Effective models are often used by scientists to understand the behavior of complicated systems with many interacting components. Typically, one desires to gather some information about the effective behavior of these systems at the expense of information about the individual components. Naturally such models arise in fields like statistical physics, chemistry, economics, finance, and social science. Classical examples of such effective models arising in statistical physics are fluid equations like the Navier-Stokes and Euler equations, and kinetic equations like the Boltzmann, Vlasov-Landau, and Fokker-Planck equations. These models are often used in lieu of direct numerical simulation of the molecular dynamics, since for instance, a simulation of a system of $10^{20}$ interacting particles is well beyond the computational limits of even the most advanced computers.

Naturally, there are two questions one can ask about an effective model:

1. Do the equations make sense? (are they well-posed?)
2. Can one justify the effective model from the underlying system? (what is the size of the error?)

There are many difficult and well-known mathematical problems in this area since many of the asymptotic problems that arise are naturally very singular, and the question of well-posedness of the effective model (for instance, Navier-Stokes) are highly non-trivial.

Much of my research is dedicated to understanding the validity of various effective models of many particle systems in classical physics, typically with the addition of a stochastic perturbation. Part of this involves studying the well-posedness of continuum models in fluid mechanics and kinetic theory from a rigorous mathematical framework. While another part of this involves studying rigorous procedure of coarse-graining many particle systems into quasi-particle systems with noise and dissipations. Broadly speaking, this means I work on a wide range of problems in partial differential equations (PDE) and stochastic analysis, arising in statistical and continuum mechanics. This includes kinetic theory, fluid mechanics, turbulence, transport theory, many particle systems, limit theorems, and stochastic partial differential equations.

The following sections outline several completed and currently active projects of mine. The first two sections are related to well-posedness of certain physical PDE with a stochastic perturbation that acts on the transport structure of the equation. Specifically, Section 1 describes a project involving the study of the Boltzmann equation under the influence of an external stochastic forcing. The resulting paper shows the global existence of renormalized martingale solutions to the equation and is set to appear in the Archives of Rational Mechanics and Analysis. Section 2 describes a more recent work on well-posedness of certain stochastic continuity equations under very weak regularity conditions on the coefficients. The work is an analogue of the theory of renormalized solutions to transport equations of DiPerna/Lions which has been submitted to Communications in PDE and is under review. The next two sections is devoted to work done in my dissertation dissertation, is describes a mesoscopic stochastic particle model that has both fluid and particle character. In contrast to the stochastic Boltzmann equation, this system models the mesoscopic behavior of a fluid where the noise takes the form of intrinsic fluctuations as opposed to external forcing. I address the questions of well-posedness, long-time behavior, and its connection to the microscopic system. Finally, in the last section I describe some recent work on systematic coarse-graining for dissipative linear systems using a general abstract framework. I introduce a notion of a strong dissipation limit where the coarse-grained system can be reduced to a hierarchy of dissipative approximations. In the last section I outline a research agenda, including several currently active projects involving stochastic kinetic equations, and SDE’s with rough diffusion coefficients.
1 Renormalized martingale solutions to the stochastic Boltzmann equation

Many models of turbulence involve forcing the equations of fluid mechanics by noise. For instance the stochastically forced incompressible Navier-Stokes equations are given by

$$\partial_t u + u \cdot \nabla u + \nabla p - \Delta u = \xi, \quad \text{div} u = 0,$$

where $\xi(t, x)$ is a certain white in time Gaussian field. From a physical perspective, this model introduces environmental fluctuations, which can contribute to the onset of turbulence. Existence of global in time weak solutions to (1) is an important first step to understanding this equation and was first shown by Bensoussan and Temam [4] under special forms of noise and has since received much attention in the mathematical literature (a relatively recent survey of the many results is given in [17]). A natural question to ask is whether this noise can be deduced from a more general form of noise at the kinetic level.

In a work with another student Scott Smith [58], we initiated a study of the Boltzmann equation with stochastic forcing,

$$\partial_t f + v \cdot \nabla_x f + \text{div}_v(f \circ \dot{W}) = B(f, f),$$

The forcing $\dot{W}(t, x, v)$ is a Gaussian noise, white in time, and colored in phase space $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$. The product between $f$ and $\dot{W}$ is interpreted in the Stratonovich sense.

