

# Curvature Computation in Normal Coordinates

Sergey Kushnarev

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## 1 Theory

Given  $M$  – Riemannian manifold,  $P \in M$ ,  $v \in T_P M$ , we define the exponential map (it is defined in the neighborhood of point  $P$ )

$$\exp_P : T_P M \rightarrow M, \quad \exp_P v = \gamma(p, v, 1),$$

where  $\gamma(p, v, t)$  is a geodesic starting at  $P$ , launched in the direction  $v$ . Note, we can always write  $\exp_P tv = \gamma(P, tv, 1) = \gamma(P, v, t)$  (for sufficiently small  $t$ ).

The derivative of this map will look like

$$(d \exp_P)_{tv} : T_{tv}(T_P M) \cong T_P M \rightarrow T_{\gamma(t)} M,$$

i.e. given  $w \in T_P M$  we will have  $(d \exp_P)_{tv} w = \partial / \partial s [\exp_P tv(s)]_{s=0}$ , where  $v(s)$  is a path in the tangent space  $T_P M$  such that  $v(0) = v$ ,  $v'_s(0) = w$ .

The *Jacobi field*  $J(t)$  along the geodesic  $\gamma(t)$  is defined as

$$J(t) = (d \exp_P)_{tv} tw,$$

and it satisfies the Jacobi equation

$$D_t^2 J - R(\gamma'(t), J(t))\gamma'(t) = 0, \quad J(0) = 0, \quad J'(0) = w.$$

Here  $R(X, Y)Z$  is the Riemannian curvature tensor.

**Nice Fact.** We know that the Jacobi field  $J(t)$  has a nice Taylor expansion, namely for  $t \rightarrow 0$

$$J(t) = P^t \left[ wt + R(\gamma'(0), w)\gamma'(0) \frac{t^3}{3!} \right] + O(t^4).$$

Here  $P^t : T_{\gamma(0)} M \rightarrow T_{\gamma(t)} M$  is parallel translation. (Easy to prove. REFERENCE)

Now we can claim: take 2 vectors  $v, w \in T_P M$ , s.t.  $\|v\| = \|w\| = 1$ ,  $\langle v, w \rangle = 0$ , consider geodesic  $\gamma(t) = \exp_P tv$ , then

$$\begin{aligned} \exp_P^*(tv) \langle w, w \rangle_P &= \langle (d \exp_P)_{tv} w, (d \exp_P)_{tv} w \rangle_{\gamma(t)} \\ &= \left\langle \frac{1}{t} J(t), \frac{1}{t} J(t) \right\rangle_{\gamma(t)} \\ &= \left\langle P^t \left[ w + R(v, w)v \frac{t^2}{3!} \right], P^t \left[ w + R(v, w)v \frac{t^2}{3!} \right] \right\rangle_{\gamma(t)} \\ &= \left\langle w + R(v, w)v \frac{t^2}{3!}, w + R(v, w)v \frac{t^2}{3!} \right\rangle_P \\ &= \langle w, w \rangle + K(v, w) \frac{t^2}{3} + O(t^3). \end{aligned} \tag{1}$$

Here  $K(v, w)$  is a sectional curvature in the directions  $v, w$ .

## 2 Practice

### 2.1 Taylor expansion of $\Phi(\theta, t)$

Take  $v(\theta, t)$  such that

$$\begin{aligned} v(\theta, t) &= v_0(\theta) + v_1(\theta)t + v_2(\theta)t^2 + \dots \\ m(\theta, t) &= m_0(\theta) + m_1(\theta)t + m_2(\theta)t^2 + \dots \end{aligned}$$

To make this path a geodesic we need to make sure that the velocity  $v(\theta, t)$  is such that it solves the EPDiff, i.e.

$$m_t = -2mv_\theta - vm_\theta, \quad \text{where } m = Lv, \quad v = G * m = L^{-1}m.$$

Plugging in the above expressions for  $v$  and  $m$  into EPDiff we get

$$\begin{aligned} m_1 + 2tm_2 + 3t^2m_3 + \dots &= -2(m_0 + tm_1 + t^2m_2 + \dots)(v'_0 + v'_1t + v'_2t^2 + \dots) \\ &\quad - (m'_0 + tm'_1 + t^2m'_2 + \dots)(v_0 + v_1t + v_2t^2 + \dots). \end{aligned}$$

