Keys to the prob. set 1. (AM 205).

1. $f_1(z) = xy + i$

$\phi = xy, \quad \psi = 1$

$\frac{\partial \phi}{\partial x} = y \qquad \frac{\partial \psi}{\partial x} = 0 \qquad \frac{\partial \phi}{\partial x} \neq \frac{\partial \psi}{\partial y}$ except for $y = 0$.

$\frac{\partial \phi}{\partial y} = x \qquad \frac{\partial \psi}{\partial y} = 0 \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$ except for $x = 0$.

$f_1(z)$ is not analytic for all $z$ except for $z = 0$.

At $z = 0$, C-R condition is satisfied, and the all 1st partial derivative of $\phi$ and $\psi$ is continuous. So $f_1(z)$ is analytic for all $z$ (including $z = 0$).

$f_2(z) = \sqrt{xy} + 100i$

$\phi = \sqrt{xy}, \quad \psi = 100.$

$\frac{\partial \phi}{\partial x} = \frac{1}{2}\sqrt{\frac{y}{x}}, \qquad \frac{\partial \psi}{\partial x} = 0$

$\frac{\partial \phi}{\partial y} = \frac{1}{2}\sqrt{\frac{x}{y}} \qquad \frac{\partial \psi}{\partial y} = 0.$

For $z \neq 0$, C-R conditions are not satisfied. It is not analytic.

At $z = 0$, the derivative of $\phi$ is undefined by the above expressions. However, $\phi(x,y) = 0$ on both line $x = 0$ and $y = 0$. Therefore both $\frac{\partial \phi}{\partial x} = 0$ (along the line $y = 0$) and $\frac{\partial \phi}{\partial y} = 0$ (along the line $x = 0$). The C-R conditions are satisfied. Since the partial derivative are not continuous, C-R conditions do not guarantte the existence of the 1st ~~derivative~~ derivative. In fact

$$\lim_{\Delta z \to 0} \frac{\Delta f_2}{\Delta z} = \lim_{\Delta z \to 0} \frac{\sqrt{\Delta x \Delta y}}{\Delta x + i\Delta y} = \frac{\sqrt{\alpha}}{1 + i\alpha} \qquad \text{for } \Delta y = \alpha \Delta x.$$

2a put the unit square in a Cartesian coordinate as shown

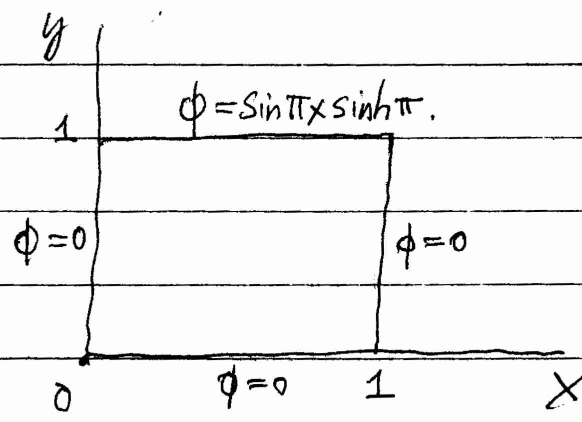
The boundary conditions are

$\phi = 0$ for $x = 0$

$\phi = 0$ for $x = 1$

$\phi = 0$ for $y = 0$.

$\phi = \sin\pi x \sinh\pi$ for $y = 1$.

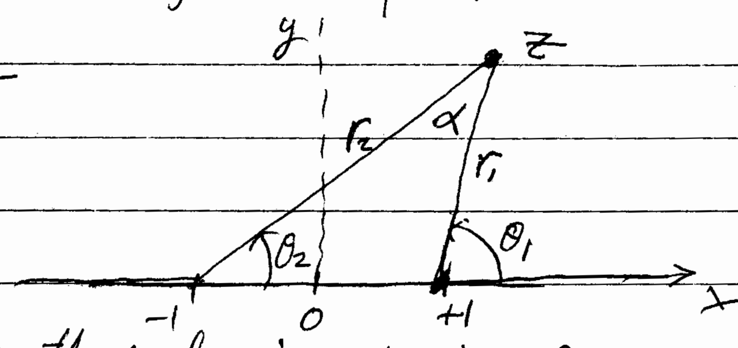



We are looking for an analytic function (say) whose real part will satisfies these conditions. Realizing $\sin\pi x = 0$ for $x=0$ and $x=1$, and $\sinh\pi y = 0$ for $y = 0$, it seems one choice of $\phi$ is

$$\phi(x,y) = \sin\pi x \sinh\pi y.$$

Which satisfies all four boundary conditions. It is the imaginary part of $\cos\pi z$, ~~w~~ which is analytic. Thus $\nabla^2\phi = 0$.

