Problem 49: Let \( f \) be continuous on \([a, b]\) and differentiable almost everywhere on \((a, b)\). Show that
\[
\int_a^b f' = f(b) - f(a)
\]
if and only if
\[
\int_a^b \left[ \lim_{n \to \infty} \text{Diff}_{1/n} f \right] = \lim_{n \to \infty} \left[ \int_a^b \text{Diff}_{1/n} f \right].
\]

Solution: Since \( f \) is continuous and differentiable a.e. on \((a, b)\), we have that
\[
\lim_{n \to \infty} \left[ \int_a^b \text{Diff}_{1/n} f \right] = \lim_{n \to \infty} (\text{Av}_{1/n} f(b) - \text{Av}_{1/n} f(a)) = f(b) - f(a),
\]
and
\[
\int_a^b \left[ \lim_{n \to \infty} \text{Diff}_{1/n} f \right] = \int_a^b f'
\]
for almost all \( x \). The result then follows immediately. \( \blacksquare \)

Problem 50: Let \( f \) be continuous on \([a, b]\) and differentiable almost everywhere on \((a, b)\). Show that if \( \{\text{Diff}_{1/n} f\} \) is uniformly integrable over \([a, b]\), then
\[
\int_a^b f' = f(b) - f(a).
\]

Solution: If \( f \) is uniformly integrable, then by Vitali’s Convergence Theorem,
\[
\int_a^b \left[ \lim_{n \to \infty} \text{Diff}_{1/n} f \right] = \lim_{n \to \infty} \left[ \int_a^b \text{Diff}_{1/n} f \right].
\]
The result follows from Problem 49. \( \blacksquare \)

Problem 51: Let \( f \) be continuous on \([a, b]\) and differentiable almost everywhere on \((a, b)\). Suppose there is a nonnegative function \( g \) that is integrable over \([a, b]\) and
\[
|\text{Diff}_{1/n} f| \leq g \quad \text{on} \ [a, b] \quad \text{for all} \ n.
\]
Show that
\[ \int_a^b f' = f(b) - f(a). \]

**Solution:** By Lebesgue dominated convergence, we know that
\[ |\text{Diff}_{1/n}f| \leq g \text{ on } [a, b] \text{ for all } n. \]

The result follows from Problem 49.

**Problem 52:** Let \( f \) and \( g \) be absolutely continuous on \([a, b]\). Show that
\[ \int_a^b f \cdot g' = f(b)g(b) - f(a)g(a) - \int_a^b f' \cdot g. \]

**Solution:** Since \( g \) is absolutely continuous \( \{\text{Diff}_{1/n}g\} \) is uniformly integrable and therefore \( \{(\text{Diff}_{1/n}g) \cdot f\} \) is uniformly integrable. Thus
\[ \int_a^b f \cdot g' = \lim_{n \to \infty} \int_a^b f \cdot \text{Diff}_{1/n}g. \]

However by direct computation we see that
\begin{align*}
\int_a^b f \cdot \text{Diff}_{1/n}g &= n \left[ \int_a^b f(x)g(x + 1/n)dx - \int_{a-1/n}^{b-1/n} f(x + 1/n)g(x + 1/n)dx \right] \\
&= n \int_a^b (f(x) - f(x + 1/n))g(x + 1/n)dx + n \int_{b-1/n}^{b} f(x + 1/n)g(x + 1/n)dx \\
&\quad - n \int_{a-1/n}^{a} f(x + 1/n)g(x + 1/n)dx \\
&= \text{Av}_{1/n}(f \cdot g)(b) - \text{Av}_{1/n}(f \cdot g)(a) - \int_a^b \text{Diff}_{1/n}f \cdot g.
\end{align*}

Since \( f \) and \( g \) are continuous,
\[ \text{Av}_{1/n}(f \cdot g)(b) \to f(b)g(b), \quad \text{Av}(f \cdot g)(a) \to f(a)g(a), \]
and \( f \) is absolutely continuous so \( \{\text{Diff}_{1/n}f \cdot g\} \) is uniformly integrable, therefore
\[ \int_a^b \text{Diff}_{1/n}f \cdot g \to \int_a^b f' \cdot g. \]

Hence we have our result.

**Problem 55:** Let \( f \) be of bounded variation on \([a, b]\) and define \( v(x) = TV(f_{[a,x]}) \) for all \( x \in [a, b] \).
(i) Show that $|f'| \leq v'$ a.e. on $[a, b]$, and infer from this that

$$\int_a^b |f'| \leq TV(f).$$

(ii) Show that the above is an equality if and only if $f$ is absolutely continuous on $[a, b]$.

(iii) Compare parts (i) and (ii) with Corollaries 4 and 12, respectively.

