Keys to the prob. set L. (Am205).

1. \( f(z) = xy + \bar{z} \)
   \[ \phi = xy, \quad \psi = \bar{z} \]
   \[ \frac{\partial \psi}{\partial x} = y, \quad \frac{\partial \psi}{\partial y} = 0 \quad \Rightarrow \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} = 2y \quad \text{except for } y = 0. \]
   \[ \frac{\partial \phi}{\partial y} = x, \quad \frac{\partial \phi}{\partial \bar{z}} = \frac{1}{2} \frac{1}{\bar{z}} \quad \Rightarrow \frac{\partial \phi}{\partial \bar{z}} = \frac{1}{2} \frac{1}{\bar{z}} \quad \text{except for } \bar{z} = 0. \]
   \( f_z(z) \) is not analytic for all \( z \) except for \( z = 0 \).

   At \( z = 0 \), C-R conditions are satisfied, and the all 1st partial derivative of \( \phi \) and \( \psi \) is continuous. So \( f(0) \) is analytic for all \( z \) (including \( z = 0 \)).

   \[ f_z(z) = \sqrt{xy} + 100z \]
   \[ \phi = \sqrt{xy}, \quad \psi = 100 \bar{z} \]
   \[ \frac{\partial \psi}{\partial x} = \frac{1}{2} \frac{1}{\sqrt{xy}} \quad \Rightarrow \frac{\partial \psi}{\partial x} = \frac{1}{2} \frac{1}{\sqrt{xy}} \quad \text{For } x \neq 0, \text{ C-R conditions are satisfied. It is not analytic.} \]
   \[ \frac{\partial \psi}{\partial y} = \frac{1}{2} \frac{1}{\sqrt{xy}} \quad \Rightarrow \frac{\partial \psi}{\partial y} = \frac{1}{2} \frac{1}{\sqrt{xy}} \]

   At \( z = 0 \), the derivative of \( f \) is undefined by the above expression.

   However, \( \phi(xy) = 0 \) at both line \( x = 0 \) and \( y = 0 \). Therefore \( \frac{\partial \phi}{\partial x} = 0 \) (along the line \( y = 0 \)) and \( \frac{\partial \phi}{\partial y} = 0 \) (along the line \( x = 0 \)). But

   The C-R conditions are satisfied. Since the partial derivatives are not continuous, C-R conditions do not guarantee the existence of the 1st derivative. In fact,

   \[ \lim_{\Delta z \to 0} \frac{\Delta f_z}{\Delta z} \]
   \[ \Delta z \to 0 \] \[ \frac{\Delta f_z}{\Delta z} = \frac{\sqrt{\Delta z \cdot \Delta y}}{\Delta \bar{z} + \Delta y} = \frac{\sqrt{1}}{1 + \Delta \bar{z}} \quad \text{for } \Delta y = \Delta x. \]
2a. Put the unit square in a Cartesian coordinate as shown

The boundary conditions are:

\[ \phi = 0 \quad \text{for} \quad x = 0 \]
\[ \phi = 0 \quad \text{for} \quad x = 1 \]
\[ \phi = 0 \quad \text{for} \quad y = 0 \]
\[ \phi = \sin \pi x \sin \pi y \quad \text{for} \quad y = 1 \]

We are looking for an analytic function whose real part will satisfy these conditions. Realizing that \( \sin \pi x = 0 \) for \( x = 0 \) and \( x = 1 \), and \( \sin \pi y = 0 \) for \( y = 0 \), it seems any choice of \( \phi \) is

\[ \phi(x, y) = \sin \pi x \sin \pi y \]

which satisfies all four boundary conditions. It is the imaginary part of \( e^{i \pi x} \), which is analytic. Thus  \( \partial \phi = 0 \).

b. In the picture on the right

\[ \alpha = \theta_1 - \theta_2 \]

The modulus of \( \alpha \) is:

\[ \alpha' = \begin{cases} 0 & \text{when } z \text{ is on the real axis and right of } x = 1, \\ \pi & \text{between } -1 \text{ and } 1, \\ 0 & \text{and left of } x = -1. \end{cases} \]

Now consider the analytic function

\[ \phi = \frac{e^{i \theta_1} - e^{i \theta_2}}{e^{i \theta_1} + e^{i \theta_2}} \]

Thus  \( \phi = \frac{\theta_1 - \theta_2}{\theta_1 + \theta_2} \) is the solution.
3. From the relation of the base vectors, one gets

\[ \frac{\partial}{\partial \theta} (\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi) = (0, \hat{e}_r, 0) \]

\[ \frac{\partial}{\partial \phi} (\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi) = \left( \frac{\hat{e}_\phi}{r}, -\hat{e}_r, 0 \right) \]

\[ \frac{\partial}{\partial z} (\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi) = (0, 0, -\hat{e}_z) \]

\[ X = r \hat{e}_r + z \hat{e}_z \]

\[ \frac{\partial}{\partial \theta} = \frac{\hat{e}_\theta}{\partial \theta} \]

\[ \frac{\partial}{\partial \phi} = \frac{\hat{e}_\phi}{\partial \phi} \]

\[ \nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \]

\[ \nabla \times V = (\hat{e}_r \frac{\partial}{\partial \theta} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \phi} + \hat{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}) \cdot (\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}) \]

Using \( \hat{e}_\theta \cdot \hat{e}_\phi = \delta_{\phi \theta} = 0 \) and the derivatives of \( \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi \),

the base vector above, one gets

\[ \nabla \cdot V = \frac{\partial V_r}{\partial r} + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial V_\phi}{\partial \phi} \]

\[ \nabla \times V = (\hat{e}_r \frac{\partial}{\partial \theta} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \phi} + \hat{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}) \times (\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}) \]