This equation describes the evolution of the one-particle phase space density $f(t, x, v)$ associated to a rarefied gas undergoing elastic binary collisions and subject to environmental noise. The elastic binary collisions are modeled by Boltzmann collision operator $f \mapsto B(f, f)$, a quadratic operator that acts pointwise $(t, x)$ and non-locally in $v$. The environmental noise $\dot{W}$ acts on the gas externally in the sense that each particle is driven by the same realization of the noise $\dot{W}$. Specifically in the absence of collisions, each particle in the gas would satisfy the same Stratonovich SDE

$$\dot{X}_t = V_t, \quad \dot{V}_t = \dot{W}(t, X_t, V_t).$$

In [58], we study the existence of global in time solutions to (1) for a general class of ‘large’ initial data in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$ with certain entropy and moment bounds. In the deterministic setting, such a result was proven by DiPerna/Lions [14] for the Boltzmann equation in the renormalized sense, and improved in subsequent works [13, 14, 39, 40]. Our main result is a proof of the existence of, global in time, probabilistically weak (in the sense of a solution to the martingale problems) solutions to (1) in the renormalized sense (the same notion of solution used in [14]). The main theorem is stated informally as follows:

**Theorem 1.1.** Let $f_0$ have finite mass, energy and entropy,

$$\|(1 + |x|^2 + |v|^2 + |\log f_0|)|f_0\|_{L^1_{x,v}} < \infty$$

and suppose that the coefficients $\{\sigma_k : k \in \mathbb{N}\}$, $\text{div}_v \sigma_k = 0$, satisfy certain regularity and summability conditions. Then for a certain class of collision operators $B(f, f)$, there exists a probabilistically weak (martingale) solution $\{f_t : t \geq 0\}$ to (1) satisfied in the renormalized sense. The process $\{f_t : t \geq 0\}$ takes values in the cone of non-negative $L^1(\mathbb{R}^n \times \mathbb{R}^n)$ functions and has bounded $p$-th moments of mass, energy, entropy, and entropy dissipation,

$$\mathbb{E}\|(1 + |x|^2 + |v|^2 + |\log f|)f\|_{L^p_x(L^1_{x,v})}^p \leq \infty, \quad \mathbb{E}\|D(f)\|_{L^1_{x,v}}^p < \infty,$$

for each $p \in [1, \infty)$, where the entropy dissipation is defined by $D(f) := -\int_{\mathbb{R}^n} (\log f) B(f, f) dv$. Moreover, $\{f_t : t \geq 0\}$ has a continuous modification with paths in $C([0, T]; L^1(\mathbb{R}^n \times \mathbb{R}^n))$.

The proof of Theorem 1.1 largely inspired by techniques laid out in [14], and more specifically on the later work by Lions [40] on the Vlasov-Maxwell-Boltzmann equation. In the deterministic case, one of the key elements of the proof is the strong compactness obtained velocity averages of solutions to the transport equation [27–29]. In our paper we prove a stochastic velocity averaging result in $L^1$ which shows, under certain conditions, that a family of solutions $\{f_n : n \in \mathbb{N}\}$ to a stochastic kinetic transport equation has the property that the laws of the velocity averages are tight on $L^1_{x,v}$. This result should be compared with other stochastic velocity averaging results in the literature [22, 41].
2 Well-posedness for stochastic continuity equations with rough coefficients

Itô stochastic differential equations (SDEs)

$$\partial_t X(t, x) = b(t, X(t, x)) + \sigma(t, X(t, x)) \dot{W}, \quad X(0, x) = x \in \mathbb{R}^d,$$

arise in many areas of mathematics, physics, statistics and finance. In certain applications, e.g. models of turbulence, complex fluids and fluid interfaces, the drift $b(t, x) \in \mathbb{R}^d$ and diffusion matrix $\sigma(t, x) \in \mathbb{R}^{n \times n}$ can be irregular (non-Lipschitz), and the classical theory of well-posedness for (2) does not apply. Never-the-less, existence and uniqueness of probabilistically strong solutions to (2) can still be obtained under significantly weaker conditions than the classical theory would allow (see [7, 64–67]). If the noise is additive, $\sigma = I$, the noise is known to have a regularizing effect on the dynamics, giving well-posedness to the SDE (2) for drifts $b$ where the ODE is known to be ill-posed, an effect known as ‘regularization by noise’. Here, well-posedness has been shown when $b \in L^\infty_{t,x}$, $b \in L^\infty_t C^\alpha_x$ [24] and $b \in L^1_t L^2_x$, $\frac{2}{q} + \frac{d}{p} < 1$ [20, 36], all of which are lower regularity than even the weakest well-posedness theories for associated ODE.