Which yields the following relations:

$$\begin{aligned} Lv_1 &= -2m_0v'_0 - m'_0v_0 \\ &= -2Lv_0 \cdot v'_0 - Lv'_0 \cdot v_0, \\ 2Lv_2 &= -2Lv_0v'_1 - 2Lv_1v'_0 - Lv'_0v_1 - Lv'_1v_0. \end{aligned} \tag{2}$$

In terms of the velocity field:

$$\begin{aligned} v_1 &= -L^{-1}(2Lv_0 \cdot v'_0 + Lv'_0 \cdot v_0), \\ v_2 &= L^{-1} \left[ Lv_0 \cdot L^{-1}(2Lv_0 \cdot v'_0 + Lv'_0 \cdot v_0)' + \frac{1}{2}Lv'_0 \cdot L^{-1}(2Lv_0 \cdot v'_0 + Lv'_0 \cdot v_0) \right. \\ &\quad \left. + v'_0(2Lv_0 \cdot v'_0 + Lv'_0 \cdot v_0) + \frac{1}{2}v_0(2Lv_0 \cdot v'_0 + Lv'_0 \cdot v_0)' \right]. \end{aligned}$$

Take a geodesic  $\Psi(\theta, t) \in \text{PSL}_2(\mathbb{R}) \backslash \mathbf{Diff}(S^1)$ . Then the vector field  $w(\theta, t)$  is given by

$$\Psi_t(\Psi^{-1}(\theta, t), t) = w(\theta, t).$$

If we instead consider  $\Phi(\theta, t) := \Psi(\theta, t)^{-1} \in \mathbf{Diff}(S^1)/\text{PSL}_2(\mathbb{R})$  (inverse is taken in  $\theta$ -variable), then we get

$$\Phi_t(\theta, t) = -w(\theta, t)\Phi_\theta(\theta, t).$$

Here vector field  $w(\theta, t)$  is the same as in the previous formula.

Consider  $\Phi(\theta, t) = \theta + t \cdot \Phi_1(\theta) + t^2 \cdot \Phi_2(\theta) + t^3 \cdot \Phi_3(\theta) + \dots$ . We have

$$\begin{aligned} \Phi_t(\theta, t) &= \Phi_1 + 2\Phi_2t + 3\Phi_3t^2 + \dots, \\ \Phi_\theta(\theta, t) &= 1 + \Phi'_1t + \Phi'_2t^2 + \Phi'_3t^3 + \dots \end{aligned}$$

Then to determine the geodesic path  $\Phi(\theta, t)$

$$\Phi_1 + 2\Phi_2t + 3\Phi_3t^2 + \dots = -(v_0 + v_1t + v_2t^2)(1 + \Phi'_1t + \Phi'_2t^2 + \dots)$$

Which yields

$$\begin{aligned}
\Phi_1 &= -v_0, \\
2\Phi_2 &= -v_0\Phi'_1 - v_1 \\
&= v_0v'_0 + L^{-1}(2Lv_0 \cdot v'_0 + Lv'_0 \cdot v_0), \\
3\Phi_3 &= -v_0\Phi'_2 - v_1\Phi'_1 - v_2 \\
&= -\frac{v_0}{2}[v_0v'_0 + L^{-1}(2Lv_0 \cdot v'_0 + Lv'_0 \cdot v_0)]' - v'_0L^{-1}(2Lv_0 \cdot v'_0 + Lv'_0 \cdot v_0) \\
&\quad - L^{-1}\left[Lv_0 \cdot L^{-1}(2Lv_0 \cdot v'_0 + Lv'_0 \cdot v_0)' + \frac{1}{2}Lv'_0 \cdot L^{-1}(2Lv_0 \cdot v'_0 + Lv'_0 \cdot v_0)\right. \\
&\quad \left.+ v'_0(2Lv_0 \cdot v'_0 + Lv'_0 \cdot v_0) + \frac{1}{2}v_0(2Lv_0 \cdot v'_0 + Lv'_0 \cdot v_0)\right].
\end{aligned} \tag{3}$$

So the geodesic  $\Phi(\theta, t)$  has the following expansion (substituting  $v$  for  $v_0$ )

$$\Phi(\theta, t) = \theta - vt + \frac{1}{2}[vv' + L^{-1}(2Lv \cdot v' + Lv' \cdot v)]t^2 + \Phi_3t^3 + O(t^4).$$

