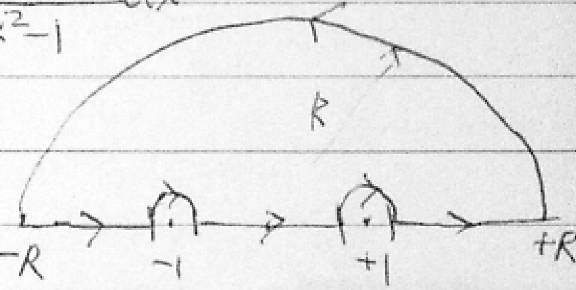
$$= \frac{4\pi}{3} \left(2e^{-m/2} - e^{-m} \right).$$

$$\int_0^{\infty} \frac{\cos mx\,dx}{4x^4+5x^2+1} = \frac{1}{8}\cdot\frac{4\pi}{3}\left(2e^{-m/2} - e^{-m}\right) = \frac{\pi}{6}\left(2e^{-m/2} - e^{-m}\right)$$

I got a factor 2 difference

(20.22) $$P\int_{-\infty}^{\infty} \frac{\cos\frac{x}{a}}{x^2-a^2}\,dx = \frac{1}{a} P\int_{-\infty}^{\infty} \frac{\cos x}{x^2-1}\,dx$$

Consider the ~~cl~~ contour C



$$0 = \oint_C \frac{dz\,e^{iz}}{z^2-1} = P\int_{-R}^{R} \frac{e^{ix}\,dx}{x^2-1} + \int_{\frown\,|z+1|=\epsilon} \frac{dz\,e^{iz}}{z^2-1} + \int_{\frown\,|z-1|=\epsilon} \frac{dz\,e^{iz}}{z^2-1}$$

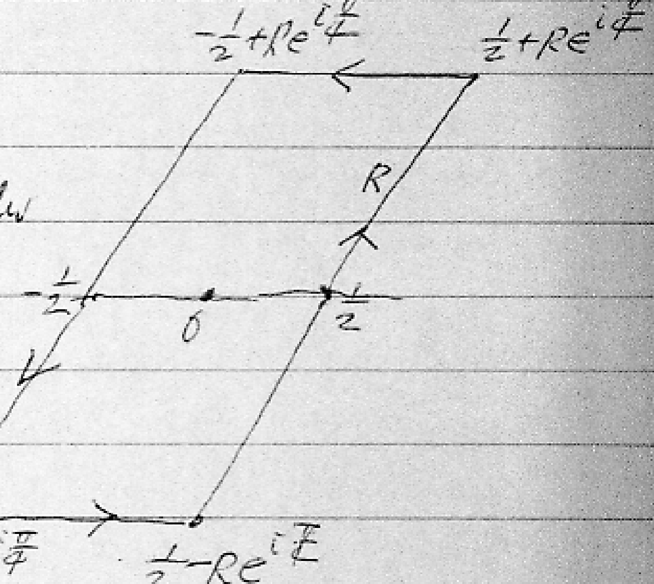
The integral over the half circle of radius R is zero by the Jordan's lemma,

$$P\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x^2-1} = \int_{\frown\,|z+1|=\epsilon} \frac{dz\,e^{iz}}{z^2-1} + \int_{\frown\,|z-1|=\epsilon} \frac{dz\,e^{iz}}{z^2-1} = \pi i \frac{e^{-i}}{-2} + \pi i \frac{e^{i}}{2} = -\pi \sin(1)$$

$$P\int_{-\infty}^{\infty}\frac{e^{ix}dx}{x^2-1} = P\int_{-\infty}^{\infty}\frac{\cos x\,dx}{x^2-1} + i\,P\int_{-\infty}^{\infty}\frac{\sin x\,dx}{x^2-1}.$$

$$P\int_{-\infty}^{\infty}\frac{\cos x\,dx}{x^2-1} = -\pi\sin(1), \qquad P\int_{-\infty}^{\infty}\frac{\sin x\,dx}{x^2-1} = 0.$$

$$P\int_{-\infty}^{\infty}\frac{\cos\frac{x}{a}}{x^2-a^2} = -\frac{\pi}{a}\sin(1)$$



(20.23) The contour C is as shown. Consider

$$\oint_C \frac{e^{i\pi z^2}}{\sin\pi z}dz = 2\pi i\cdot\underbrace{\frac{1}{\pi}}_{\text{the residue at } z=0} = 2i$$

on the horizontal part $\cancel{z = x \pm iR\sin\frac{\pi}{4} = x \pm i\frac{R}{\sqrt{2}}}$

$$z = x \pm Re^{i\frac{\pi}{4}} \quad \text{for } -\tfrac{1}{2}\le x\le\tfrac{1}{2}$$

