1. Approximation of functions in $L^p(R)$ by smooth functions.

A function $f: R \to R$ is said to be $C^\infty$ if it has derivatives of all orders. A function $f: R \to R$ is said to be analytic if it has a power series expansion on all of $R$, so

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ for all } x$$

An analytic function is $C^\infty$. Is this obvious?

The simplest type of analytic function is a polynomial. Weierstrass proved the remarkable fact that if $f: [a, b] \to R$ is continuous and $\epsilon > 0$, there is a polynomial $r$ such that

$$|f(x) - r(x)| < \epsilon \text{ for all } x \text{ in } [a, b].$$

This is called the Weierstrass Approximation Theorem. One can phrase it in terms of the Banach space $C([a, b])$, with the max norm. The polynomials are dense in $C([a, b])$, in the sense that if $f$ belongs to $C([a, b])$ and $\epsilon > 0$, there is a polynomial $p$ such that

$$\|f - p\|_{\max} < \epsilon.$$

There is the exact same approximation result in $L^p[a, b]$. For $1 \leq p < \infty$, the polynomials are dense in $L^p[a, b]$, in the sense that if $f$ belongs to $L^p[a, b]$ and $\epsilon > 0$, there is a polynomial $r$ such that

$$\|f - r\|_{L^p[a, b]} < \epsilon.$$

The following problem tells you of a nice way to approximate characteristic functions of intervals by smooth functions, with respect to the $L^p(R)$ norm.

2. Problem

Let $a < b$ and $\epsilon > 0$. Consider the function

$$f_\epsilon(x) = \frac{1}{e^{\frac{(x-a)(x-b)}{\epsilon}} + 1}$$

(i) Show that as $\epsilon \to 0$, $f_\epsilon(x) \to 1$ if $a < x < b$, and $f_\epsilon(x) \to 0$ for $x$ outside $[a, b]$. Thus

$$\lim_{\epsilon \to 0} f_\epsilon(x) = \chi_{[a, b]}(x) \text{ for almost all } x \in R.$$

(ii) For $1 \leq p < \infty$, prove that

$$\lim_{\epsilon \to 0} \|f_\epsilon(x) - \chi_{[a, b]}\|_{L^p(R)} = 0.$$

Hint: Use the Dominated Convergence Theorem.