Problem 1: Prove that if $A$ and $B$ are two sets in $\mathcal{A}$ with $A \subseteq B$, then $m(A) \leq m(B)$.

Proof: Since $A \subseteq B$ we can split up $B$ into a union of two disjoint sets $B = A \cup (B \sim A)$. Using the countable additivity of $m$ we find

$$m(B) = m(A \cup (B \sim A)) = m(A) + m(B \sim A) \geq m(A),$$

since $m(B \sim A) \geq 0$.

Problem 3: Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $m(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k)$.

Proof: Using the collection $\{E_k\}_{k=1}^{\infty}$ we construct a countable disjoint collection of sets $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{A}$ such that $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} E_k$ in the following way. Choose $A_1 = E_1$ and define for $n > 1$,

$$A_n = E_n \sim \bigcup_{k=1}^{n-1} E_k.$$

It is easy to see that $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ are pairwise disjoint, $\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} A_k$, and $A_k \subseteq E_k$. Therefore using countable additivity and monotonicity,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m(A_k) \leq \sum_{k=1}^{\infty} m(E_k).$$

Problem 5: By using the properties of an outer measure, prove that the interval $[0,1]$ is not countable.

Solution: We know the outer measure of an interval is it’s length, so $m^*([0,1]) = 1$, however we also know that the outer measure of a countable set if 0. Therefore $[0,1]$ cannot be countable.

Problem 8: Let $B$ be the set of rational numbers in the interval $[0,1]$, and let $\{I_k\}_{k=1}^{n}$ a finite collection of open intervals that covers $B$. Prove that $\sum_{k=1}^{n} m^*(I_k) \geq 1$.

Solution: Since the rationals are dense in $\mathbb{R}$ we see that the closure $\overline{B} = [0,1]$. By the property of closure $\overline{B} \subseteq \bigcup_{k=1}^{n} I_k$. However, since the closure of the union is just the union
of the closure for finite unions, we have $B \subseteq \bigcup_{k=1}^{n} T_k$. Therefore by the finite subadditivity of $m^*$,

$$1 = m^*([0, 1]) = m^*(B) \leq m^*\left( \bigcup_{k=1}^{n} T_k \right) \leq \sum_{k=1}^{n} m^*(T_k) = \sum_{k=1}^{n} m^*(I_k),$$

where we have used the fact that $I_k$ are intervals, so $m^*(I_k) = l(I_k) = m^*(I_k)$.

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**Problem 11:** Prove that if a $\sigma$-algebra of subsets of $\mathbb{R}$ contains intervals of the form $(a, \infty)$, then it contains all intervals.

**Solution:** Let $\Sigma$ be the $\sigma$-algebra. By complement $\Sigma$ must contain intervals of the form $(-\infty, a]$. Also intervals of the form $(-\infty, a)$ are in $\Sigma$ since they can be given by a countable union $(-\infty, a) = \bigcup_{k=1}^{\infty} (-\infty, a - 1/n]$ and so by complement $[a, \infty)$ is also in $\Sigma$. We conclude that for any $a, b$ finite,

\begin{align*}
(a, b) &= (-\infty, b) \cap (a, \infty) \\
(a, b] &= (-\infty, b] \cap (a, \infty) \\
[a, b) &= (-\infty, b) \cap [a, \infty) \\
[a, b] &= (-\infty, b] \cap [a, \infty)
\end{align*}

Therefore $\Sigma$ contains all intervals.

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**Problem 14:** Show that if a set $E$ has positive outer measure, then there is a bounded subset of $E$ that also has positive outer measure.

**Proof:** We prove by contradiction. Suppose that every bounded subset of $E$ has outer measure zero. Define $I_k = [k, k+1]$, to be a countable collection of disjoint bounded intervals that decompose $\mathbb{R}$. We can then decompose $E$ as a countable union of bounded subsets of $E$

$$E = \bigcup_{k \in \mathbb{Z}} E \cap I_k,$$

and so by the hypothesis $m^*(E \cap I_k) = 0$. Therefore by finite sub-additivity of $m^*$,

$$0 < m^*(E) = m^*\left( \bigcup_{k \in \mathbb{Z}} E \cap I_k \right) \leq \sum_{k \in \mathbb{Z}} m^*(E \cap I_k) = 0$$

which is a contradiction.

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**Problem 15:** Show that if $E$ has finite measure and $\epsilon > 0$, then $E$ is the disjoint union of a finite number of measurable sets, each of which has measure at most $\epsilon$.

**Solution:** Since $E$ has finite measure, we may choose countable collection of open intervals $\{I_k\}$ such that $\sum_{k=1}^{\infty} l(I_k) < m(E) + 1$. Since this series converges, there exists an $N$ such that $\sum_{k=N}^{\infty} l(I_k) < \epsilon$. Define

$$E_0 = E \cap \bigcup_{k=N}^{\infty} I_k,$$

(1)
then \( m(E_0) < \epsilon \). Furthermore we see that \( \mathcal{O} \equiv \bigcup_{k=1}^{N-1} I_k \) is a bounded open set covering \( E \sim E_0 \). We let \([a, b]\) be any bounded interval containing \( \mathcal{O} \). Choose an integer \( M \) such that \( M\epsilon > b - a \), then clearly we can break up \([a, b]\) into \( M \) disjoint intervals \( \{[a_k, b_k]\}_{k=1}^{M} \) with \( a = a_1 < b_1 = a_2 \ldots b_{M-1} = a_M < b_M = b \) and \( b_k - a_k < \epsilon \). Finally if we define

\[
E_k = (E \sim E_0) \cap [a_k, b_k] \quad \text{for } 1 \leq k \leq M,
\]

then it is clear that \( \{E_k\}_{k=0}^{M} \) is a finite collection of disjoint measurable sets with \( m(E_k) < \epsilon \) and

\[
E = \bigcup_{k=0}^{M} E_k.
\]