Real Analysis HW 7 Solutions

Problem 37: Let \( f \) be an integrable function on \( E \). Show that for each \( \epsilon > 0 \), there is a natural number \( N \) for which if \( n \geq N \), then \( \left| \int_{E_n} f \right| < \epsilon \), where \( E_n = \{ x \in E | |x| \geq n \} \).

Solution: Note that \( \{ E_n \} \) are a descending collection of sets such that \( \bigcap_{k=1}^{\infty} E_k = \emptyset \) and therefore
\[
m \left( \bigcap_{k=1}^{\infty} E_k \right) = m(\emptyset) = 0.
\]
Therefore since \( f \) is integrable on \( E \), by the continuity of integration
\[
\lim_{n \to \infty} \int_{E_n} f = \int_{\bigcap_{k=1}^{\infty} E_k} f = 0.
\]
The result follows. \( \blacksquare \)

Problem 38: For each of the two functions \( f \) on \([1, \infty)\) defined below, show that \( \lim_{n \to \infty} \int_{1}^{n} f \) exists while \( f \) is not integrable over \([1, \infty)\). Does this contradict the continuity of integration?

(i) Define \( f(x) = (-1)^n/n \) for \( n \leq x \leq n + 1 \).

(ii) Define \( f(x) = (\sin(x))/x \) for \( 1 \leq x < \infty \).

Solution: Neither of these cases contradict the continuity of integration since continuity assumes that \( f \) integrable over a set \( E \) containing each of the sets in the collection \( \{ E_k \} \), which we do not have in either of these cases.

(i) We can easily see that
\[
\lim_{n \to \infty} \int_{1}^{n} f = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}
\]
converges. However
\[
\lim_{n \to \infty} \int_{1}^{n} |f| = \sum_{k=1}^{\infty} \frac{1}{k}
\]
diverges. So \( f \) is not integrable.
(ii) Note that since \( \sin(x)/x \) is continuous and bounded so we may use the Riemann integral. It follows from the fundamental theorem of calculus for Riemann integrals

\[
\int_1^n \frac{\sin x}{x} \, dx = - \int_1^n \left( \frac{d}{dx} \frac{\cos x}{x} + \frac{\cos x}{x^2} \right) \, dx
\]

\[
= \cos 1 - \frac{\cos n}{n} - \int_1^n \frac{\cos x}{x^2} \, dx
\]

It follows that if \( n, m > 0 \),

\[
\left| \int_m^n \frac{\sin x}{x} \, dx \right| \leq \frac{2}{\min\{n, m\}} + \int_m^n \frac{1}{x^2} \, dx
\]

\[
\leq \frac{4}{\min\{n, m\}} \to 0
\]

as \( m, n \) goes to infinity. Therefore \( \int_1^n \frac{\sin x}{x} \, dx \) is Cauchy and converges. In particular we have the well known result

\[
\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2},
\]

which can be gotten by complex integration or the Laplace transform. However \( \sin(x)/x \) is not Lebesgue integrable since

\[
\int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} \, dx \geq \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin x| \, dx = \frac{2}{(k+1)\pi}
\]

and therefore

\[
\lim_{n \to \infty} \int_1^n \frac{|\sin x|}{x} \, dx \geq \frac{2}{\pi} \sum_{k=2}^{\infty} \frac{1}{k} = \infty
\]

diverges.

\[\square\]

**Problem 40:** Let \( f \) be integrable over \( \mathbb{R} \). Show that the function \( F \) defined by

\[ F(x) = \int_{-\infty}^x f \quad \text{for all } x \in \mathbb{R} \]

is properly defined and continuous. Is it necessarily Lipschitz?

**Solution:** Clearly \( F \) is properly defined since \( f \) is integrable and therefore \( f \chi_{(-\infty, x]} \) is integrable for every \( x \). To see continuity, consider a sequence \( \{x_n\} \to x \) and define \( f_n = f \chi_{(-\infty,x_n]} \). Note that \( f_n \to f \chi_{(-\infty,x]} \) and \( |f_n| \leq f \). Therefore by dominated convergence,

\[
\lim_{n \to \infty} F(x_n) = \lim_{n \to \infty} \int_{-\infty}^x f_n = \int_{-\infty}^x f = F(x).
\]
For $F$ to be Lipschitz, we would need $f$ to be in $L^\infty$. Since $f$ is merely integrable we cannot guarantee that $F$ will be Lipschitz. A counter-example is $f = \frac{1}{\sqrt{x}}\chi_{(0,\infty)}(x)$. Using the Riemann integral we can show that

$$F(x) = \int_{-\infty}^{x} f = \int_{0}^{x} \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\chi_{(0,\infty)}(x),$$

which is not Lipschitz as the derivative is unbounded near 0.

\[ \square \]

\textbf{Problem 43:} Let $\{h_n\}$ and $\{g_n\}$ be uniformly integrable over $E$. Show that for any $\alpha$ and $\beta$, the sequence of linear combinations $\{\alpha f_n + \beta g_n\}$ are also uniformly integrable over $E$.