## 2.2 Taylor expansion of $\Psi(\theta, t) \in \text{PSL}_2(\mathbb{R}) \setminus \mathbf{Diff}(S^1)$

Given  $\Phi(\theta, t) := \Psi^{-1}(\theta, t) = \theta + \Phi_1(\theta)t + \Phi_2(\theta)t^2 + O(t^3)$ . We have the following:

$$\begin{aligned}
\frac{\partial}{\partial t}\Psi^{-1}(\Psi(\theta, t), t) &= 0, \\
\Psi_\theta^{-1}(\Psi(\theta, t), t)\Psi_t(\theta, t) + \Psi_t^{-1}(\Psi(\theta, t), t) &= 0, \\
\Psi_t(\theta, t) &= -\frac{\Psi_\theta^{-1}(\Psi(\theta, t), t)}{\Psi_t^{-1}(\Psi(\theta, t), t)}.
\end{aligned}$$

$$\begin{aligned}
\frac{\Psi_t^{-1}(\theta, t)}{\Psi_\theta^{-1}(\theta, t)} &= \frac{\Phi_1 + 2\Phi_2t + 3\Phi_3t^2 + O(t^3)}{1 + \Phi'_1t + \Phi'_2t^2 + O(t^3)} \\
&= (\Phi_1 + 2\Phi_2t + 3\Phi_3t^2 + O(t^3))(1 - \Phi'_1t - \Phi'_2t^2 + \Phi_1'^2t^2 + O(t^3)) \\
&= \Phi_1 + t(2\Phi_2 - \Phi_1\Phi'_1) + t^2(3\Phi_3 - 2\Phi'_1\Phi_2 - \Phi_1\Phi'_2 + \Phi_1\Phi_1'^2) + O(t^3).
\end{aligned}$$

Combining two expressions we have

$$\begin{aligned}
\Psi_t(\theta, t) &= -\frac{\Psi_\theta^{-1}(\Psi(\theta, t), t)}{\Psi_t^{-1}(\Psi(\theta, t), t)} \\
&= -\left[\Phi_1(\theta) + t(2\Phi_2(\theta) - \Phi_1(\theta)\Phi'_1(\theta)) + t^2(3\Phi_3 - 2\Phi'_1\Phi_2 - \Phi_1\Phi'_2 + \Phi_1\Phi_1'^2) + O(t^3)\right]_{\theta=\Psi(\theta, t)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Psi_t(\theta, t=0) &= v_0, \\
\Psi_{tt}(\theta, t=0) &= v_0v'_0 - L^{-1}(2Lv_0 \cdot v'_0 + Lv'_0 \cdot v_0), \\
\Psi_{ttt}(\theta, t=0) &= v_0(v_0v'_0)' - 2v_0L^{-1}(2Lv_0 \cdot v'_0 + Lv'_0 \cdot v_0)' - v'_0L^{-1}(2Lv_0 \cdot v'_0 + Lv'_0 \cdot v_0) + 2v_2. \tag{4}
\end{aligned}$$

Using the series for  $\Phi(\theta, t)$  we obtain series for  $\Psi(\theta, t) \in \text{PSL}_2(\mathbb{R}) \setminus \mathbf{Diff}(S^1)$ :

$$\Psi(\theta, t) = \theta + vt + \frac{1}{2}[vv' - L^{-1}(2Lv \cdot v' + Lv' \cdot v)]t^2 + \Psi_3t^3 + O(t^4),$$

where  $\Psi_3 = \frac{1}{6}\Psi_{ttt}$ .

### 2.3 Exponential maps

We have two exponential maps, right and left exponential maps:

$$\begin{aligned}\exp^L -tv &= \Psi(\theta, -v, t) \in \mathbf{Diff}(S^1)/\mathrm{PSL}_2(\mathbb{R}), \\ \exp^R tv &= \Phi(\theta, v, t) \in \mathrm{PSL}_2(\mathbb{R}) \setminus \mathbf{Diff}(S^1).\end{aligned}$$

We have explicit formulae for them:

$$\begin{aligned}\exp^L -tv &= \theta - vt + \frac{1}{2} [vv' + L^{-1}(2Lv \cdot v' + Lv' \cdot v)] t^2 + \Phi_3 t^3, \\ \exp^R tv &= \theta + vt + \frac{1}{2} [vv' - L^{-1}(2Lv \cdot v' + Lv' \cdot v)] t^2 + \Psi_3 t^3,\end{aligned}\tag{5}$$

where  $\Phi_3, \Psi_3$  are defined as in (3) and (4).

