TEICHONS: SOLITON-LIKE GEODESICS ON UNIVERSAL TEICHMÜLLER SPACE

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Abstract. This paper studies EPDiff(S^1) equation (Euler-Poincaré equation for diffeomorphisms of S^1) with the Weil-Petersson metric on the coset space $\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$. This coset space is known as the universal Teichmüller space. It has another realization as the space of smooth simple closed curves mod translations and scalings. EPDiff(S^1) admits a class of soliton like solutions (Teichons) in which the "momentum" m is a distribution. The solutions of this equation can also be thought of as paths in the space of simple closed plane curves which minimize a certain energy. In this paper we study the solution in the special case where m is expressed as a sum of four delta functions. We prove the existence of the solution for infinite time and find bounds on its longterm behavior, showing that it is asymptotic to a one-parameter subgroup in $\mathbf{Diff}(S^1)$. We then show a series of numerical experiments on solitons with more delta functions and make some conjectures about these.

Contents

1. Introduction

Geodesic equations of groups of diffeomorphisms on a manifold X were first studied in Arnold's ground-breaking paper [Arnold 66]. Taking the tangents to such a geodesic and translating them back to the Lie algebra, i.e. the space of vector fields on X, we get a time varying vector field $\vec{u}(x,t)$ on X from which we can recover the geodesic by integrating. The geodesic equation now becomes a differential equation for $\vec{u}$, *first order* in t. Arnold considered in particular the group of volume preserving diffeomorphisms of Euclidean space in its L^2 metric and found the geodesic equation for the vector field $\vec{u}(\vec{x},t)$ to be Euler's fluid flow equation (see [Arnold and Khesin 98] for a full exposition). Since then, many other examples have been looked at. For example, both the periodic Korteweg-deVries equation (KdV) for a function $u(\theta,t)$:

$$u_t = -3u.u_\theta - u_{\theta\theta\theta}$$

and the periodic Camassa-Holm equation or C-H (see [Camassa and Holm 93])

$$m_t = -2m.u_\theta - u.m_\theta - u_{\theta\theta\theta}, \text{ where } m = u - u_{\theta\theta}$$

have been found to be the geodesic equations on the Virasoro group, a central extension by S^1 of the group $\mathbf{Diff}(S^1)$ of the diffeomorphisms of S^1. These are two completely integrable partial differential equations and have soliton solutions. More recently, Holm and collaborators have found that quite generally the geodesic equation on $\mathbf{Diff}(\mathbf{R}^n)$ admits special solutions with many of the properties of solitons: for each fixed time, they are diffeomorphisms which are largely localized in space and, as time varies, they retain their general shape and can interact somewhat like solitons for KdV [Holm and Marsden 04]. There are not, however, infinitely many conserved quantities so they are not true solitons. A discussion of EPDiff and solitons in the case of template matching in computational anatomy can be found in [Holm et al 04].

This paper studies a new example closely related to KdV and C-H. We consider the Weil-Petersson (WP) metric on the coset space $\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$. This coset space (or its completion in the WP metric or in the Teichmüller topology) is known as the universal Teichmüller space and is well-known in many contexts: in the classification of Riemann surfaces [Hubbard 06], conformal and quasi-conformal maps [Lehto 87], string theory [Bowick and Rajeev 87] and most recently computer vision [Sharon and Mumford 06]. Its completion in the WP metric is an infinite dimensional homogeneous complex Kähler-Hilbert manifold ([Takhtajan and Teo 06]). Again the geodesic equation lifts to the Lie algebra, the space of vector fields on the circle mod the subspace spanned by 1, cos and sin, but now it is an integro-differential equation involving not only derivatives but the periodic Hilbert transform $\mathcal{H}$ (defined by convolution with $\frac{1}{2\pi}\mathrm{ctn}(\theta/2)$ instead of with $1/x$). The equation is:

$$m_t = -2m.u_\theta - u.m_\theta, \text{ where } m = -\mathcal{H}(u_\theta + u_{\theta\theta\theta}).$$

Here, we may invert the relationship between m and u and write:

$$u(\theta,t) = \int_{S^1} G(\theta-\xi)m(\xi,t)d\xi = G * m.$$

The integral kernel, or Green's function, $G(\theta)$ turns out to be given in the Fourier domain by

$$G(\theta) = 2\sum_{n=2}^{\infty} \frac{\cos(n\theta)}{(n^3-n)}.$$

Note that $m(\cdot,t)$ is always orthogonal to 1, cos and sin.