Using the right-handedness of the base vectors \( \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi \)

and \( \hat{e}_x \cdot \hat{e}_y = \hat{e}_y \cdot \hat{e}_z = \hat{e}_z \cdot \hat{e}_x = 0 \), one gets

\[ \nabla \times V = \frac{1}{r} \left( \frac{\partial^2 V_\phi}{\partial \theta^2} - \frac{\partial^2 V_\theta}{\partial \phi^2} \right) + \frac{\partial}{\partial \theta} \left( \frac{V_\phi}{r} - \frac{V_\theta}{r} \right) + \frac{\partial}{\partial \phi} \left( \frac{V_\theta}{r} - \frac{V_\phi}{r} \right) \]

4. From the relation expression of the base vector, one gets

the 1st derivatives of the base vectors as follows...
\[ \frac{\partial}{\partial \tau} (e_r, e_\theta, e_\phi) = (0, 0, 0) \]
\[ \frac{\partial}{\partial \theta} (e_r, e_\theta, e_\phi) = (\hat{e}_\theta, -e_r, 0) \]
\[ \frac{\partial}{\partial \phi} (e_r, e_\theta, e_\phi) = (\sin \theta \hat{e}_\theta, \cos \theta \hat{e}_r, -\sin \theta \hat{e}_\phi - \cos \theta \hat{e}_\theta) \]

\[ \hat{d}X = \frac{\partial X}{\partial r} dr + \frac{\partial X}{\partial \theta} d\theta + \frac{\partial X}{\partial \phi} d\phi \quad \text{where} \quad X = r \hat{e}_r \]

\[ = \hat{e}_r dr + r \hat{e}_\theta d\theta + r \hat{e}_\phi d\phi \]

\[ = \hat{e}_r dr + \hat{e}_\theta d\theta + \hat{e}_\phi d\phi \]

\[ \nabla \cdot \mathbf{V} = \left( \frac{\partial}{\partial r} \hat{e}_r + \frac{\partial}{\partial \theta} \hat{e}_\theta + \frac{\partial}{\partial \phi} \hat{e}_\phi \right) \cdot \left( e_r \hat{e}_r + e_\theta \hat{e}_\theta + e_\phi \hat{e}_\phi \right) \]

\[ = e_\theta \frac{\partial}{\partial r} \left( \frac{e_\theta}{r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( e_\theta \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left( e_\phi \sin \theta \right) \]

\[ = \frac{1}{r^2} \frac{\partial}{\partial \phi} (r^2 \phi) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( e_\theta \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left( e_\phi \sin \theta \right) \]

\[ \nabla \times \mathbf{V} = \left( \frac{\partial}{\partial \theta} \hat{e}_r + \frac{\partial}{\partial \phi} \hat{e}_\theta + \frac{\partial}{\partial \phi} \hat{e}_\phi \right) \times \left( e_r \hat{e}_r + e_\theta \hat{e}_\theta + e_\phi \hat{e}_\phi \right) \]

\[ = \frac{\hat{e}_r}{r} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( e_\phi \right) - \frac{\partial}{\partial \phi} \left( e_\theta \right) \right) + \frac{\hat{e}_\theta}{r} \left( \frac{\partial}{\partial r} \left( e_\phi \right) - \frac{\partial}{\partial \phi} \left( e_r \right) \right) + \frac{\hat{e}_\phi}{r} \left( \frac{\partial}{\partial r} \left( e_\theta \right) - \frac{\partial}{\partial \theta} \left( e_r \right) \right) \]

\[ + \frac{\hat{e}_r}{r} \left( \frac{\partial}{\partial \phi} \left( e_\theta \right) - \frac{\partial}{\partial \theta} \left( e_\phi \right) \right) + \frac{\hat{e}_\theta}{r} \left( \frac{\partial}{\partial \phi} \left( e_r \right) - \frac{\partial}{\partial \phi} \left( e_\phi \right) \right) + \frac{\hat{e}_\phi}{r} \left( \frac{\partial}{\partial \theta} \left( e_r \right) - \frac{\partial}{\partial \phi} \left( e_\phi \right) \right) \]
5. Let the elementary area \( A = r(r+\Delta r) \phi \Delta \theta \).

Change of area in \( \phi \)-direction:

\[
\Delta A = \frac{\Delta r}{\Delta \theta} \frac{\sin \theta}{r^2} \left[ \frac{q(r+\Delta r) - q(r)}{\Delta \theta} \right] \Delta \theta \Delta r
\]

In the \( \phi \)-direction:

\[
\Delta A = \frac{\Delta \phi}{\Delta \theta} \frac{\sin \theta}{r^2} \left[ \frac{q(r+\Delta r) - q(r)}{\Delta \theta} \right] \Delta \theta \Delta \phi
\]

Total change:

\[
\frac{\Delta A}{\Delta \theta} = \frac{q(r+\Delta r) - q(r)}{\Delta \theta} \frac{\sin \theta}{r^2} \Delta r
\]

Compare this with \( V \nu \) in prob 3.

Vector Analysis:

\[
V = \hat{e}_\phi \nu \Delta \phi + \frac{\phi}{\Delta \phi} \nu \Delta \phi
\]

\[
\Delta V = \frac{\partial V}{\partial r} \Delta r + \frac{\partial V}{\partial \theta} \Delta \theta
\]

Sonic rotation:

\[
\frac{1}{2}(a_{11} \Delta r + a_{12} \Delta \theta) = -\frac{1}{2}(a_{21} - a_{22}) \Delta r \Delta \theta + a_{21} \Delta r + \frac{1}{2}(a_{11} + a_{22}) \Delta \theta
\]

Compare this with the \( \phi \)-component of \( \Delta V \) in prob 3.