Research Statement

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1 Introduction

The study of 2D shapes and their similarities is a central problem in the field of computer vision. It arises in the task of characterizing and classifying objects from their observed silhouette. Defining natural distance between 2D shapes creates a metric on the infinite-dimensional space of shapes. In my research I am investigating one particular metric, which comes from the conformal mapping of the 2D shapes, via the theory of Teichmüller spaces. In this space every simple closed curve (or a 2D shape) is represented by a smooth self-map of a circle.

2 Background

2.1 Shapes as Diffeomorphisms of the Circle

To study 2D shapes with mathematical means we need to formulate the problem in a certain framework. There is a convenient way to do so which comes from complex analysis. By "shape" we mean a simple closed smooth curve Γ in the complex plane $\mathbb{C}$. It is possible to associate with any curve, mod translations and scalings, a *fingerprint*, an orientation preserving diffeomorphism of the circle S^1 onto itself. This fingerprint is defined up to an action of a Möbius subgroup $\mathrm{PSL}_2(\mathbb{R})$ from the left, i.e. it is a member of the coset space $\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$ (also known as the universal Teichmüller space). In other words we have the following isomorphism

$$\boxed{\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1) \cong \text{ set of shapes/translations, scalings.}}$$

For example, the fingerprint of the eye-shape is shown in Fig.1. To reconstruct the shape given the diffeomorphism one needs to use the process called 'conformal welding' (for details see [8]).

2.2 Weil-Petersson Norm

Thus the space of all 2D shapes, mod translations and scalings, becomes an infinite-dimensional space of right cosets $\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$. The metric that this space comes equipped with is called the Weil-Petersson metric (WP). It is well-known in many contexts: in the classification of Riemann surfaces [5], conformal and quasi-conformal maps [6], string theory [3] and most recently computer vision [8]. The metric is defined on the tangent space to the coset space $\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$, which is $psl_2(\mathbb{R})\backslash\mathbf{Vec}(S^1)$, i.e. smooth vector fields on the circle mod the vector fields of the type $a + b\sin + c\cos$. For $u, v \in psl_2(\mathbb{R})\backslash\mathbf{Vec}(S^1)$:

$$\langle u, v\rangle = \int_{S^1} Lu\, v\; d\theta,$$

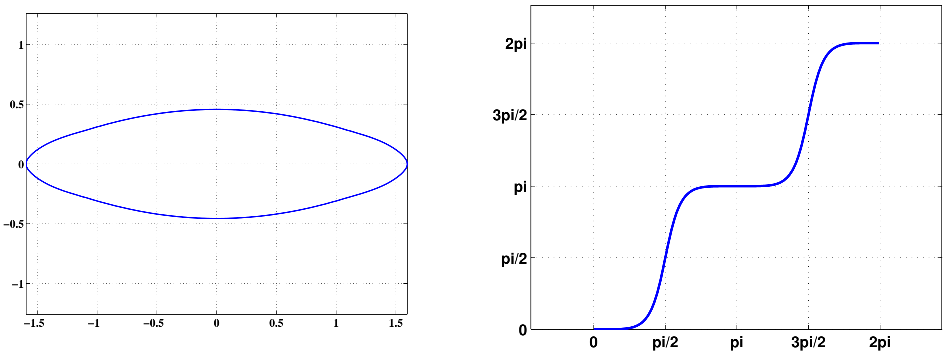

Figure 1: Example of the eye-shape and its fingerprint ψ.

where L is the WP operator. For $u = \sum u_n e^{in\theta} \frac{\partial}{\partial \theta}$ the operator L has a simple form:

$$Lu = \sum |n^3 - n| u_n, \text{ or } Lu = -\mathcal{H}(u_\theta + u_{\theta\theta\theta}),$$

where $\mathcal{H}$ is a Hilbert transform, defined on the circle by convolution with $\frac{1}{2\pi}\text{ctn}(\theta/2)$. Notice that this metric has a null space of the form $(a + b\cos\theta + c\sin\theta)\partial/\partial\theta$, which is precisely the Lie algebra $psl_2(\mathbb{R})$. It is known from differential geometry that the above WP norm on vector fields, gives the right-invariant *WP-Riemannian metric* on the coset space $\text{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$.

The choice of the WP metric for the space of shapes is dictated by the following fact known from algebraic geometry: all sectional curvatures are negative. That means if we choose any two vectors in the tangent space $psl_2(\mathbb{R})\backslash\mathbf{Vec}(S^1)$ the curvature of the space of shapes in the 2-dimensional plane spanned by these vectors will be negative. It is generally believed that this would provide us with some beautiful properties: unique length-minimizing geodesics and unique so-called intrinsic means (or Karcher means), to name a few. The Karcher mean, for example, is used in computational anatomy applications for computing mean shape of a diseased tissue for computer diagnosis.

2.3 EPDiff and Teichons

EPDiff(X) stands for the Euler-Poincare equation on the group of diffeomorphism of the manifold X. It is a geodesic equation on the group of diffeomorphisms. EPDiff(X) was first studied in Arnold's ground-breaking paper [1]. Taking the tangents to such a geodesic and translating them back to the Lie algebra, i.e. the space of vector fields on X, we get a time varying vector field $\vec{u}(x,t)$ on X from which we can recover the geodesic by integrating. The geodesic equation now becomes a differential equation for $\vec{u}$, *first order* in t. Arnold considered in particular the group of volume preserving diffeomorphisms of Euclidean space in its L^2 metric and found the geodesic equation for the vector field $\vec{u}(\vec{x},t)$ to be Euler's fluid flow equation (see [2] for a full exposition). Other examples include the periodic Korteweg-deVries equation (KdV), and the periodic Camassa-Holm equation or C-H (see [4]). They are the geodesic equations on the Virasoro group, a central extension by S^1 of the group $\mathbf{Diff}(S^1)$ of the diffeomorphisms of S^1. These are two completely integrable partial differential equations. They have soliton solutions, i.e. for each fixed time, they are diffeomorphisms which are largely localized in space and, as time varies, they retain their general shape and can interact somewhat like solitons for KdV.

