Problem 1: (20 points)

(i) Let $E$ be a subset of $\mathbb{R}$ that has finite outer measure. Prove that there is a $G_\delta$ subset $A$ of $\mathbb{R}$ such that $E \subseteq A$ and $m^*(E) = m^*(A)$.

(ii) Prove that if $m^*(A \sim E) = 0$, then $E$ is measurable.

Solution:

(i) Since $E$ has finite outer measure, there are open sets $\{O_k\}$ containing $E$ such that $m^*(O_k) - m^*(E) < 1/k$, therefore $A \equiv \bigcap_k O_k$ is $G_\delta$ and contains $A$ such that $m^*(A) - m^*(E) = 0$.

(ii) If $m^*(A \sim E) = 0$, then $A \sim E$ is measurable and therefore $E = A \sim (A \sim E)$ is measurable.

Problem 2: (20 points)

(i) State the Lebesgue Dominated Convergence Theorem and the Vitali Convergence Theorem.

(ii) Deduce the Lebesgue Dominated Convergence Theorem from the Vitali Convergence Theorem.

Solution:

(i) **Lebesgue Dominated Convergence:** Let $\{f_n\}$ be a sequence extended real-valued measurable functions over $X \subseteq \mathbb{R}$ such that for some $f$, $\{f_n\} \to f$ pointwise a.e. in $X$. If there exists a nonnegative and integrable $g$ such that $|f_n| \leq g$ for all $n$, then $f$ is integrable and

$$\lim_{n \to \infty} \int_X f_n = \int_X f.$$
Vitali Convergence: Let \( \{f_n\} \) be a sequence extended real-valued measurable functions over \( X \subseteq \mathbb{R} \) such that for some \( f \), \( \{f_n\} \to f \) pointwise a.e. in \( X \). If \( \{f_n\} \) are uniformly integrable, i.e. there exists a \( \delta > 0 \) such that if \( m(A) < \delta \) then

\[
\int_A |f_n| < \epsilon
\]

for all \( n \), and tight, i.e. there exists a measurable set \( E \subset X \) for which

\[
\int_{X \sim E} |f_n| < \epsilon
\]

for all \( n \), then \( f \) is integrable over \( X \) and

\[
\lim_{n \to \infty} \int_X f_n = \int_X f.
\]

(ii) Suppose \( \{f_n\} \to \{f\} \) pointwise a.e. and suppose \( g \) dominates \( \{f_n\} \) and is integrable. It is not hard to see that \( \{f_n\} \) are uniformly integrable and tight since by the integrability of \( g \) we may always find \( \delta \) so that if \( m(A) < \delta \) then

\[
\int_A |f_n| \leq \int_A g < \epsilon
\]

and we can also find a set \( E \) of finite measure such that

\[
\int_{X \sim E} |f_n| \leq \int_{X \sim E} g < \epsilon.
\]

Therefore we may apply Vitali Convergence.

Problem 3: (20 points)

(i) Let \( f: (0, 1] \to \mathbb{R} \) be continuous. Suppose the following finite limit exists:

\[
\lim_{n \to \infty} \int_{1/n}^1 f(x) \, dx.
\]

Is \( f \) Lebesgue integrable over \((0, 1]\)?

(ii) If \( f \) is nonnegative, is \( f \) Lebesgue integrable over \((0, 1]\)?

Solution:
(i) This is not true, consider the functions \( f(x) = \sin(2\pi/x)/x^2 \). Since \( f \) is continuous we may change variables in the Riemann integral to find that
\[
\lim_{n \to \infty} \int_{1/n}^{1} \frac{\sin \left( \frac{2\pi}{x} \right)}{x^2} \, dx = \lim_{n \to \infty} \int_{1}^{n} \sin(2\pi x) \, dx = 0.
\]
However, \( f \) is not Lebesgue integrable, since
\[
\int_{1/n}^{1} |f(x)| \, dx = \int_{1}^{n} |\sin(2\pi x)| \, dx = 2n \to \infty.
\]

(ii) If \( f \) is nonnegative, then \( f_n = f\chi_{[1/n,1]} \) is nonnegative and \( \{f_n\} \to f \) in a monotone increasing fashion. Therefore by monotone convergence,
\[
\lim_{n \to \infty} \int_{1/n}^{1} f(x) \, dx = \int_{0}^{1} f \, dx < \infty,
\]
and so \( f \) is integrable.

\[\blacksquare\]

**Problem 4: (20 points)** A real valued function on \( \mathbb{R} \) is said to be Lipschitz provided there is a constant \( c > 0 \) such that
\[
|f(u) - f(v)| \leq c|u - v| \quad \text{for all} \quad u, v.
\]

(i) Show that a Lipschitz functions takes sets of measure zero to sets of measure zero.