It is not known if this new equation is completely integrable but it admits a class of soliton like solutions which we study here, namely the solutions in which m is a distribution. In fact, we want $m(\cdot,t)$ to be a weighted sum of delta functions for one and hence all t. Because m is orthogonal to 1, cos and sin, there must be at least 4 delta functions. Following a suggestion of Darryl Holm, we call these and their corresponding geodesics in Teichmüller space *Teichons*.

In general, solutions to the above equation integrate to geodesics on the universal Teichmüller space $\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$. As we explain in §2 below, the universal Teichmüller space has another realization, namely as the space of smooth simple closed curves mod translations and scalings. Therefore the solutions of this equation can also be thought of as paths in the space of simple closed plane curves which minimize a certain energy. The soliton property means that, in a certain sense, their momentum is concentrated at a finite set of points.

In this paper we study the solution in the special case where the momentum m is expressed as a sum of four delta functions. We prove the existence of the solution for infinite time and find bounds on its longterm behavior, showing that it is asymptotic to a one-parameter subgroup in $\mathbf{Diff}(S^1)$. We then show a series of experiments on solitons with more delta functions and make some conjectures about these.

I would like to thank my adviser, David Mumford, and Peter Michor for valuable comments and insightful discussions. I especially grateful to Darryl Holm for introducing me to the idea of studying these remarkable special solutions of the geodesic equations.

2. Shapes as Diffeomorphisms of the Circle

In this paper 'shape' means a simple closed smooth curve Γ in the plane which is associated with the complex plane $\mathbb{C}$; also we need an extended complex plane (or Riemann sphere) which is denoted by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In this section, we review how the universal Teichmüller space is isomorphic to a space of shapes, that is:

$$\boxed{\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1) \cong \text{ set of shapes/translations, scalings.}}$$

Denote the interior of the unit disk by $\mathbb{D}_{\mathrm{int}} = \{z \mid |z| \leq 1\}$ and the infinite region outside (including ∞) by $\mathbb{D}_{\mathrm{ext}} = \{z \mid |z| \geq 1\}$. For every simple closed curve Γ in $\mathbb{C}$ denote by Γ_{int} its union with the region enclosed by it, and denote by Γ_{ext} its union with the infinite region outside of Γ (including ∞).

Then by Riemann mapping theorem, for all Γ there exists a conformal map

$$\phi_{\mathrm{int}} : \mathbb{D}_{\mathrm{int}} \to \Gamma_{\mathrm{int}},$$

unique up to replacing ϕ_{int} by $\phi_{\mathrm{int}} \circ A$ for any Möbius transformation $A : \mathbb{D}_{\mathrm{int}} \to \mathbb{D}_{\mathrm{int}}$, where A defined as

$$A(z) = \frac{az+b}{\bar{b}z+\bar{a}}, \ |a|^2 - |b|^2 = 1.$$

This subgroup of Möbius group of selfmaps of the circle is denoted $\mathrm{PSL}_2(\mathbb{R})$.

Similarly we get a conformal map of the exteriors

$$\phi_{\mathrm{ext}} : \mathbb{D}_{\mathrm{ext}} \to \Gamma_{\mathrm{ext}}.$$

ϕ_{ext} is also unique up to any Möbius transformation as above. But in this case we normalize: we choose a unique Möbius map A, such that $\phi_{\text{ext}} \circ A$ maps ∞ to ∞, and that its differential carries the real positive axis of the D-plane at infinity to the real positive axis of the Γ-plane at infinity. Thus the ambiguity in the choice of ϕ_{ext} is eliminated for every Γ.