\textbf{Solution:} Let $\varepsilon > 0$ and choose $\delta_1$ and $\delta_2$ so if $m(A) < \delta_1$ then $\int_A |f_n| < \varepsilon/2|\alpha|$ and if $m(A) < \delta_2$, $\int_A |g_n| < \varepsilon/2|\beta|$. Choose $\delta = \min\{\delta_1, \delta_2\}$, then if $m(A) < \delta$,

$$\int_A |\alpha f_n + \beta g_n| \leq |\alpha| \int_A |f_n| + |\beta| \int_A |g_n| < \varepsilon.$$

\[ \square \]

\textbf{Problem 44:} Let $f$ be integrable over $\mathbb{R}$ and $\varepsilon > 0$. Establish the following three approximation properties.

(i) There is a simple function $\eta$ on $\mathbb{R}$ which has finite support and $\int_{\mathbb{R}} |f - \eta| < \varepsilon$

(ii) There is a step function $s$ on $\mathbb{R}$ which vanishes outside a closed, bounded interval and $\int_{\mathbb{R}} |f - s| < \varepsilon$.

(iii) There is a continuous function $g$ on $\mathbb{R}$ which vanishes outside a bounded set and $\int_{\mathbb{R}} |f - g| < \varepsilon$.

\textbf{Solution:}

(i) As shown in Problem 24, if $f$ is non-negative we may find an increasing sequence of non-negative simple functions $\{\phi_n\}$ with finite support such that $\phi_n \to f$ pointwise. It follows by monotone convergence that we can find a $\phi$ such that

$$\int_{\mathbb{R}} |f - \phi| = \int_{\mathbb{R}} f - \phi < \varepsilon.$$

For general $f$ we write $f = f^+ - f^-$, and find $\eta_1$ and $\eta_2$ simple and of finite support such that $\int_{\mathbb{R}} |f^+ - \eta_1| < \varepsilon/2$ and $\int_{\mathbb{R}} |f^- - \eta_2| < \varepsilon/2$. Since $f^+$ and $f^-$ have disjoint support, we see that $\eta_1$ and $\eta_2$ must also have disjoint support, therefore $\eta = \eta_1 - \eta_2$ is also simple with finite support and it follows that

$$\int_{\mathbb{R}} |f - \eta| \leq \int_{\mathbb{R}} |f^+ - \eta_1| + \int_{\mathbb{R}} |f^- - \eta_2| < \varepsilon.$$
By part (i), since we can approximate by simple functions, by the triangle inequality it suffices to show that the characteristic function $\chi_E$ of a bounded measurable set $E$ can be approximated by step functions. Note that since $E$ is measurable, we can find a disjoint collection of open intervals $\{I_k\}_{k=1}^{\infty}$ such that $O = \bigcup_{k=1}^{\infty} I_k$, and $m(O \sim E) < \epsilon/2$. Since $O$ must have finite measure we can find an $N$ large enough such that $m(\bigcup_{k=N+1}^{\infty} I_k) < \epsilon/2$. Therefore
\[ s = \sum_{k=1}^{N} \chi_{I_k} \]
is a step function and
\[ \int_{\mathbb{R}} |\chi_E - s| \leq \sum_{k=1}^{N} \int_{\mathbb{R}} |\chi_{E \cap I_k} - \chi_{I_k}| + \sum_{k=N+1}^{\infty} \int_{\mathbb{R}} \chi_{E \cap I_k} \]
\[ \leq m \left( \bigcup_{k=1}^{N} I_k \sim E \right) + m \left( \bigcup_{k=N+1}^{\infty} I_k \cap E \right) \]
\[ \leq m(O \sim E) + m \left( \bigcup_{k=N+1}^{\infty} I_k \right) < \epsilon. \]

(iii) Using part (ii), once again we see by the triangle inequality that it suffices to show that any characteristic function of a bounded interval $\chi_{[a,b]}$ can be approximated by a continuous function. Let $g$ be the continuous function which is 1 on $[a + \epsilon/2, b - \epsilon/2]$ and linearly interpolated to 0 outside of $[a, b]$, then
\[ \int_{\mathbb{R}} |\chi_{[a,b]} - g| < m([a, a + \epsilon/2] \cup (b - \epsilon/2, b]) = \epsilon. \]