One approach to studying the SDE (2) when $b$ and $\sigma$ have low regularity is to study weak solutions to the associated stochastic continuity equation

$$\partial_t f + \text{div}(b f) - \text{div}(\text{div}(a f) + \text{div}(\sigma f \dot{W})) = 0, \quad f|_{t=0} = f_0 \in L^p,$$

where $a = \frac{1}{2} \sigma \sigma^\top$. For smooth coefficients, a theory of Kunita the solution to this equation can be uniquely represented by solutions to the SDE (2), by push forward under the stochastic flow $x \mapsto X(t,x)$,

$$f(t, \cdot) = X(t, \cdot) \# f_0.$$

Linear stochastic PDE of the type (2) arise in the Kraichnan model of turbulence advection as well as stochastic kinetic equations with random external forces. As with the SDE, when $\sigma = I$ there have been a number of works devoted to the well-posedness of weak solution to (2).

This problem was studied by Flandoli, Gubinelli and Priola [24, 25], as well as by [6, 53].

In a paper [56], I develop a general well-posedness theory for weakly differentiable $\sigma$ and $b$ analogous to the theory of renormalized solutions developed by DiPerna/Lions [12]. The theory is related to the works of [32, 45, 46, 49] on turbulent advection of passive scalars.

More importantly, the PI showed that this precise renormalizability allows on to treat the additive case $\sigma^k = e_k$ when $b$ is in $L^1_t(L^p)$ satisfying the condition $\frac{2}{q} + \frac{2}{p} < 1$, with out using regularity of the stochastic flow or a duality method as in [21] and [3].

The hope is to generalize the DiPerna/Lions theory for the deterministic transport equation [12] to one for the stochastic transport equation. When $\sigma$ is rough and degenerate, a version of the DiPerna/Lions theory for the associated Kolmogorov equation has been developed by Figalli [23] and by Lions/Le Bris [38]. However, there appear to be few results in the literature concerning solutions to the stochastic continuity (2).

Some preliminary work on this has already been done in the context of stochastic kinetic equations in our paper on stochastic Boltzmann [58], where the uniqueness problem was studied in the context of renormalized solutions to the transport equation. We employed the usual commutator estimates used in [12], along with a new double commutator that arises due to the stochastic term. Interestingly, in [58] we were only able to obtain uniqueness for solutions in $L^p$ for $p > 2$, when $\sigma \in W^{1,2p/p-2}$ and $\text{div} \ a \in W^{1,2p/p-2}$. The existence and uniqueness of (probabilistically) strong solutions in $L^p$ for $p \in [1,2]$ when $\sigma$ is rough appears to be rather non-trivial. There are potential applications for understanding (2) in the $L^1$ setting in the theory of kinetic equations with stochastic transport.

I plan to investigate stochastic transport equations with rough $\sigma$ in a later work.
3 Coarse-graining of a one-dimensional particle system

Since the advent of modern computing, it is increasingly of interest to try to model the system of $10^{20}$ particles using a system involving a much smaller number of particles, say $10^8$, a number which makes simulation accessible to modern computers. One approach in this direction, often referred to as ‘coarse-graining’, involves simulating a collection of $10^8$ ‘pseudo-particles’ each representing a collection of roughly $10^{12}$ particle of the system of interest. In order to be of any use, these pseudo-particles must evolve according to their own equations of motion, while effectively capturing the large scale behavior of the original system. In general, these equations of motion will be stochastic and involve some dissipative phenomena even though these effects may be absent in the original equations of motion. Understanding the emergence of noise and dissipation as well as the validity of the associated coarse-grained model is fundamental to the understanding of the emergence of dissipative phenomena as well as the efficient simulation of large systems. An example of such a model is ‘Dissipative Particle Dynamics’ (DPD), [18, 19, 31, 34, 47, 59], used to describe complex fluids and polymeric fluids. Rigorously justifying these quasi-particle models is a difficult task. Derivations of such models typically involve poorly defined projection operator methods along ad-hoc approximations.

As part of my dissertation [55] I studied the problem of justifying quasi-particle models from a rigorous mathematical perspective.

We will consider a Hamiltonian system of $N$ particles in one dimension with positions $x = (x_1, \ldots, x_N) \in \mathbb{T}^N$ and velocities $v = (v_1, \ldots, v_N) \in \mathbb{R}^N$, satisfying periodic boundary conditions and interacting through nearest neighbors. The particles are governed by the Hamiltonian $H = \sum_{i=1}^N \frac{1}{2}v_i^2 + V(x_i - x_{i-1})$, and the potential $V(r)$ is singular enough at the origin so that particles cannot cross. The particles evolve according to Hamilton’s equations

$$\dot{x}_i = v_i, \quad \dot{v}_i = -V'(x_{i+1} - x_i) + V'(x_i - x_{i-1}).$$

It is useful to introduce the deformation coordinates $r_i = x_i - x_{i-1}$ and view the particle system $(r, v) \in \mathbb{R}_+^N \times \mathbb{R}^N$ as a lattice system on the periodic lattice $T_N = \mathbb{Z} \backslash \mathbb{N}$. Naturally this system conserves total length $L = \sum_i r_i$, momentum $P = \sum_i v_i$ and total energy $H = \sum_i \frac{1}{2}v_i^2 + V(r_i)$.