The goal of this construction is to define the map called the 'fingerprint' of the shape

$$\psi = \phi_{\text{int}}^{-1} \circ \phi_{\text{ext}} \in \mathrm{PSL}_2(\mathbb{R}) \backslash \mathbf{Diff}(S^1),$$

which is defined on the unit circle S^1 (note that $\phi_{\text{int}}(S^1) = \Gamma,\ \phi_{\text{ext}}^{-1}(\Gamma) = S^1$). The fingerprint $\psi : S^1 \to S^1$ is a real-valued orientation-preserving diffeomorphism. ψ is a uniquely identifying fingerprint of the shape Γ. The fingerprint of the eye-shape is shown in Fig.4. From Möbius transformation ambiguity in the choice of ϕ_{int} we can see by construction that ψ is a member of the right coset space $\mathrm{PSL}_2(\mathbb{R}) \backslash \mathbf{Diff}(S^1)$, where $\mathrm{PSL}_2(\mathbb{R})$ is a group of Möbius maps.

Note that one can equally define the fingerprint to be

$$\phi_{\text{ext}}^{-1} \circ \phi_{\text{int}} \in \mathbf{Diff}(S^1)/\mathrm{PSL}_2(\mathbb{R})$$

and this is just the inverse of our fingerprint. This alternate version is the definition used in [Sharon and Mumford 06]. However, in this paper, we stick to *right* cosets and put the Möbius ambiguity on the left.

The inverse map from diffeomorphisms to shapes is defined as follows: starting with ψ, construct an abstract Riemann surface by 'welding' the boundaries of $\mathbb{D}_{\text{int}}$ and $\mathbb{D}_{\text{ext}}$ via ψ. The resulting Riemann surface must be conformally equivalent to the Riemann sphere. Choose a conformal map ϕ from the welded surface to the sphere taking $\infty \in \mathbb{D}_{\text{ext}}$ to itself and having real positive derivative there. Let $\Gamma = \phi(S^1)$ (for details see [Sharon and Mumford 06]).

3. WEIL-PETERSSON NORM ON THE LIE ALGEBRA OF $\mathbf{Diff}(S^1)$

3.1. **The Norm.** The Lie algebra of the group $\mathbf{Diff}(S^1)$ is given by the vector space $\mathbf{Vec}(S^1)$ of smooth periodic vector fields $v(\theta)\partial/\partial\theta$ on the circle.

Expanding such a v in a Fourier series $v(\theta) = \sum_{n=-\infty}^{\infty} v_n e^{in\theta}$ (where $\overline{v_n} = v_{-n}$ for the vector field to be real) we can define the Weil-Petersson norm on $\mathbf{Vec}(S^1)$ as:

$$\|v\|_{WP}^2 = \sum_{n \in \widehat{\mathbb{Z}}} |n^3 - n||v_n|^2. \tag{1}$$

Here $\widehat{\mathbb{Z}} = \mathbb{Z} \backslash \{n = 0, \pm 1\}$.

The null space of this norm is given by the vector fields whose only Fourier coefficients are v_{-1}, v_0 and v_1, i.e. vector fields of the type $(a + b\cos\theta + c\sin\theta)\partial/\partial\theta$. These vector fields are exactly in the Lie algebra $psl_2(\mathbb{R})$ of the Lie group $\mathrm{PSL}_2(\mathbb{R})$.

The motivation for this particular definition is the fact that for all $A \in \mathrm{PSL}_2(\mathbb{R})$ and $v \in \mathbf{Vec}(S^1)$ one can verify that

$$\|\mathrm{Ad}_A(v)\|_{WP} = \|v\|_{WP}.$$

3.2. Extending the WP Metric to $\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$.

Consider any Lie group G, a subgroup H, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their corresponding Lie algebras. Quite generally, any norms $\|\cdot\|$ on the Lie algebra of G which are zero on the Lie subalgebra of H and which satisfy $\|\mathrm{Ad}_h(v)\| = \|v\|$ for all $h \in H$ induce a Riemannian metric on coset spaces $H\backslash G$ which are invariant by all right multiplication maps $R_g : H\backslash G \to H\backslash G$, $g \in G$.

In particular this applies to $G = \mathbf{Diff}(S^1), H = \mathrm{PSL}_2(\mathbb{R})$ and the above WP norm on vector fields, hence it gives the right-invariant *WP-Riemannian metric* on the coset space $\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$.