### 3 Concise way

For any  $u, v \in \mathfrak{g}$  we have the transpose of the adjoint map:

$$\begin{aligned}\mathrm{ad}_u^\dagger v &: \mathfrak{g} \rightarrow \mathfrak{g}, \\ v &\mapsto \mathrm{ad}_u^\dagger v, \\ \langle \mathrm{ad}_u^\dagger v, w \rangle &= \langle v, \mathrm{ad}_u w \rangle \\ &= \langle v, [u, w] \rangle.\end{aligned}$$

We will derive the formula for the  $\mathrm{ad}^\dagger$  using integration by parts:

$$\begin{aligned}\langle \mathrm{ad}_u^\dagger v, w \rangle &= \langle v, [u, w] \rangle \\ &= \int Lv(uw' - wu') dx \\ &= \int -Lv \cdot u'w + Lv \cdot uw' dx \\ &= \int (-2Lv \cdot u' - Lv' \cdot u) w dx.\end{aligned}$$

Therefore:

$$\mathrm{ad}_u^\dagger v = -L^{-1}(2Lv \cdot u' + Lv' \cdot u).$$

(Notice, that  $\mathrm{ad}_u^\dagger v = \beta(v, u)$  in Arnold's notation.)

In terms of the transpose of the adjoint map exponential maps (6) take the form (substituting  $v$  for  $tv$ , with small  $\|v\|$ ):

$$\begin{aligned}\exp^R v &= \theta + v + \frac{1}{2} [vv' + \mathrm{ad}_v^\dagger v] \\ &\quad + \frac{1}{6} [v(vv')' + v' \mathrm{ad}_v^\dagger v + 2v(\mathrm{ad}_v^\dagger v)' + \mathrm{ad}_v^\dagger(\mathrm{ad}_v^\dagger v) + \mathrm{ad}_{\mathrm{ad}_v^\dagger v}^\dagger v] + O(\|v\|^4), \\ \exp^L v &= \theta + v + \frac{1}{2} [vv' - \mathrm{ad}_v^\dagger v] \\ &\quad + \frac{1}{6} [v(vv')' - v(\mathrm{ad}_v^\dagger v)' - 2v' \mathrm{ad}_v^\dagger v + \mathrm{ad}_v^\dagger(\mathrm{ad}_v^\dagger v) + \mathrm{ad}_{\mathrm{ad}_v^\dagger v}^\dagger v] + O(\|v\|^4).\end{aligned}$$

Now computing  $\tilde{u} = D \exp_v^R u$ :

$$\begin{aligned} \tilde{u} = D \exp_v^R u = u + \frac{1}{2} [(uv)' + \text{ad}_u^\dagger v + \text{ad}_v^\dagger u] + \frac{1}{6} \left[ u(vv')' + v(uv')' + v(vu')' \right. \\ \left. + u' \text{ad}_v^\dagger v + v' (\text{ad}_u^\dagger v + \text{ad}_v^\dagger u) + 2u (\text{ad}_v^\dagger v)' + 2v (\text{ad}_u^\dagger v + \text{ad}_v^\dagger u)' \right. \\ \left. + \text{ad}_u^\dagger \text{ad}_v^\dagger v + \text{ad}_v^\dagger (\text{ad}_u^\dagger v + \text{ad}_v^\dagger u) + \text{ad}_{\text{ad}_u^\dagger v + \text{ad}_v^\dagger u}^\dagger v + \text{ad}_{\text{ad}_v^\dagger v}^\dagger u \right]. \quad (6) \end{aligned}$$

Translating back  $\tilde{u}$  to the tangent space at the identity  $T_{Id} \text{PSL}_2(\mathbb{R}) \setminus \mathbf{Diff}(S^1)$  and keeping all the terms upto  $O(\|v\|^2)$  we will get a rather long formula:

$$\begin{aligned} \tilde{u}(\exp^R(v)^{-1}) = \tilde{u}(\exp^L(-v)) = \\ u - vv' + \frac{1}{2} [vv'u' - u' \text{ad}_v^\dagger v + v^2 u''] + \frac{1}{2} [(uv)' + \text{ad}_u^\dagger v + \text{ad}_v^\dagger u] - \frac{1}{2} v [(uv)' + \text{ad}_u^\dagger v + \text{ad}_v^\dagger u]' \\ + \frac{1}{6} \left[ u(vv')' + v(uv')' + v(vu')' + u' \text{ad}_v^\dagger v + v' (\text{ad}_u^\dagger v + \text{ad}_v^\dagger u) + 2u (\text{ad}_v^\dagger v)' + \right. \\ \left. 2v (\text{ad}_u^\dagger v + \text{ad}_v^\dagger u)' + \text{ad}_u^\dagger \text{ad}_v^\dagger v + \text{ad}_v^\dagger (\text{ad}_u^\dagger v + \text{ad}_v^\dagger u) + \text{ad}_{\text{ad}_u^\dagger v + \text{ad}_v^\dagger u}^\dagger v + \text{ad}_{\text{ad}_v^\dagger v}^\dagger u \right]. \quad (7) \end{aligned}$$