Solution:

(i) Suppose we take a partition $P = \{u, v\}$, with $u, v \in [a, b]$

$$|f(v) - f(u)| = V(f, P) \leq TV(f_{[u,v]}) = T(f_{[a,v]}) - T(f_{[a,u]})$$

therefore

$$\frac{|f(v) - f(u)|}{v - u} \leq \frac{T(f_{[a,v]}) - T(f_{[a,u]})}{v - u},$$

since $f$ is of bounded variation, and $TV(f_{[a,x]})$ is absolutely continuous, the limits as $u \to v$ exists pointwise a.e., so $|f'| \leq v'$ a.e. However,

$$\int_a^b |f'| \leq \int_a^b v' \leq v(b) - v(a) = TV(f).$$

(ii) If $f$ is absolutely continuous, then we know that for any $(u, v) \subseteq [a, b]$

$$\int_u^v f' = f(v) - f(u).$$

Also $f$ is of bounded variation, therefore for any partition $P$ of $[a, b]$, $V(f, P) = \sum_{k=1}^n |f(b_k) - f(a_k)| = \sum_{k=1}^n \left| \int_{a_k}^{b_k} f' \right| \leq \sum_{k=1}^n \int_{a_k}^{b_k} |f'| = \int_a^b |f'|.$$

Therefore

$$TV(f) \leq \int_a^b |f'|$$

and so by the previous inequality,

$$TV(f) = \int_a^b |f'|.$$

To show the converse, suppose that we have equality. We can see from the work in part (i) that this must imply that

$$\int_a^b v' = v(b) - v(a).$$
Since \( v \) is strictly increasing, this means that \( v \) is absolutely continuous on \([a, b]\). Let \( \delta > 0 \) be such that if \( \{(a_k, b_k)\} \) is a collection of disjoint open intervals then
\[
\sum_{k=1}^{n} |TV(f_{[a,b_k]}) - TV(f_{[a,a_k]})| = \sum_{k=1}^{n} TV(f_{[a_k,b_k]}) < \epsilon
\]
whenever \( \sum_{k=1}^{n} (b_k - a_k) < \delta \). By the definition of \( TV(f_{[a_k,b_k]}) \) have
\[
|f(b_k) - f(a_k)| \leq TV(f_{[a_k,b_k]}).
\]
Therefore
\[
\sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq \sum_{k=1}^{n} TV(f_{[a_k,b_k]}) < \epsilon
\]
whenever \( \sum_{k=1}^{n} (b_k - a_k) < \delta \).

(iii) If \( f \) is increasing as in Corollaries 4 and 12, then we know that total variation is \( TV(f) = f(b) - f(a) \), and \( f' \) is positive, Part (i) says that
\[
\int_{a}^{b} f' \leq f(b) - f(a)
\]
which is equivalent to Corollary 4, and part (ii) says that \( f \) is absolutely continuous if and only if
\[
\int_{a}^{b} f' = f(b) - f(a)
\]
which is equivalent to Corollary 12.

\[\blacksquare\]

**Problem 56:** Let \( g \) be strictly increasing and absolutely continuous on \([a, b]\).

(i) Show that for any open subset \( \mathcal{O} \) of \((a, b)\),
\[
m(g(\mathcal{O})) = \int_{\mathcal{O}} g'(x)dx.
\]

(ii) Show that for any \( G_\delta \) subset \( E \) of \((a, b)\),
\[
m(g(E)) = \int_{E} g'(x)dx.
\]

(iii) Show that for any subset \( E \) of \([a, b]\) that has measure 0, its image \( g(E) \) also has measure 0, so that
\[
m(g(E)) = 0 = \int_{E} g'(x)dx.
\]
(iv) Show that for any measurable subset $A$ of $[a, b]$,
\[
m(g(A)) = \int_A g'(x)dx.
\]

(v) Let $c = g(a)$, and $d = g(b)$. Show that for any simple function $\phi$ on $[c, d]$,
\[
\int_c^d \phi(y)dy = \int_a^b \phi(g(x))g'(x)dx.
\]

(vi) Show that for any nonnegative integrable function $f$ over $[c, d]$,
\[
\int_c^d f(y)dy = \int_a^b f(g(x))g'(x)dx.
\]

(vii) Show that part (i) follows from (vi) in the case that $f$ is the characteristic function of $O$ and the composition is defined.

Solution:

(i) Write $O$ countable collection of disjoint open intervals, $O = \bigcup_{k=1}^{\infty} (a_k, b_k)$, since $g$ is increasing and continuous, we can also write $g(O) = \bigcup_{k=1}^{\infty} (g(a_k), g(b_k))$, where $\{(g(a_k), g(b_k))\}$ is also a countable collection of disjoint open intervals. Thus we see
\[
m(g(O)) = \sum_{k=1}^{\infty} [g(b_k) - g(a_k)]
\]
which by the absolute continuity of $g$ gives,
\[
m(g(O)) = \sum_{k=1}^{\infty} \left[ \int_{a_k}^{b_k} g' \right] = \int_{O} g'.
\]