In my work I study a new example closely related to KdV and C-H. We consider the Weil-Petersson metric on the coset space $\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$. The geodesic equation on the space becomes:

$$m_t = -2m.u_\theta - u.m_\theta, \text{ where } m = -\mathcal{H}(u_\theta + u_{\theta\theta\theta}).$$

It is not known if this new equation is completely integrable but it admits a class of soliton like solutions which I study in my thesis, namely the solutions in which the momentum m is a distribution. Following the suggestion of Darryl Holm, we call these and their corresponding geodesics in Teichmüller space *Teichons*.

In general, solutions to the above equation integrate to geodesics on the universal Teichmüller space $\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$. As I have explained above, the universal Teichmüller space has another realization, namely as the space of smooth simple closed curves mod translations and scalings. Therefore the solutions of this equation can also be thought of as paths in the space of simple closed plane curves which minimize a certain energy. My research addresses two parts of the above statement: firstly, I investigate singular solutions of the geodesic equation, and, secondly, I compute geodesics between shapes numerically by minimizing a Weil-Petersson energy.

3 Main Contribution

- Existence for infinite time and bounds of an anzatz example of a Teichon. Conjectured heuristics for approximating any 2D shape.

  I have proven the existence for infinite time and bounds for a certain example of a singular solution of a geodesic equation, a 4-Teichon, in the special case when the momentum m is represented as a sum of four delta functions. I have done numerical experiments with randomly initialized Teichons and stated heuristics for approximating any smooth shape with a small, say twenty, number of solitons. This will allow us to have a finite-dimensional approximation of any smooth contour by a low-dimensional object, a 20-Teichon.

- Solving the geodesic equation numerically

  I have implemented various methods for solving the geodesic equation numerically. In particular, direct energy minimization via the gradient with respect to the path; forward solution of the full EPDiff equation; symplectic solver for the EPDiff in the case of Teichon solution. I have compared these methods of finding geodesic. Also I was able to demonstrate curvature properties of WP using the above methods.

- A novel way of obtaining an established expression [7],[9],[11] for the sectional curvatures of $\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$.

  In my work I have shown that the formula for the curvature of the Lie group G established by Arnold in the seminal paper [1] is valid in our case of the quotient group $\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$ with the WP metric which has a null space of the form $a + b\cos + c\sin$. I have obtained the formula for sectional curvature in the directions $u, v \in psl_2(\mathbb{R})\backslash\mathbf{Vec}(S^1)$. I am currently working on the proof of the negativity of the sectional curvature, it will be a different proof of the fact established in [7],[9],[11].

- Currently I am working on obtaining geodesics in the WP metric on a dataset of medical shapes. The dataset includes images of the saggital section of corpus collosum from 394 subjects. I am approximating geodesics with the Teichon solutions. Given the low-dimensional nature of Teichons and the simplicity of the numerical implementation of geodesic equations in this case, the approach looks promising.

4 Future Work

Throughout my course of study with Prof. Mumford at Brown University I have studied other metrics on the space of curves and worked on various applications and statistical analysis. I am looking forward to collaborating on other research projects in vision or related fields. There is also an immediate continuation of my research. Since my work draws from a variety of mathematical fields, therefore there is a multitude of directions the research can follow:

- Further refine numerical methods for finding geodesics. Shapes with large protrusions and big concavities are represented by fingerprints which have intervals of extremely large and extremely low derivatives. The way to overcome this problem would be to do optimization on the non-uniform grid, adjusting the grid's step sizes at every iteration of the minimization procedure to ensure proper rendering of large and small derivatives. Another interesting possibility would be to incorporate sectional curvature. Since curvature gives us information on the second derivatives along the geodesic (or, equivalently, on the behavior of the Jacobi vector fields along the geodesic path) we can use the expression for sectional curvature in the Newton optimization of the geodesic finding problem.
- Prove long-term existence of the general N-Teichon. Provide an answer to the question: given a specific shape, what is the N-Teichon that approximates this shape with an ε-error? Provide bounds on the "goodness" of the approximation of the shape by an N-Teichon.
- Investigate the question of complete integrability of the full EPDiff and the Teichon EPDiff.
- Develop statistics on curved manifolds that takes into the account the curvature. Currently, the statistics on the curved spaces is done via the linearization. A set of images is mapped to a common template, a Karcher mean ϕ_m. Then, each image is represented as a tangent vector in $T_{\phi_m}M$. Provided the vector, one solves the geodesic equation with the starting shape ϕ_m thus recovering the original shape. Effectively, the shape space is linearized in the tangent space $T_{\phi_m}M$. This allows to do statistics on this space, compute means, perform clustering etc. But current approach does not account for the curvature. Sectional curvature dictates the spread of the geodesics. In the negatively curved spaces, for example, geodesics with initial vectors that span the plane with a lower negative curvature c_1 will spread out more than the ones in the plane with the sectional curvature c_2 higher ($c_1 < c_2 < 0$). Thus, accounting for it will allow us to do better clustering and compute better mean shapes.
- Refine the shooting algorithm and apply it to various database, e.g. to an MPEG7 database.
- Prove Arnold's formula for the sectional curvature and develop the theory of the EPDiff equation in the case of a general quotient group $H\backslash G$.

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