Problem 28: Let $f$ be a step function on $[a, b]$. Find a formula for its total variation.

Solution: Without loss of generality we may assume that $f$ is right continuous, however then proof works for more general step functions. This means that there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ so that $f$ may be written as

$$f = \sum_{k=1}^{n} c_k \chi_{[x_{k-1}, x_k)}$$

where any two consecutive values of $\{c_k\}$ are distinct. It follows that

$$V(f, P) = \sum_{k=1}^{n} |c_k - c_{k-1}|.$$

However, for any other partition $Q$, we can always adjoin $P$, so that $Q' = Q \cup P$ is a partition of $[a, b]$. We may then divide this partition up as $Q' = \bigcup_{k=1}^{n} Q_k$, where

$$Q_k = \{q \in Q' \mid x_{k-1} \leq q < x_k\}.$$

Define $q^*_k = \max\{Q_k\}$. We point out that if $q \in Q_k$ then $f(q) = c_k$ and therefore $V(f_{[x_{k-1}, q^*_k]}, Q_k) = 0$. We then conclude that

$$V(f, Q) \leq V(f, Q') = \sum_{k=1}^{n} V(f_{[x_{k-1}, q^*_k]}, Q_k) + \sum_{k=1}^{n} |f(x_k) - f(q^*_k)| = \sum_{k=1}^{n} |c_k - c_{k-1}|.$$

Since this upper bound on $V(f, Q)$ is achieved at $Q = P$, we conclude

$$TV(f) = \sum_{k=1}^{n} |c_k - c_{k-1}|.$$

Problem 29:

(i) Define

$$f(x) = \begin{cases} x^2 \cos(1/x^2) & \text{if } x \neq 0, x \in [-1, 1] \\ 0 & \text{if } x = 0. \end{cases}$$

Is $f$ of bounded variation on $[-1, 1]$?
(ii) Define
\[ g(x) = \begin{cases} 
  x^2 \cos \left(\frac{1}{x}\right) & \text{if } x \neq 0, x \in [-1, 1] \\
  0 & \text{if } x = 0.
\end{cases} \]

Is \( g \) of bounded variation on \([-1, 1]\)?

Solution:

(i) It suffices to consider \( f \) on the interval \([0, \pi^{-1/2}]\). Let \( n \) be some natural number, and consider the partition
\[ P_n = \left\{ 0, \frac{1}{\sqrt{n\pi}}, \frac{1}{\sqrt{(n-1/2)\pi}}, \ldots, \frac{1}{\sqrt{2\pi}}, 1 \right\}. \]

We see that
\[ V(f, P_n) = 1 + 1/2 + \ldots + 1/n \]

is divergent and so \( f \) is not of bounded variation.

(ii) Note that
\[ g'(x) = 2x \cos \left(\frac{1}{x}\right) - \sin \left(\frac{1}{x}\right). \]

and that for \(-1 < x < 1\)
\[ |g'(x)| \leq 3. \]

It follows that \( g \) is Lipschitz and therefore of bounded variation.

\[ \square \]

**Problem 35:** For \( \alpha \) and \( \beta \) positive numbers, define the function \( f \) on \([0, 1]\) by
\[ f(x) = \begin{cases} 
  x^\alpha \sin \left(\frac{1}{x^\beta}\right) & \text{for } 0 < x \leq 1 \\
  0 & \text{for } x = 0.
\end{cases} \]

Show that if \( \alpha > \beta \), then \( f \) is of bounded variation on \([0, 1]\), by showing that \( f' \) is integrable over \([0, 1]\). Then show that if \( \alpha \leq \beta \), then \( f \) is not of bounded variation on \([0, 1]\).

**Solution:** If \( \alpha > \beta \), then
\[ f'(x) = \alpha x^{\alpha-1} \sin \left(\frac{1}{x^\beta}\right) - \beta x^{\alpha-\beta-1} \cos \left(\frac{1}{x^\beta}\right). \]

Since \( f \) is \( C^1 \) and bounded on \((0, 1)\) we can use the fundamental theorem of calculus for Riemann integrals to conclude that for any partition \( P \)
\[ V(f, P) = \sum_{k=0}^{n} |f(x_k) - f(x_{k-1})| \leq \int_0^1 |f'| \]
and therefore
\[ TV(f) \leq \int_0^1 |f'| \leq \int_0^1 \alpha x^{\alpha-1} + \beta x^{\alpha-\beta-1} < \infty, \]
since \( \alpha > 0 \) and \( \alpha - \beta > 0 \).