2b. In the ~~sk~~ picture on the right




$$\alpha = \theta_1 - \theta_2$$

~~The value of α is:~~

$$\alpha = \begin{cases} 0 & \text{when } z \text{ is on the real axis and right of } x=1. \\ \pi & \text{" " " " " " " between } -1 \text{ and } 1 \\ 0 & \text{" " " " " " " and left of } x=-1. \end{cases}$$

~~Thus~~

Now consider the analytic function $\frac{1}{\pi}\ln\frac{z-1}{z+1} = \frac{1}{\pi}\ln\frac{r_1}{r_2} + i\,\frac{\theta_1-\theta_2}{\pi}$.

Thus $\phi = \frac{\theta_1-\theta_2}{\pi}$ is the solution.

3. From the relation of the base vectors, one gets

$$\frac{\partial}{\partial r}(\hat{e}_r, \hat{e}_\theta, \hat{e}_z) = (0, 0, 0)$$

$$\frac{\partial}{\partial \theta}(\hat{e}_r, \hat{e}_\theta, \hat{e}_z) = (\hat{e}_\theta, -\hat{e}_r, 0)$$

$$\frac{\partial}{\partial z}(\hat{e}_r, \hat{e}_\theta, \hat{e}_z) = (0, 0, 0)$$

$$X = r\hat{e}_r + z\hat{e}_z$$

$$dX = \frac{\partial X}{\partial r}dr + \frac{\partial X}{\partial \theta}d\theta + \frac{\partial X}{\partial z}dz$$

$$= \hat{e}_r dr + r\frac{\partial \hat{e}_r}{\partial \theta}d\theta + \hat{e}_z dz = \hat{e}_r dr + \hat{e}_\theta r d\theta + \hat{e}_z dz.$$

$$\nabla = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z\frac{\partial}{\partial z}.$$

$$\nabla \cdot \underset{\sim}{V} = \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z\frac{\partial}{\partial z}\right)\cdot\left(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{e}_z V_z\right)$$

Using $\hat{e}_\alpha \cdot \hat{e}_\beta = \delta_{\alpha\beta} = \begin{cases} 0, & \alpha \neq \beta \\ 1, & \alpha = \beta. \end{cases}$ and the derivatives of the base vector above, one gets

$$\nabla \cdot V = \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}.$$

$$\nabla \times V = \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z\frac{\partial}{\partial z}\right)\times\left(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{e}_z V_z\right)$$

Using the right-handedness of the base vectors $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$ and $\hat{e}_r \times \hat{e}_r = \hat{e}_\theta \times \hat{e}_\theta = \hat{e}_z \times \hat{e}_z = 0$, One gets

$$\nabla \times V = \hat{e}_r\left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}\right) + \hat{e}_\theta\left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right) + \hat{e}_z\left(\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right).$$

4. From the ~~relation~~ expression of the base vector, one get the 1st derivatives of the base vectors as follows

$$\frac{\partial}{\partial r}(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi) = (0, 0, 0)$$

$$\frac{\partial}{\partial \theta}(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi) = (\hat{e}_\theta, -\hat{e}_r, 0)$$

$$\frac{\partial}{\partial \phi}(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi) = (\sin\theta\, \hat{e}_\phi, \cos\theta\, \hat{e}_\phi, -\sin\theta\, \hat{e}_r - \cos\theta\, \hat{e}_\theta).$$

$$dX = \frac{\partial X}{\partial r}dr + \frac{\partial X}{\partial \theta}d\theta + \frac{\partial X}{\partial \phi}d\phi \quad \text{, where } X = r\hat{e}_r$$

$$= \hat{e}_r dr + r\frac{\partial \hat{e}_r}{\partial \theta}d\theta + r\frac{\partial \hat{e}_r}{\partial \phi}d\phi.$$

$$= \hat{e}_r dr + \hat{e}_\theta r d\theta + \hat{e}_\phi r\sin\theta\, d\phi.$$

$$\nabla = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}.$$

$$\nabla\cdot V = \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\cdot\left(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{e}_\phi V_\phi\right)$$

$$= \frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{2V_r}{r} + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi} + \frac{V_\theta\cos\theta}{r\sin\theta}.$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}.$$

$$\nabla\times V = \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\times\left(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{e}_\phi V_\phi\right)$$