Naturally this problem lends itself to a statistical description. Instead of dealing with trajectories of individual particles, we study the distribution function $f^N(t, r_1, \ldots, r_N, v_1, \ldots, v_N)$ satisfying Liouville’s equation

$$\partial_t f^N + \sum_{i \in T_N} (v_i - v_{i-1}) \partial_{r_i} f^N + V'(r_i)(\partial_{v_i} - \partial_{v_{i-1}})f^N = 0.$$ 

The distribution function $f^N$ is symmetric with respect to cyclic permutations of the particles. The system has a family of invariant (equilibrium) Gibbs densities $\{g^N_\alpha : \alpha \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+\}$ given by

$$g^N_\alpha = \frac{1}{Z(\alpha)} \exp \{-\alpha_1 L - \alpha_2 P - \alpha_3 H\}$$

To define quasi-particles, we partition the periodic lattice $T_N$ into $M$ disjoint cells $T_N = \bigcup_i \Lambda_i$ of size $K = N/M$. For each configuration $(r, v)$ we define the length, momentum, and energy of the cell $\Lambda_i$ by the averages of the locally conserved variables over the cells

$$\ell_i = \frac{1}{K} \sum_{j \in \Lambda_i} r_j, \quad \rho_i = \frac{1}{K} \sum_{j \in \Lambda_i} v_j, \quad e_i = \frac{1}{K} \sum_{j \in \Lambda_i} \frac{1}{2}v_j^2 + V(r_j).$$

Naturally the dynamics of the averages $U_i(t) = (\ell_i(t), \rho_i(t), e_i(t))$ still conserve length, momentum and energy. However they do not satisfy a closed evolution equation. The goal is to approximate the dynamics of the averages by closed system which capture the large scale structure of the original system and conserves the length energy and momentum. If one defines a map $\Pi : (\mathbb{R}_+ \times \mathbb{R})^N \to (\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+)^M$ by

$$\Pi_k(\ell_1, \rho_1, e_1, \ldots, \ell_N, \rho_N, e_N) = \Pi(\ell_1, \rho_1, \ldots, \rho_N, v_N, e_N),$$

then this closure problem is equivalent to approximating $\dot{f}^M = \Pi \dot{f}^N$ by a closed evolution equation.

In [55]

However, the evolution will not be closed, that is, the evolution at any time cannot be determined from just the state (or potentially the entire history) of $(\ell_i(t), \rho_i(t), e_i(t))$. In order to obtain a
As part of my dissertation [55] I have introduced a modification of DPD in one-dimension, which I refer to fluid-particle model. As with DPD, it can be viewed as a discretization of the compressible Landau-Lifshitz-Navier-Stokes equations of fluctuating hydrodynamics in Lagrangian coordinates. It is a stochastic interacting particle system that conserves volume, momentum and energy and models the effects of bulk viscosity and thermal conductivity. In one dimension, since the interactions are only through nearest neighbors, and the dynamics preserve particle ordering, the fluid particle model has the benefit of also being lattice model, similar to a one-dimensional anharmonic chain with dissipation.

The model consists of a collection of $N$ parcels $U = \{(\ell_i, u_i, e_i) : i \in T_N\}$ periodically arranged on the discrete torus $T_N$ with phase space $\Gamma^N = (\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+)^{TN}$. Each parcel $U_i = (\ell_i, u_i, e_i)$ is defined by its volume $\ell_i$, velocity $u_i$, and internal energy $e_i$. The parcels are governed by a concave entropy function $S(\ell, e) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, which satisfies the first law of thermodynamics

$$\partial_\ell S(\ell, e) = T(\ell, e)^{-1} > 0, \quad \partial_e S(\ell, e) = P(\ell, e),$$

where $T(\ell, e)$ is the temperature and $P(\ell, e)$ is the pressure function. Denote by $S_i, T_i, P_i$ the entropy, temperature, and pressure of the $i$th parcel given by evaluating these functions on the pair $(\ell_i, e_i)$. The parcels evolve according to a diffusion process $\{U_t : t \geq 0\}$ in $\Gamma^N$. The generator of this process is given formally by

$$\mathcal{L} = \sum_{i \in T_N} (u_i - u_{i-1}) \partial_{\ell_i} + P_i \mathcal{X}_i - \eta(u_i - u_{i-1}) \mathcal{X}_i - \kappa(T_i - T_{i-1}) \mathcal{Y}_i + \eta T_i \mathcal{X}_i^2 + T_i T_{i-1} \mathcal{Y}_i^2.$$

where $\{X_i : i \in T_N\}$ and $\{Y_i : i \in T_N\}$ are two families of differential operators representing vector fields tangent to certain manifolds defining pairwise momentum and energy exchange. They are given by

$$X_i = \partial u_i - \partial u_{i-1} - (u_i - u_{i-1}) \partial e_i, \quad Y_i = \partial e_i - \partial e_{i-1}.$$