Problem 46: (Riemann-Lebesgue) Let $f$ be integrable over $(-\infty, \infty)$. Show that
\[ \lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) \cos nx dx = 0. \]

Solution: Using the result from problem 44, we know there exists a step function $s$, vanishing outside of a closed bounded interval such that $\int_{\mathbb{R}} |f - s| < \epsilon/2$. Let $s$ be the step function which we write in canonical form as
\[ s = \sum_{k=1}^{K} s_k \chi_{(a_k, b_k)}, \]
where $\{(a_k, b_k)\}$ are a disjoint collection of bounded open intervals and $\{s_k\}$ are distinct. Note that it doesn’t matter that we don’t define $s$ at the end points of the intervals since
they are a set of measure 0. We see that
\[
\left| \int_{-\infty}^{\infty} s(x) \cos nx \, dx \right| \leq \sum_{k=1}^{K} |s_k| \left| \int_{a_k}^{b_k} \cos nx \, dx \right|
\]
\[
= \sum_{k=1}^{K} \frac{|s_k|}{n} \left| \sin nb_k - \sin na_k \right|
\]
\[
\leq \frac{2K \max\{|s_i|\}}{n}.
\]
Therefore if \( n > N \equiv 4K \max\{|s_i|\}/\epsilon \), we conclude
\[
\left| \int_{-\infty}^{\infty} f(x) \cos nx \, dx \right| \leq \int_{-\infty}^{\infty} |f(x) - s(x)| \, dx + \int_{-\infty}^{\infty} s(x) \cos nx \, dx
\]
\[
< \epsilon/2 + \epsilon/2 = \epsilon.
\]

\textbf{Problem 49:} Let \( f \) be integrable over \( \mathbb{R} \). Show that the following four assertions are equivalent:

(i) \( f = 0 \) a.e. on \( \mathbb{R} \).

(ii) \( \int_{\mathbb{R}} fg = 0 \) for every bounded measurable function \( g \) on \( \mathbb{R} \).

(iii) \( \int_A f = 0 \) for every measurable set \( A \).

(iv) \( \int_{\mathcal{O}} f = 0 \) for every open set \( \mathcal{O} \).

\textbf{Solution:}

- For (i) \( \Rightarrow \) (ii) note that if \( f = 0 \) a.e., then \( f \cdot g = 0 \) a.e. for every bounded measurable function, and so \( \int_{\mathbb{R}} fg = 0 \).

- For (ii) \( \Rightarrow \) (iii), choose \( g = \chi_A \) for a measurable function \( A \).

- For (iii) \( \Rightarrow \) (iv), choose \( A = \mathcal{O} \) an open set.

- For (iv) \( \Rightarrow \) (i), note that since \( \{ f > 0 \} \) is measurable, then for any \( \delta > 0 \) we may find an open set \( \mathcal{O}_\delta \supseteq \{ f > 0 \} \) such that \( m(\mathcal{O}_\delta \sim \{ f > 0 \}) < \delta \). We may write

\[
0 = \int_{\mathcal{O}_\delta} f = \int_{\{ f > 0 \}} f + \int_{\mathcal{O}_\delta \sim \{ f > 0 \}} f = \int_{\{ f > 0 \}} f - \int_{\mathcal{O}_\delta \sim \{ f > 0 \}} |f|
\]

Since \( f \) is integrable we may choose \( \delta \) so that \( \int_{\mathcal{O}_\delta \sim \{ f > 0 \}} |f| < \epsilon \), therefore

\[
0 \leq \int_{f > 0} f = \int_{\mathcal{O}_\delta \sim \{ f > 0 \}} |f| < \epsilon.
\]
Since this holds for every $\epsilon > 0$, $\int_{f>0} f = 0$. However since $\{f > 1/n\} \subseteq \{f > 0\}$ we see that

$$0 = \int_{\{f>0\}} f \geq \int_{\{f > 1/n\}} f \geq \frac{1}{n} m(\{f > 1/n\}) \geq 0$$

and therefore $m(\{f > 1/n\}) = 0$. It follows that

$$m(\{f > 0\}) = \lim_{n \to \infty} m(\{f > 1/n\}) = 0.$$ 

Therefore $f \leq 0$ a.e. Making the same argument for $\{f < 0\}$, we conclude that $f = 0$ a.e.