After some simplification the formula turns into

$$\begin{aligned} \tilde{u}(\exp^R(v)^{-1}) = u + \underbrace{\frac{1}{2} [u, v] + \frac{1}{2} [\text{ad}_u^\dagger v + \text{ad}_v^\dagger u]}_{I_1} \\ + \frac{1}{6} \underbrace{[uv'^2 - uvv'' - vv'u' + v^2 u'' - 2u' \text{ad}_v^\dagger v - v (\text{ad}_u^\dagger v + \text{ad}_v^\dagger u)' + v' (\text{ad}_u^\dagger v + \text{ad}_v^\dagger u) + 2u (\text{ad}_v^\dagger v)']}_{I_2} \\ + \frac{1}{6} \left[ u(vv')' + v(uv')' + v(vu')' + u' \text{ad}_v^\dagger v + v' (\text{ad}_u^\dagger v + \text{ad}_v^\dagger u) + 2u (\text{ad}_v^\dagger v)' + \right. \\ \left. 2v (\text{ad}_u^\dagger v + \text{ad}_v^\dagger u)' + \text{ad}_u^\dagger \text{ad}_v^\dagger v + \text{ad}_v^\dagger (\text{ad}_u^\dagger v + \text{ad}_v^\dagger u) + \text{ad}_{\text{ad}_u^\dagger v + \text{ad}_v^\dagger u}^\dagger v + \text{ad}_{\text{ad}_v^\dagger v}^\dagger u \right]. \quad (8) \end{aligned}$$

One can check that

$$\begin{aligned} \langle I_1, I_1 \rangle &= \|u\|^2 + \frac{1}{4} \|[u, v]\|^2 + \frac{1}{4} \|\text{ad}_u^\dagger v + \text{ad}_v^\dagger u\|^2 + \frac{1}{2} \langle [u, v], \text{ad}_u^\dagger v + \text{ad}_v^\dagger u \rangle, \\ \langle u, I_2 \rangle &= -\frac{1}{6} \langle [u, v], \text{ad}_v^\dagger u \rangle + \frac{1}{3} \langle \text{ad}_u^\dagger u, \text{ad}_v^\dagger v \rangle - \frac{1}{6} \langle \text{ad}_v^\dagger u, \text{ad}_v^\dagger u + \text{ad}_u^\dagger v \rangle, \\ \langle u, I_3 \rangle &= -\frac{1}{6} \langle [u, v], \text{ad}_v^\dagger u + \text{ad}_u^\dagger v \rangle - \frac{1}{6} \langle \text{ad}_u^\dagger u, \text{ad}_v^\dagger v \rangle - \frac{1}{6} \langle \text{ad}_u^\dagger v, \text{ad}_v^\dagger u + \text{ad}_u^\dagger v \rangle. \end{aligned}$$

Combining the above one gets (again up to the terms quadratic in  $v$ ):

$$\begin{aligned}
& \langle \tilde{u}(\exp^R(v)^{-1}), \tilde{u}(\exp^R(v)^{-1}) \rangle \\
& \quad = \langle I_1, I_1 \rangle + 2\langle u, I_2 + I_3 \rangle = \|u\|^2 + \frac{1}{3}K(u, v) \\
& \quad = \|u\|^2 + \frac{1}{4}\|[u, v]\|^2 - \frac{1}{12}\|\text{ad}_u^\dagger v + \text{ad}_v^\dagger u\|^2 + \frac{1}{6}\langle [u, v], \text{ad}_u^\dagger v - \text{ad}_v^\dagger u \rangle + \frac{1}{3}\langle \text{ad}_u^\dagger u, \text{ad}_v^\dagger v \rangle. \quad (9)
\end{aligned}$$

Finally, the formula for the sectional curvature  $K(u, v)$  in the directions  $u, v$  has the form:

$$K(u, v) = \frac{3}{4}\|[u, v]\|^2 - \frac{1}{4}\|\text{ad}_u^\dagger v + \text{ad}_v^\dagger u\|^2 + \frac{1}{2}\langle [u, v], \text{ad}_u^\dagger v - \text{ad}_v^\dagger u \rangle + \langle \text{ad}_u^\dagger u, \text{ad}_v^\dagger v \rangle. \quad (10)$$

The expression (10) coincides with Arnold's formula in [?].