The constants $\eta, \kappa > 0$ play the role of bulk-viscosity and thermal-conductivity in the model, but can be taken to be varying. The functions $L = \sum_{i \in T_N} \ell_i, P = \sum_{i \in T_N} u_i, H = \sum_{i \in T_N} \frac{1}{2} u_i^2 + e_i$, corresponding to total length, momentum, and energy, are in the null space of $\mathcal{L}$, and therefore conserved by the dynamics. The SDE associated to $\mathcal{L}$ is an Itô type drift-diffusion process with degenerate diffusion and unbounded coefficients.

The total entropy $S = \sum_{i \in T_N} S_i$ is not strictly dissipated as, as one might expect being a discrete version of Navier-Stokes. This barrier to dissipation is due to the noise (the same behavior is observed, for instance, in stochastic gradient dynamics). In general this can lead to problems of well posedness for the fluid-particle model, often finite time blow up in the form of parcel volumes or energies collapsing to 0. However, certain assumptions on the concavity on $S(\ell, e)$ allow one to obtain enough dissipation of $S$ to show existence and uniqueness of a process $\{U_t : t \geq 0\}$ which stays in the interior of $\Gamma^N$. More precisely we have,

**Theorem 3.1.** Suppose that the entropy function $S(\ell, e)$ approaches $-\infty$ when either $e$ or $\ell$ approach 0, grows sub-linearly when either $e$ or $\ell$ approach $\infty$ and satisfies the lower bound

$$\partial^2_s S(\ell, e) \geq (1 - \gamma) T(\ell, e)^{-2}$$

for some $\gamma \in (0, 1)$. Then for any $N$ and $U_0$ in the interior of $\Gamma^N$, the SDE associated to $\mathcal{L}$ has a unique, strong solution $\{U_t : t \geq 0\}$ with continuous modification and which remains in the interior of $\Gamma^N$ for all $t \geq 0$.

The process $\{U_t : t \geq 0\}$ has a family of invariant probability measures $\{\nu_\alpha : \alpha \in \Gamma\}$ on $\Gamma^N$ given by

$$d\nu_\alpha = \frac{1}{Z(\alpha)} \exp \left\{ -\alpha_1 L - \alpha_2 P - \alpha_3 H + S \right\} \prod_{i \in T_N} T(\ell_i, e_i)^{-1} d\ell_i du_i de_i,$$

where $Z(\alpha)$ is a normalization factor. The measures $\{\nu_\alpha : \alpha \geq 0\}$ play the role of the canonical ensemble often used in classical particle systems. However, they are not ergodic, since they may be decomposed into a convex combination of so-called micro-canonical measures $\{\nu_a : a \in \Gamma\}$, defined by conditioning any of the canonical measures $\nu_a$ to the manifold $\mathcal{G}_a = \{(L, P, H) = a\}$ for each $a \in \Gamma$. These micro-canonical measures are precisely the ergodic invariant measures for the dynamics.
One would expect that for each $a \in \Gamma$, $\nu(a)$ is the unique invariant probability measure for $\{U_t : t \geq 0\}$ with $U_0 \in \mathcal{G}_a$, and that the law of $\{U_t : t \geq 0\}$, converges to $\nu(a)$ at a certain rate. However, this is non-trivial since the noise is degenerate, even when constrained to the manifold $\mathcal{G}_a$. Therefore certain difficult hypoelliptic estimates are are required. However, if one can show smoothness of the transition probabilities, then the entropy $S$ can be used to construct a Lyapunov function that allows one to obtain explicit rates of convergence for the law. Some preliminary work has been done on this in my dissertation [55] for a special case of the entropy function $S(\ell, e)$, specifically in the \textit{ideal gas case}

$$S(\ell, e) = a \log e + \log \ell, \quad a > 1.$$ 

One of the long term goals for this work is to justify the use of the fluid particle model (or a similar system) from the dynamics of a one-dimensional system of classical particles. Particularly, if one considers a ‘block averaging’ procedure which averages conserved quantities over large cells and takes a suitable many-particle limit. There is some evidence that the dynamics in (3) might naturally arise in a coarse-graining procedure of a many particle system. Indeed, the energy dynamics with generator

$$\mathcal{L}_{GL} = \sum_i -(T_i - T_{i-1}) Y_i + T_i T_{i-1} Y_i^2,$$

is of \textit{Ginzburg Landau} type and has arisen recently in the study of the energy dynamics of \textit{weakly coupled} systems [5, 15, 43] and studied further in [44]. The invariant measures obtained in these works are precisely of the form $\exp\{-\gamma \sum_i U(e_i)\} \text{ where } U(e) \sim \log(e)$, which corresponds to the ideal-gas-like entropy $S(e) = a \log e$ considered above. Of course this motivates the study of \textit{viscosity dynamics} that exchange kinetic and internal energy according to

$$\mathcal{L}_V = \sum_i -(u_i - u_{i-1}) X_i + T_i X_i^2,$$

particularly in the context of weakly coupled systems.