(ii) Write a $G_\delta$ sets $E$ as the intersection of a countable collection of open sets $E = \bigcap_{k=1}^{\infty} O_k$. Since for any $n$, $E_n = \bigcap_{k=1}^{n} O_k$ is open and descending, we have
\[
m(g(E_n)) = \int_{E_n} g'.
\]
Since $g$ is strictly increasing and continuous, we know $g(E_n)$ is also open and descending, therefore,
\[
m(g(E)) = \int_{E} g'.
\]

(iii) Problem 40.
(iv) Since $A$ is measurable, we may write it as $A = E \cup F$, a union of a $G_\delta$ set $E$ and $F$ a set of measure 0. Since $g(E \cup F) = g(E) \cup g(F)$, and $g(F)$ is measure 0 and $g(E)$ is $G_\delta$,

$$m(g(A)) = m(g(E)) = \int_E g' = \int_A g'.$$

(v)* We first show this for an indicator function $\chi_E$ of a measurable set $E \subset [c, d]$. That is, we would like to show that

$$m(E) = \int_a^b (\chi_E \circ g) \cdot g'.$$

Note, the difficulty here is that we do not know the measurability of $\chi_E \circ g \equiv \chi_{g^{-1}(E)}$, since the inverse of a strictly increasing absolutely continuous function may not map measurable sets to measurable sets. This can be seen in a counter-example by Silvia Spataru where an absolutely continuous strictly increasing function $f$ is constructed so that $f' = 0$ a.e. on a ‘fat’ cantor set $B$, and $f$ maps $B$ to a set of measure 0. Hence $g^{-1}$ behaves like the Cantor-Lebesgue function, mapping a set of measure 0 to a set of positive measure.

However, it turns out that while $\chi_E \circ g$ may not be measurable, $(\chi_E \circ g) \cdot g'$ will always be measurable. We will show this by approximation with measurable functions. Denote $h := (\chi_E \circ g) \cdot g'$ and note that since $g$ is continuous, then for any open set $O$ and compact $K$, $(\chi_O \circ g)$ and $(\chi_K \circ g)$ are both measurable functions. In particular this implies by (iv) that

$$m(O) = \int_a^b (\chi_O \circ g) \cdot g', \quad \text{and} \quad m(K) = \int_a^b (\chi_K \circ g) \cdot g'.$$

We now choose $\{O_n\}$ a decreasing sequence of open sets containing $E$, and $\{K_n\}$ an increasing sequence of compact sets contained in $E$ so that,

$$m(O_n) \to m(E) \quad \text{and} \quad m(K_n) \to m(E).$$

Define $\phi_n := (\chi_K \circ g) \cdot g'$ and $\psi_n := (\chi_O \circ g) \cdot g'$. We see that for every $n$,

$$\phi_1 \leq \ldots \leq \phi_n \leq h \leq \psi_n \leq \ldots \leq \psi_1.$$ 

Since $g' \geq 0$ a.e., $\{\phi_n\}$ and $\{\psi_n\}$ are monotone sequences of functions bounded by $g'$. Thus they have pointwise limits a.e. In particular since $g'$ is integrable, dominated convergence gives

$$\int_a^b \lim_{n \to \infty} (\psi_n - \phi_n) = m(E) - m(E) = 0.$$

Since $\psi_n - \phi_n \geq 0$ we conclude

$$\lim_{n \to \infty} (\psi_n - \phi_n) = 0.$$
pointwise. Therefore

$$\lim_{n \to \infty} |h - \phi_n| \leq \lim_{n \to \infty} (\psi_n - \phi_n) = 0,$$

pointwise. So $h$ is measurable, being the pointwise limit of measurable functions $\phi_n$.

We conclude by dominated convergence again that

$$m(E) = \lim_{n \to \infty} \int_a^b (\chi_{K_n} \circ g) \cdot g' = \int_a^b (\chi_E \circ g) \cdot g'.$$

Now suppose that $\varphi$ is a simple function over $(c, d)$ given in canonical form by

$$\varphi = \sum_{k=1}^n s_k \chi_{E_k},$$

where $\{E_k\}$ is a finite disjoint collection of measurable sets such that $\bigcup_{k=1}^n E_k = (c, d)$.

We find

$$\int_{g(a)}^{g(b)} \varphi = \sum_{k=1}^n s_k \cdot m(E_k) = \sum_{k=1}^n s_k \int_a^b (\chi_{E_k} \circ g) \cdot g' = \int_a^b (\varphi \circ g) \cdot g'.$$

(vi) If $f$ is a non-negative integrable function over $[c, d]$ there is a monotone sequence of simple functions $\varphi_n$ that converge to $f$ p.w. Therefore by the previous problem

$$\int_c^d \varphi_n = \int_a^b (\varphi_n \circ g) \cdot g'$$

so by the monotone convergence theorem,

$$\int_c^d f = \int_a^b (f \circ g) \cdot g'.$$

(vii) Choose $f$ to be the characteristic function of $g(O)$, where $O \subseteq (a, b)$. Then

$$m(g(O)) = \int_a^b (\chi_{g(O)} \circ g)g' = \int_a^b \chi_O g' = \int_O g'.$