$$z^2 = x^2 + R^2e^{i\frac{\pi}{2}} \pm 2xRe^{i\frac{\pi}{4}}$$

$$= x^2 + iR^2 \pm \sqrt{2}xR(1+i) = x^2+\sqrt{2}xR + i(R^2+\sqrt{2}xR)$$

$$i\pi z^2 = i\pi(x^2+\sqrt{2}xR) - \pi(R^2+2xR).$$

$$\left|e^{i\pi z^2}\right| = e^{-\pi(R^2+\sqrt{2}xR)}.$$

$$\sin\pi z = \sin\pi(x\pm Re^{i\frac{\pi}{4}}) = \sin\pi\left(x\pm\tfrac{R}{\sqrt{2}} \pm i\tfrac{R}{\sqrt{2}}\right)$$

$$= \sin\pi(x\pm R/_2)\cosh\pi\tfrac{R}{\sqrt{2}} \pm i\cos\pi(x\pm R/_{\sqrt{2}})\sinh\pi\tfrac{R}{\sqrt{2}}.$$

$$|\sin\pi z| = \sqrt{\tfrac{1}{2}\left[(\cosh\sqrt{2}\pi R) - \cos 2\pi(x\pm R/_{\sqrt{2}})\right]} \simeq \cosh\pi\tfrac{R}{\sqrt{2}} \quad \text{for large } R$$

$$\frac{|e^{i\pi z^2}|}{|\sin\pi R|} = \frac{e^{-\pi(R^2+\sqrt{2}xR)}}{\cosh\pi R/_{\sqrt{2}}} \to 0 \quad \text{as } R\to\infty,$$

No contribution from the horizontal part.

On the slant parts. $z = \pm\frac{1}{2} + re^{i\frac{\pi}{4}}$

$$dz = dre^{i\frac{\pi}{4}}$$

$$z^2 = \frac{1}{4} + ir^2 \pm re^{i\frac{\pi}{4}}$$

$$e^{i\pi z^2} = e^{i\pi(\frac{1}{4} \pm re^{i\frac{\pi}{4}}) - \pi r^2}$$

$\pi re^{i\frac{\pi}{4}}$

$$\sin\pi z = \sin\pi(\pm\tfrac{1}{2} + re^{i\frac{\pi}{4}}) = \pm\sin(\tfrac{\pi}{2} \pm \pi re^{i\frac{\pi}{4}}) = \pm\cos\pi re^{i\frac{\pi}{4}}$$

$$= \pm\cos\theta, \quad \text{where } \theta \equiv \pi re^{i\frac{\pi}{4}}.$$

Now the two integrals: one up on the right and down on the left.

$$\int_{\frac{1}{2}-Re^{i\frac{\pi}{2}}}^{\frac{1}{2}+Re^{i\frac{\pi}{4}}} dz\,\frac{e^{i\pi z^2}}{\sin\pi z} + \int_{-\frac{1}{2}+Re^{i\frac{\pi}{4}}}^{-\frac{1}{2}-Re^{i\frac{\pi}{4}}} dz\,\frac{e^{i\pi z^2}}{\sin z}$$

$$= \int_{-R}^{R} dr\, e^{-\pi r^2}\left(\frac{e^{i(\frac{\pi}{2}+\theta)}}{\cos\theta} + \frac{e^{i(\frac{\pi}{2}-\theta)}}{\cos\theta}\right) = 2i\int_{-R}^{R} dr\, e^{-\pi r^2}$$

$$\therefore \int_{-\infty}^{\infty} dr\, e^{-\pi r^2} = 1.$$

(20.29) (a) $f(z) = \dfrac{\pi\cot\pi z}{z^2+a^2} = \dfrac{\pi\cos\pi z}{(z^2+a^2)\sin\pi z}$

Zeros of $\sin\pi z$ are $\begin{cases} y = 0 \\ x = 0, \pm1, \pm2, \cdots \pm n \end{cases}$

i.e. all the integral points on the real axis.

Assuming that a is not an integral, then all the poles are simple

Residue at $z = \pm ia$; $\dfrac{\pi\cot(\pi\pm ia)}{\pm 2ia} = -\dfrac{\pi\coth\pi a}{2a}$

at each zero ; $\sin\pi z = \sin\pi n + \pi\cos\pi n\,(z-n)$

The residue at $z = n$; $\dfrac{\pi\cos\pi n}{\pi\cos\pi n}\,\dfrac{1}{n^2+a^2} = \dfrac{1}{n^2+a^2}$

(b) Take the contour as a circle centered at $z=0$ with a radius of $N+\frac{1}{2}$, wher N is an integer. As $N\to\infty$, it is easy to verify that

$$f(z) \to \frac{1}{R^2} \quad \text{as } N\to 0 \quad \text{wher } R=N+\frac{1}{2}.$$

Sum of all the residue in R equals to zero i.e.