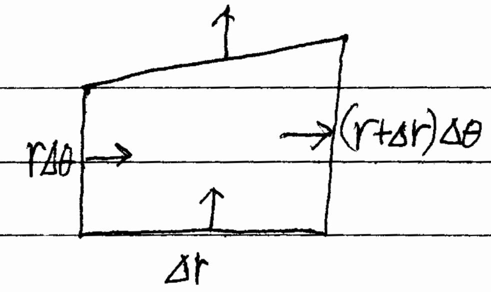
$$= \hat{e}_r\left(\frac{1}{r}\frac{\partial V_\phi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial \phi}\right) + \hat{e}_\theta\left(\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial \phi} - \frac{\partial V_\phi}{\partial r}\right) + \hat{e}_\phi\left(\frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right)$$

$$\hat{e}_\theta \times \frac{\partial \hat{e}_\theta}{\partial \theta}\frac{V_\theta}{r} + \frac{V_\phi}{r\sin\phi}\hat{e}_\phi \times \frac{\partial \hat{e}_\phi}{\partial \phi}$$

$$= \hat{e}_r\left(\frac{1}{r}\frac{\partial V_\phi}{\partial \theta} + \frac{\cos\theta}{r\sin\theta}V_\phi - \frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial \phi}\right)$$

$$+ \hat{e}_\theta\left(\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial \phi} - \frac{\partial V_\phi}{\partial r} - \frac{V_\phi}{r}\right) + \hat{e}_\phi\left(\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right)$$

5. Let the elementary area $A = r\Delta r\Delta\theta$.




Change of area in $r$-direction

$$= \Delta t\left[v_r(r+\Delta r)(r+\Delta r)\Delta\theta - v_r(r)r\Delta\theta\right]$$

$$= \Delta t\Delta\theta\left[r\frac{\partial v_r}{\partial r}\Delta r + v_r\Delta r\right] = \Delta t\, A\left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right)$$

In the $\theta$-direction $= \Delta t\left[v_\theta(\theta+\Delta\theta)\Delta r - v_\theta(\theta)\Delta r\right]$

$$= \Delta t A\,\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}.$$

Total change,

$$\frac{1}{A}\frac{\Delta A}{\Delta t} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta},$$

Compare this with $\nabla\cdot v$ in prob 3

$$V = \hat{e}_r v_r + \hat{e}_\theta v_\theta$$

$$\Delta V = \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial\theta}\Delta\theta = \hat{e}_r\left[\frac{\partial v_r}{\partial r}\Delta r + \left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r}\right)r\Delta\theta\right]$$

$$+\hat{e}_\theta\left[\frac{\partial v_\theta}{\partial r}\Delta r + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right)r\Delta\theta\right]$$

$$\boxed{\begin{array}{ll} a_{11} = \frac{\partial v_r}{\partial r}, & a_{12} = \frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r} \\ a_{21} = \frac{\partial v_\theta}{\partial r}, & a_{22} = \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r} \end{array}}$$

$$= \hat{e}_r(a_{11}\Delta r + a_{12}r\Delta\theta)$$

$$\hat{e}_\theta(a_{21}\Delta r + a_{22}r\Delta\theta)$$

$$(\Delta v)_r = a_{11}\Delta r + a_{12}r\Delta\theta = -\frac{1}{2}(a_{21}-a_{12})r\Delta\theta + a_{11}\Delta r + \frac{1}{2}(a_{21}+a_{12})r\Delta\theta$$

$$(\Delta v)_\theta = a_{21}\Delta r + a_{22}r\Delta\theta = \frac{1}{2}(a_{21}-a_{12})\Delta r + a_{22}r\Delta\theta + \frac{1}{2}(a_{21}+a_{12})\Delta r$$

Solid rotation $\doteq \frac{1}{2}(a_{21}-a_{12})$

$$= \frac{1}{2}\left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right).$$

Compare this with the $z$-component of $\nabla\times\underset{\sim}{v}$ in prob 3.