(ii) Show that a Lipschitz function on a closed, bounded interval is of bounded variation.

**Solution:**

(i) Since \( f \) is Lipschitz, it maps intervals to intervals, and so for any open interval \( I \)
\[
\ell(f(I)) = \sup_{x,y \in I} |f(x) - f(y)| \leq c \cdot \sup_{x,y \in I} |x - y| = \ell(I).
\]
Furthermore if \( \{I_i\}_{i=1}^{\infty} \) is an open cover of a measureable set \( A \), then \( \{f(I_i)_{\epsilon/2^{i+1}}\}_{i=1}^{\infty} \) is an open cover of \( f(A) \), where \( f(I)_{\epsilon} \) is an \( \epsilon \) thickening of \( I \) given by
\[
f(I)_{\epsilon} = \bigcup_{x \in f(I)} (x - \epsilon, x + \epsilon).
\]
It follows that for any \( \epsilon > 0 \) we can choose the open covering so that
\[
m(f(A)) \leq \sum_{i=1}^{\infty} \ell(f(I_i)_{\epsilon/2^{i+1}}) \leq c \cdot \sum_{i=1}^{\infty} \left( \ell(I_i) + \frac{\epsilon}{2^i} \right) < c \cdot (m(A) + 2\epsilon).
\]
Taking \( \epsilon \to 0 \) we conclude
\[
m(f(A)) \leq c \cdot m(A).
\]
Therefore \( m(f(A)) = 0 \) whenever \( m(A) = 0 \).
(ii) Let \( \{x_i\}_{i=0}^N \) be the end points of a finite partition \( \mathcal{P}_I \) of the interval \( I = [a, b] \). If \( f \) is Lipschitz, then

\[
\sum_{i=1}^N |f(x_i) - f(x_{i-1})| \leq c \cdot \sum_{i=1}^N |x_i - x_{i-1}| = c \cdot (b - a) < \infty
\]

Since this bound is independent of the partition, we conclude that \( f \) is bounded variation.

Problem 5: (20 points)

(i) Define what it means for a sequence \( \{f_n : \mathbb{R} \rightarrow \mathbb{R}\} \) of integrable functions to be uniformly integrable.

(ii) Show that a sequence of nonnegative, integrable functions \( \{f_n : \mathbb{R} \rightarrow \mathbb{R}\} \) is uniformly integrable and tight if and only if the following property holds:

for each \( \epsilon > 0 \), there is a constant \( C \) such that for all \( n \)

\[
\int_{\{x \in \mathbb{R} | f_n(x) > c\}} |f_n| < \epsilon.
\]

Solution:

(i) A sequence \( \{f_n\} \) is uniformly integrable if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) so that if \( A \) is measurable and satisfies \( m(A) < \delta \), then

\[
\int_A |f_n| \leq \epsilon
\]

uniformly in \( n \).

(ii) Suppose \( \{f_n\} \) is uniformly integrable. Let \( \epsilon > 0 \) and choose \( \delta > 0 \) as in the uniformly integrability condition. The claim is that there is a \( c_0 \) such that for all \( n \)

\[
m(\{\mathbb{R} | f_n \geq c_0\}) < \delta.
\]

Using the claim it easily follows by uniform integrability that

\[
\int_{\{\mathbb{R} | f_n > c_0\}} f_n < \epsilon,
\]

uniformly in \( n \). To see the claim, suppose not, that is suppose for every \( c > 0 \) there exists an \( n_0 \) such that

\[
m(\{\mathbb{R} | f_{n_0} \geq c\}) > \delta.
\]
If this is indeed the case, then upon choosing $c > 2\epsilon/\delta$ we may find a set $A_{n_0} \subseteq \{ \mathbb{R} \mid f_{n_0} \geq c \}$ such that

$$\delta \leq m(A_{n_0}) < \delta.$$ 

By Chebyshev and uniform integrability of $\{f_n\}$, we conclude that

$$\epsilon < \frac{c\delta}{2} \leq c \cdot m(A_{n_0}) \leq \int_{A_n} f_n < \epsilon,$$

which is a contradiction.

For the converse, choose $c$ so that $\int_{\{x \in \mathbb{R} \mid f_n > c\}} f_n < \epsilon/2$. Then for any measurable set $A \subseteq \mathbb{R}$ we may write

$$\int_A f_n \leq \int_{\{x \in \mathbb{R} \mid f_n > c\}} f_n + \int_{\{x \in A \mid f_n \leq c\}} f_n$$

$$\leq \epsilon/2 + c \cdot m(A)$$

uniformly in $n$. Choosing $m(A) < \epsilon/2c$, we conclude

$$\int_A f_n < \epsilon,$$

uniformly in $n$.  ■