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} - 2\,\frac{\pi\coth\pi a}{2a} = 0$$

$$\frac{1}{a^2}+2\sum_{n=1}^{\infty}\frac{1}{n^2+a^2} - \pi\,\frac{\coth\pi a}{a} = 0$$

or $$\sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = \frac{1}{2}\left(\frac{\pi\coth\pi a}{a} - \frac{1}{a^2}\right)$$

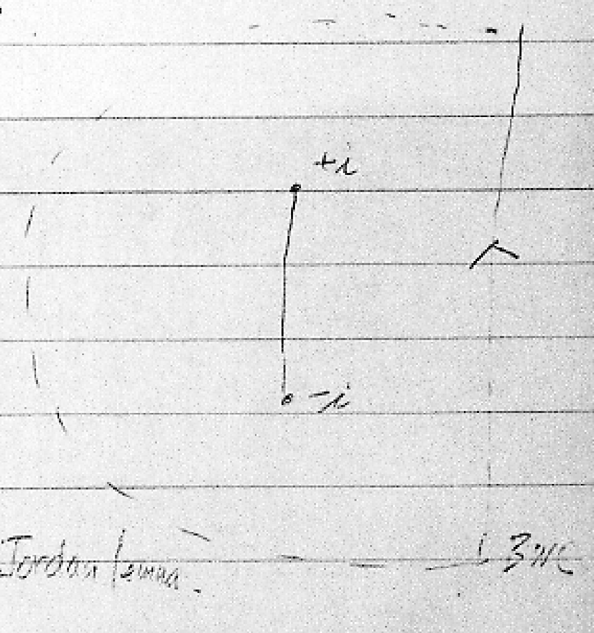
(c) $$\frac{\pi\coth\pi a}{a} = \frac{\pi}{a}\frac{\cosh\pi a}{\sinh\pi a} = \frac{\pi}{a}\,\frac{1+\pi^2a^2/2}{\pi a+\frac{\pi^3a^3}{6}} = \frac{1}{a^2}\,\frac{1+\frac{\pi^2a^2}{2}}{1+\pi^2a^2/6}$$

$$= \frac{1}{a^2}\left(1+\frac{\pi^2a^2}{2}\right)\left(1-\frac{\pi^2a^2}{6}\right) = \frac{1}{a^2}\left(1+\frac{\pi^2a^2}{3}\right)$$

as $a\to 0$. $$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{2}\,\frac{\pi^2}{3}$$

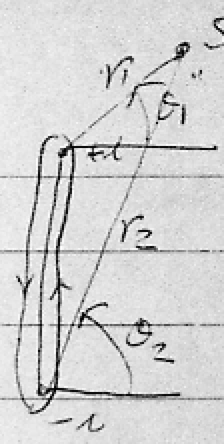
(20.33)

$$f(t) = \frac{1}{2\pi i}\int_{BWC} e^{st}F(s)\,ds$$



$$F(s) = \frac{1}{2i}\ln\frac{s+i}{s-i}$$

(a) As long as $|F(s)|\to 0$ as $|s|\to\infty$, one can close the BWC by the semi-circle, the integral there is zero by Jordan lemma.



(b) Use Cauchy theorem, the integral can be replaced by ~~the~~ an integral around the branch cut from $-i$ to i. To effect this cut we let

$$s+i = r_2 e^{i\theta_2} \quad \text{where } -\frac{3\pi}{2} < \theta_2 < \frac{\pi}{2}$$

$$s-i = r_1 e^{i\theta_1} \quad \text{and } -\frac{3\pi}{2} < \theta_1 < \frac{\pi}{2}$$

$$\ln\frac{s+i}{s-i} = \ln\frac{r_2}{r_1} + i(\theta_2 - \theta_1)$$

On the right side of the cut:

$$F_R(s) = \frac{1}{2i}\left[\ln\frac{r_2}{r_1} + i\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)\right]$$

$$= \frac{1}{2i}\left[\ln\frac{r_2}{r_1} + i\pi\right]$$

$$= \frac{1}{2i}\ln\frac{r_2}{r_1} + \frac{\pi}{2}.$$

On the left side of the cut.

$$F_L(s) = \frac{1}{2i}\left[\ln\frac{r_2}{r_1} + i\left(-\frac{3\pi}{2} - \left(-\frac{\pi}{2}\right)\right)\right]$$

$f(t) =$

$$= \frac{1}{2i}\ln\frac{r_2}{r_1} - \frac{\pi}{2}$$

$$\frac{1}{2\pi i}\oint ds\, e^{st} F(s) = \frac{1}{2\pi i}\left[\int_{-i}^{i} ds\, e^{st} F_R(s) + \int_{i}^{-i} ds\, e^{st} F_L(s)\right]$$

$$= \frac{1}{2\pi i}\int_{-i}^{i} ds\, e^{st}\left[F_R(s) - F_L(s)\right] \qquad s = iy, \quad ds = i\,dy$$

$$= \frac{1}{2\pi}\int_{-1}^{1} dy\, e^{iyt}\cdot\pi = \frac{1}{2}\,\frac{e^{it} - e^{-it}}{it} = \frac{\sin t}{t}$$