If \( \alpha \leq \beta \), choose a partition \( P_n = \{0, a_n, a_{n-1}, \ldots, a_1\} \), where
\[ a_n = \left( \frac{n\pi}{2} \right)^{-1/\beta} \]
then we see that
\[ V(f, P_n) = \sum_{k=1}^{n} \left( \frac{k\pi}{2} \right)^{-\alpha/\beta} \]
Note that this series diverges as \( n \to \infty \) since \( \alpha/\beta \leq 1 \). Therefore
\[ \lim_{n \to \infty} V(f, P_n) \leq TV(f) = \infty. \]

Problem 37: Let \( f \) be a continuous function on \([0, 1]\) that is absolutely continuous on \([\epsilon, 1]\) for each \( 0 < \epsilon < 1 \).

(i) Show that \( f \) may not be absolutely continuous on \([0, 1]\).

(ii) Show that \( f \) is absolutely continuous on \([0, 1]\) if it is increasing.

(iii) Show that the function \( f \) on \([0, 1]\), defined by \( f(x) = \sqrt{x} \) for \( 0 \leq x \leq 1 \), is absolutely continuous, but not Lipschitz, on \([0, 1]\).

Solution:

(i) Consider
\[ f(x) = \begin{cases} 
  x \sin \left( \frac{1}{x} \right) & \text{if } 0 < x \leq 1 \\
  0 & \text{if } x = 0
\end{cases} \]
Note that on \([\epsilon, 1]\)
\[ |f'(x)| = \left| \sin(1/x) - \frac{1}{x} \cos(1/x) \right| \leq 1 + \frac{1}{\epsilon} \]
Therefore \( f \) is Lipschitz on \([\epsilon, 1]\) and hence absolutely continuous on \([\epsilon, 1]\). However, we know from Problem 35 that \( f \) is not BV on \([0, 1]\) and therefore not absolutely continuous on \([0, 1]\).
(ii) Suppose \( f \) is increasing, let \( \eta > 0 \) and choose \( \epsilon \) so that
\[
f(\epsilon) - f(0) < \eta/2.
\]
Since \( f \) is absolutely continuous on \([\epsilon, 1]\) choose \( \delta > 0 \) in response to \( \eta/2 \) in the absolute continuity condition on \([\epsilon, 1]\). Now suppose that \( \{(a_k, b_k)\}_{k=1}^{N} \) is a collection of disjoint open intervals such that \( \sum_{k=1}^{N} |b_k - a_k| < \delta \). We may assume that \( \epsilon \) is not contained in any of the intervals since we may always split any such interval \((a_{k_0}, b_{k_0})\) into two consecutive intervals \((a_{k_0}, \epsilon) \cup (\epsilon, b_{k_0})\) such that, by the fact that \( f \) is increasing,
\[
|f(b_{k_0}) - f(\epsilon)| + |f(\epsilon) - f(a_{k_0})| = |f(b_{k_0}) - f(a_{k_0})|.
\]
It follows that we may divide the set of intervals into \( n_- \) intervals to the left of \( \epsilon \), \( \{(a_k^-, b_k^-)\}_{k=1}^{n_-} \) and \( n_+ \) intervals to the right of \( \epsilon \), \( \{(a_k^+, b_k^+)\}_{k=1}^{n_+} \). We observe that
\[
\sum_{k=1}^{n_+} |b_k^+ - a_k^-| < \delta
\]
and so by uniform integrability on \([\epsilon, 1]\),
\[
\sum_{k=1}^{n_+} |f(b_k^+) - f(a_k^+)| < \eta/2.
\]
Also, since \( f \) is increasing,
\[
\sum_{k=1}^{n_-} |f(b_k^-) - f(a_k^-)| \leq f(\epsilon) - f(0) < \eta/2.
\]
Therefore
\[
\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \eta.
\]

