Problem 23: Find an example of a Cauchy sequence of numbers that is not rapidly Cauchy.

Solution: Consider the \( \left\{ \frac{(-1)^n}{n} \right\}_{n=1}^{\infty} \). This is sequence converges to 0 and is therefore Cauchy. However
\[
\left| \frac{(-1)^n}{n} - \frac{(-1)^{n+1}}{n+1} \right| = \frac{n+2}{n(n+1)} \geq \frac{1}{n}
\]
and is therefore not rapidly Cauchy. ■

Problem 28: Assume \( E \) has finite measure and \( 1 \leq p < \infty \). Suppose \( \{f_n\} \) is a sequence of measurable functions that converges pointwise a.e. on \( E \) to \( f \). For \( 1 \leq p < \infty \), show that \( \{f_n\} \to f \) in \( L^p(E) \) if there is a \( \theta > 0 \) such that \( \{f_n\} \) belongs to and is bounded as a subset of \( L^{p+\theta}(E) \).

Solution: Suppose that there exists a \( \theta > 0 \) such that
\[
\|f_n\|_{L^{p+\theta}} \leq C
\]
uniformly in \( n \), then \( \|f\|_{L^{p+\theta}} \leq C \) by Fatou’s Lemma. We see by an application of Hölder’s inequality that
\[
\int_E |f - f_n|^p \leq m(E)^{\theta/(p+\theta)} \int_E |f - f_n|^{p+\theta} \leq 2C^{p+\theta}m(E)^{\theta/(p+\theta)}.
\]
This clearly implies that \( \{|f - f_n|^p\} \) is uniformly integrable and so by Vitali convergence \( \{f_n\} \to f \) in \( L^p(E) \). ■

Problem 36: Let \( S \) be a subset of a normed linear space \( X \). Show that \( S \) is dense in \( X \) if and only if each \( g \in X \) is the limit of a sequence in \( S \).

Solution: Suppose \( S \) is dense and \( g \in X \), then by definition of density for each \( n \geq 1 \) we may find a \( s_n \in S \) so that
\[
\|g - s_n\| < 1/n.
\]
It follows that \( \{s_n\} \to g \) in \( X \).

For the converse suppose let \( g \in X \) and choose a sequence \( \{s_n\} \subseteq S \) converging to \( g \). Clearly for any \( \epsilon > 0 \) we may choose \( n_0 \) large enough so that
\[
\|g - s_{n_0}\| < \epsilon
\]
and so $S$ is dense.

**Extra Problem:** Let $a < b$ and $\epsilon > 0$. Consider the function

$$f_\epsilon(x) = \frac{1}{e^{\frac{(x-a)(x-b)}{\epsilon}} + 1}$$

(i) Show that as $\epsilon \to 0$, $f_\epsilon(x) \to 1$ if $a < x < b$, and $f_\epsilon(x) \to 0$ for $x$ outside $[a, b]$. Thus

$$\lim_{\epsilon \to 0} f_\epsilon(x) = \chi_{[a,b]}(x) \text{ for almost all } x \in \mathbb{R}.$$  

(ii) For $1 \leq p < \infty$, prove that

$$\lim_{\epsilon \to 0} \|f_\epsilon(x) - \chi_{[a,b]}\|_{L^p(\mathbb{R})} = 0.$$

**Solution:**

(ii) If $a < x < b$ then $(x-a)(x-b)/\epsilon \to -\infty$ as $\epsilon \to 0$ and therefore $f_\epsilon(x) \to 1$. Also if $x$ is outside $[a, b]$ then $(x-a)(x-b)/\epsilon \to +\infty$ as $\epsilon \to 0$ and so $f_\epsilon(x) \to 0$. Therefore

$$\lim_{\epsilon \to 0} f_\epsilon(x) = \chi_{[a,b]}(x) \text{ for all } x \neq a, b.$$

(ii) Note that for $\epsilon < 1$,

$$|f_\epsilon(x) - \chi_{[a,b]}(x)| \leq e^{-(x-a)(x-b)}$$

and since $e^{-(x-a)(x-b)}$ decays faster than any inverse power of $x$ as $x \to \pm \infty$ it is in $L^p(\mathbb{R})$ for $1 \leq p < \infty$. Therefore by Dominated Convergence

$$\lim_{\epsilon \to 0} \|f_\epsilon(x) - \chi_{[a,b]}(x)\|_{L^p(\mathbb{R})} = 0.$$  

\[ \blacksquare \]