Both the Ginzburg-Landau and viscosity dynamics described above likely have some interesting hydrodynamics scaling limits to non-linear diffusion equations, similar to to [30, 62]. However, as mentioned in [44], previous results do not apply in this case due to the logarithmic ‘single-site potential’ $U(e) \sim \log e$ and the associated difficulties with showing a spectral gap.

Some of this work has been explored in my thesis [55]. I intend to pursue this problem and it’s many related questions in future works.

4 Markov approximations of coarse-grained linear systems

A popular framework for coarse-graining of the equations of classical mechanics is the \textit{projection operator formalism}. Although the idea is likely much older, it’s development is often attributed to Mori [51] and Zwanzig [69, 70]. The Mori-Zwanzig approach is to recast the coarse-graining problem into one involving projections onto a certain spaces of ‘slow-variables’, and close the evolution to obtain a certain, ‘non-Markovian’ or ‘generalized Langevin’ equation. The projection operator formalism is often applied directly to the Liouville equation associated to a classical particle system to obtain a ‘kinetic’ Markovian equation with noise and dissipation included. Usually this is justified by sending a certain parameter, representing the interaction strength between particles, to zero, the so-called \textit{weak coupling limit}. This is, for instance, the case with the Boltzmann-Grad limit, considered in Lanford’s derivation of the Boltzmann equation from a system of hard spheres [37], where one sends (among other things) the diameter of the spheres to zero. However, the derivation of such kinetic equations from classical mechanics is a very difficult mathematical problem, and most derivations either formal, or (in the case of Lanford’s proof) only valid for very short times.

It is natural to wonder whether one might have better success using the projection operator formalism starting from a system that has some dissipation. This is often the approach taken when deriving the equations of fluid mechanics, either from a Hamiltonian system that has some noise added (for instance [54]), or from the Boltzmann equation. Such approaches have been very successful over the years, since the dissipation usually provides some form of ergodicity and a mechanism for equilibration that can be used to show closeness to hydrodynamic behavior.
In a forthcoming work [57] with C.D. Levermore, we study the projection-operator framework when the underlying system has some dissipation. We approach this from the point of view of $C_0$ contraction semi-groups on Hilbert spaces. Specifically, consider the abstract Cauchy problem in a Hilbert space $H$,

$$\frac{df}{dt}(t) = Lf(t), \quad f(0) = f_0,$$

where $L$ is a linear, densely defined, maximally dissipative operator, in the sense that

$$\langle f, Lf \rangle_H \leq 0 \quad \text{for all} \quad f \in D(L), \quad \text{and} \quad \text{Ran}(I - L) = H.$$

Of course, this implies that $L$ generates a $C_0$ contraction semi-group, $\{e^{tL} : t \geq 0\}$, and that (4) has a unique solution given by $f(t) = e^{ct}f_0$.

In the projection operator formalism one considers an orthogonal projection $P$ onto a space $\hat{H}$ of ‘relevant variables’, along with the complement projection $\tilde{P} = I - P$ onto the space $\tilde{H}$ of ‘irrelevant variables’.

One may think of (4) as the (backward) Kolmogorov equation associated to a Markov process with generator $L$. If the process has an equilibrium measure $\mu$ then we can take $H$ to be $L^2(\mu)$. This also applies to the linearized Boltzmann equations in the $L^2$ setting with $\mu$ being a Maxwellian. In this setting, a natural choice of orthogonal projection is the average with respect to the equilibrium measure $\mu$ conditioned on a particular value of a ‘slow variable’.