(iii) Clearly \( \sqrt{x} \) is not Lipschitz on \([0, 1]\) since its derivative is unbounded as \( x \to 0 \). However, \( \sqrt{x} \) is increasing and is Lipschitz on \([\epsilon, 1]\) for any \( \epsilon > 0 \). Therefore by (ii), \( f \) is absolutely continuous on \([0, 1]\).

\[\blacksquare\]

**Problem 39:** Use the preceding problem to show that if \( f \) is continuous and increasing on \([a, b]\), then \( f \) is absolutely continuous on \([a, b]\) if and only if for each \( \epsilon \), there is a \( \delta > 0 \) such that for a measurable subset \( E \) of \([a, b]\),
\[
m^*(f(E)) < \epsilon \text{ if } m(E) < \delta.
\]

**Solution:** Suppose that \( f \) is absolutely continuous, and let \( \delta > 0 \) be chosen in response to \( \epsilon > 0 \) in the absolute continuity condition as generalized in Problem 38. Suppose that
$E \subseteq [a, b]$ is measurable and $m(E) < \delta/2$. We can find a countable cover of disjoint open intervals $\{(a_k, b_k)\}_{k=1}^{\infty}$ of $E$ so that
\[
m\left(\bigcup_{k=1}^{\infty} I_k\right) < \delta/2 + m(E) < \delta
\]
and therefore
\[
\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \epsilon.
\]
Since $f$ is increasing and continuous $\{f(I_k)\}_{k=1}^{\infty}$ is an open cover of $f(E)$ and $m(f(I_k)) = f(b_k) - f(a_k)$. By the definition of outer measure we conclude that
\[
m^*(f(E)) \leq \sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \epsilon.
\]
For the converse, let $\delta > 0$ be chosen to satisfy the converse condition with $\epsilon > 0$ and let $\{(a_k, b_k)\}_{k=1}^{n}$ be a finite collection of disjoint open interval such that $E = \bigcup_{k=1}^{n} (a_k, b_k)$ and $m(E) < \delta$. Since $f$ is increasing we see that $\{f((a_k, b_k))\}_{k=1}^{n}$ are disjoint and by continuity $m(f((a_k, b_k))) = f(b_k) - f(a_k)$. By the converse condition
\[
\sum_{k=1}^{n} |f(b_k) - f(a_k)| = m\left(\bigcup_{k=1}^{n} (a_k, b_k)\right) < \epsilon.
\]

**Problem 40:** Use the preceding problem to show that an increasing absolutely continuous function $f$ on $[a, b]$ maps sets of measure zero onto sets of measure zero. Conclude that the Cantor-Lebesgue function $\varphi$ is not absolutely continuous $[0, 1]$ since the function $\psi$, defined by $\psi = x + \varphi(x)$ for $0 \leq x \leq 1$, maps the Cantor set to a set of measure 1.

**Solution:** Problem 39 obviously shows that an increasing absolutely continuous function maps sets of measure 0 to sets of measure 0.

By construction, the map $\psi$ maps the Cantor set to a set of measure 1 and therefore cannot be absolutely continuous. Since $x$ is absolutely continuous and the sum of two absolutely continuous functions is continuous we conclude that $\varphi$ cannot be continuous.

**Problem 41:** Let $f$ be an increasing absolutely continuous function on $[a, b]$. Use (i) and (ii) below to conclude that $f$ maps measurable sets to measurable sets.

(i) Infer from the continuity of $f$ and the compactness of $[a, b]$ that $f$ maps closed sets to closed sets and therefore maps $F_\sigma$ sets to $F_\sigma$ sets.

(ii) The preceding problem tells us that $f$ maps sets of measure zero to sets of measure zero.
**Solution:** Since $f$ is continuous, $f$ maps compact sets to compact sets. In particular since $[a, b]$ is compact $f$ maps closed subsets of $[a, b]$ to closed sets. Also since

$$f \left( \bigcup_{k=1}^{\infty} A_k \right) = \bigcup_{k=1}^{\infty} f(A_k)$$

we conclude then that $f$ maps $F_\sigma$ sets to $F_\sigma$ sets. Now since any measurable set $A$ can be written as $A = F \cup E$ where $F$ is an $F_\sigma$ set and $E$ is measure zero, we conclude that

$$f(A) = f(F) \cup f(E)$$

is measurable, being the union of an $F_\sigma$ set and a set of measure zero.

■