The aim is to study the evolution of the relevant part of the dynamics $\hat{f}(t) = Pf(t)$ in $\hat{H}$. Upon decomposing the evolution of $f(t)$ into a relevant part $\hat{f}(t)$ and irrelevant part $\tilde{f}(t) = P\tilde{f}(t)$, one may formally apply Duhammel’s principle to obtain a closed equation for $\hat{f}$,

$$\frac{d\hat{f}}{dt}(t) = P L P \hat{f}(t) + \int_0^t \Psi(t-s)\hat{f}(s)ds, \quad \Psi(t) = P L P e^{tL}P L P,$$

where we have defined, for brevity, $\tilde{L} := \tilde{P} L P$. The equation (4), known as the generalized master equation, was first derived by Nakajima [52] and Zwanzig [69]. The family of operators $\{\Psi(t) : t \geq 0\}$ are often referred to as the ‘Brussels school’ collision operator (not to be confused with the Boltzmann collision operator), and appears as a non-local (in time) contribution in the equation (4), accounting for ‘non-Markovian’ memory effects in the evolution of $\hat{f}$.

The applicability of equation (4) is limited due to the complicated nature of $\Psi(t)$ and the non-local (in time) behavior. A common approach to remedy this is to make a so-called ‘Markov approximation’, that produces an autonomous equation, heuristically corresponding to another Markov process. This usually involves estimating $\Psi(t)$ to be ‘white in time’

$$\Psi(t) \approx \Psi_0 \delta(t),$$

for some operator $\Psi_0$. Such Markov approximation often justified in the weak coupling limit, where the generator takes the form $L = L_0 + \alpha L_1$, with $L_0$ a skew adjoint operator and $H_0$ is an invariant subspace of $L_0$, and $\alpha > 0$ a small coupling parameter. There are several interesting rigorous results in this direction by Davies [8–10], when $L_1$ is bounded.

Our main focus, however, is not to obtain a dissipative approximation starting from classical mechanics, but instead to obtain further dissipative approximations starting from a system that already has some dissipation. If we denote $S$ the symmetric (dissipative part) of $L$ and $A$ the anti-symmetric part, then, roughly speaking, we are interested in the case where the generator takes the form $\tilde{L} = A + \alpha^{-1}S$, where $\alpha > 0$ is a small parameter. Instead of a weak coupling limit, we instead expect this to be valid in a strong dissipation limit, analogous to sending the Knudsen number to 0 in the Boltzmann equation.

In [57], we propose a Markov approximation of the form,

$$\Psi_0 = P L \tilde{P} (-S)^{-1} \tilde{P} L P,$$

where $\tilde{S}$ denotes the symmetric (dissipative) part of the generator $\tilde{L}$. One of the benefits of the approximation given in (4) is that the operator $L_\Psi := P L P + \Psi_0$ is still a dissipative operator, $\langle f, L_\Psi f \rangle \leq 0$, which we obtain from a Green-Kubo-like variational formula.
The approximation (4), can be viewed as an approximation of the more general ‘long-time’ Markov approximation

\[ \Psi_0 = \mathcal{P} \mathcal{L} \tilde{\mathcal{P}} (\mathcal{L}^{-1} \tilde{\mathcal{P}} \mathcal{L}) = \int_0^\infty \Psi(s) \, ds. \]

Indeed, under the assumption that \( \tilde{\mathcal{A}} \) is ‘small’ relative to \( \tilde{\mathcal{S}} \), one can use this to obtain ‘higher-order’ approximations by truncating the formal Neumann series,

\[ \Psi_0 = -\sum_{k \geq 0} \mathcal{P} \mathcal{L} \tilde{\mathcal{P}} \tilde{\mathcal{S}}^{-1} (\tilde{\mathcal{A}} \tilde{\mathcal{S}}^{-1})^k \tilde{\mathcal{P}} \mathcal{L} \mathcal{P}. \]

It is important to remark that not every truncation of (4) will lead to a generator \( \mathcal{L}_\Psi \) which is dissipative. Specifically in [57] we show that only truncations at the \((4n + 1)\)th term in the series (4) will lead to dissipative operators. This is analogous to the Chapman Enskog expansion of the Boltzmann equation, where certain truncations lead to the Burnett and super-Burnett equations, which are generally ill-posed and non-dissipative.

Of course, if \( \tilde{\mathcal{S}} \) has a non-trivial null space, it is not clear that the operator \( \Psi_0 \), defined in (4) is well defined on a dense subset of \( H \). In [57], we discuss several conditions on \( \mathcal{L} \), particularly a strong sector condition, that lead to a well-defined approximation. We consider several examples related to Markov semigroups, and make connections to the literature on the central limit theorem for Markov processes [33, 35, 63].

5 Research agenda and future work

In this section I outline a research agenda and discuss various future works. Some of these are a continuations of the work described above, while others are of independent interest.

5.1 Problems related to stochastic Boltzmann

5.1.1 Hydrodynamic limit

As mentioned in Section 1, one of the primary motivations for studying the stochastic Boltzmann equation in [58] was to understand the kinetic origin of certain stochastically forced models in fluid mechanics, particularly the stochastic Navier-Stokes equations. Naturally, one would like to show that this stochastic model for the Boltzmann equation converges to, in a suitable scaling limit, a stochastically forced fluid model, particularly to a weak (in the analytic and probabilistic sense) solution. Indeed, one of the reasons for considering such weak (in the renormalized and probabilistic sense) solutions to the stochastic Boltzmann equation is with a hydrodynamic limit in mind.

The study of the hydrodynamic limit from the Boltzmann equation to weak solutions of a fluid equation was first initiated by Bardos, Golse and Levermore [1, 2] and subsequently several important hydrodynamic limits were proven rigorously starting from a renormalized solution to the Boltzmann equation, and converging to a weak solution of various fluid equations, including the Stokes-Fourier system, the acoustic system, the incompressible Navier-Stokes system, and the incompressible Euler system (see the book of Laure Saint-Raymond [60] for a recent review of such results).

In a collaboration with S. Smith, we are studying the hydrodynamic limit problem for the stochastically forced Boltzmann equation starting from the solutions we obtained in [58]. In the incompressible Navier-Stokes limit, we expect to obtain convergence (in law) to Leray-type martingale solutions of the stochastic Navier-Stokes equations obtained by Flandoli/Gatarek [26]. Interestingly when the stochastic forcing on the Boltzmann equation conserves energy, i.e. \( \sigma_\tilde{S}(x, v) \cdot v = 0 \), we expect the limiting stochastic Navier Stokes equation to have multiplicative noise.

5.1.2 Questions of uniqueness and pathwise solutions

The solutions to the stochastic Boltzmann studied in [58] are considered in a general \( L^1 \) framework, and as a result are very weak. Not only are they analytically weak, interpreted in the renormalized sense, but they are also probabilistically weak, interpreted as a solution to a martingale problem. In the context of solutions
arising from general ‘large’ $L^1$ initial data, it is not clear whether global solutions exist which are both classical and probabilistically strong (pathwise). Indeed in the deterministic case, global classical solutions in the $L^1$ setting are a major open problem. Accordingly, any questions of uniqueness for such general solutions are currently out of reach. In fact, given the connections to the equations of fluid mechanics, it is not at all surprising.

There are, however, situations where one might expect to obtain unique pathwise solutions to the stochastic Boltzmann equation. These are the so-called perturbative solutions in the ‘$L^\infty$ setting’ [61] and the ‘$L^2$-setting’ [42], where there has been significant success in proving existence of unique global classical solutions.

It is natural to expect that in either the $L^\infty$ or $L^2$ setting, one might obtain unique global pathwise solutions to the stochastic Boltzmann equation. At the moment it is not clear whether such $L^\infty$ solutions can exist for general noise coefficients $\{\sigma_k(x, v) : k \geq 0\}$ as there is no reason for the solution to remain uniformly close to a (deterministic) Maxwellian. However, there is evidence in the $L^2$ setting that such strong solutions can be obtained by perturbing about a stochastic local Maxwellian. This is done, for instance in my ongoing work on the hydrodynamic limit. I intend to investigate the feasibility of such solutions to the stochastic Boltzmann equation in later works.

### 5.1.3 Linearized stochastic Boltzmann equation

A natural first step towards understanding the long-time behavior of the stochastic Boltzmann equation is to instead consider the stochastic linearized Boltzmann equation in the $L^2$ setting,

$$\partial_t f + v \cdot \nabla_x f + \nabla_v f \circ \dot{X} - \frac{1}{2}(vf) \circ \dot{X} = \mathcal{L}f,$$

where $\dot{X}(t, x)$ is a Gaussian noise that is white in time and colored in space (does not depend on velocity) and $\mathcal{L}$ is a linearized Boltzmann collision operator

$$\mathcal{L}f = M^{-1/2}B(M, M^{1/2}f), \quad M(v) = (2\pi)^{-n/2} \exp\left(-\frac{1}{2}|v|^2\right),$$

In the case of a deterministic, time dependent forcing $\dot{X}(t, x)$, this problem was studied in [16], in terms of the decay rates of the solution in $L^2$. In fact the analysis presented in [16] also applies to the nonlinear Boltzmann equation with time-dependent forcing in the $L^2$ setting.

In the stochastic case, when $\mathcal{L}$ is the Fokker-Planck operator, this problem has been investigated in a recent paper by De Moor, Rodruigez, and Vovelle [11]. Here, the authors are able to obtain unique pathwise solutions, prove existence and uniqueness of an invariant measure, and show exponentially fast convergence towards the invariant measure.

I expect that similar results are obtainable for the stochastic linearized Boltzmann equation (5.1.3) and intend to investigate this in a future work. Such problems should shed some light on the analogous problem for the non-linear stochastic Boltzmann equation.

### References