3.3. The WP Green's Function.

The Weil-Petersson norm can also be defined via a differential operator L:

$$\begin{aligned} \|v\|^2_{WP} &= \langle Lv, v\rangle, \\ L &= -\mathcal{H}(\partial_\theta^3 + \partial_\theta), \end{aligned}$$

where $\mathcal{H}$ is the periodic Hilbert transform defined by convolution with $\frac{1}{2\pi}\mathrm{ctn}(\theta/2)$. This is because in Fourier series, $\mathcal{H}$ is multiplication by $-i.\mathrm{sign}(n)$ hence

$$-\mathcal{H}(\partial_\theta^3 + \partial_\theta)e^{in\theta} = i\mathrm{sign}(n)((in)^3 + in)e^{in\theta} = |n^3 - n|e^{in\theta}.$$

For later purposes we need to find an inverse of operator L, i.e. its Green's function $G(\theta)$. Using representation of L in Fourier basis (1) we need to find

$$G(\theta) = \sum_{n\in\widehat{\mathbb{Z}}} \frac{e^{in\theta}}{|n^3 - n|}.$$

Using the following series from [Gradshteyn and Ryzhik 00]

$$\sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k} = \frac{1}{2}\ln\frac{1}{2(1-\cos\theta)}, \text{ for } \theta \in (0, 2\pi).$$

And employing the decomposition $\frac{1}{n^3-n} = \frac{1}{2}\frac{1}{n+1} + \frac{1}{2}\frac{1}{n-1} - \frac{1}{n}$ we find that the Green's function of the WP operator L has the form

$$G(\theta) = (1 - \cos\theta)\log[2(1 - \cos\theta)] + \frac{3}{2}\cos\theta - 1. \tag{2}$$

One can easily verify that $LG(\theta) = \delta_0(\theta) - 1 - 2\cos\theta$, the projection of δ_0 onto the subspace of distributions orthogonal to $1, \cos(\theta), \sin(\theta)$. The profile of Green's function could be seen on fig.1.

4. EPDiff

The EPDiff equation (which stands for Euler-Poincaré equation for diffeomorphisms) is a variant of Euler's fluid flow equation. It describes geodesics on the group of diffeomorphisms of $\mathbb{R}^n$ in any invariant metric given on vector fields by

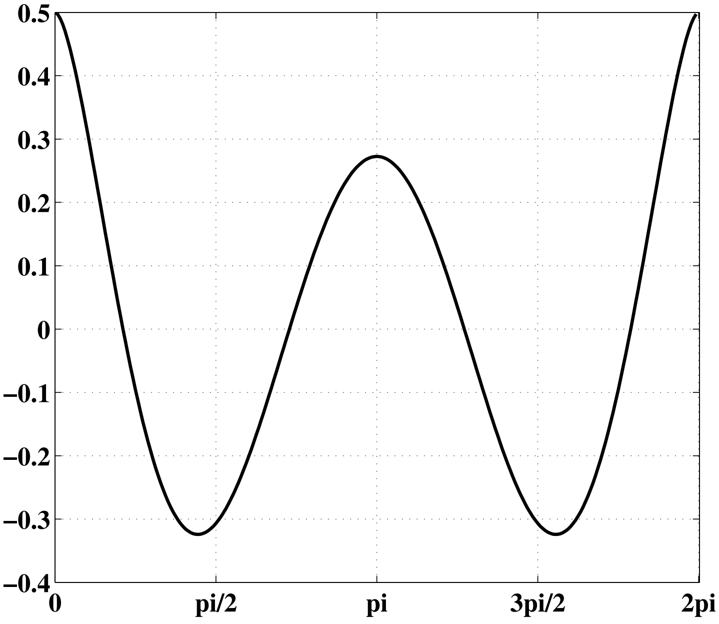

Figure 1. Green's function (2) of the Weil-Peterson operator L_{WP}. It is smooth except for a log-pole in its *second* derivative at $\theta = 0$.

$\|v\|^2 = \int_{\mathbb{R}^n} < Lv, v > dx$ for some positive definite self-adjoint operator L. The general EPDiff follows from [Arnold 66] and has the form

$$\frac{\partial}{\partial t} Lv + (v \cdot \nabla)(Lv) + \mathrm{div} v . Lv + Dv^t \cdot Lv = 0,$$

It can also be derived (as in the 1998 Insitut Henri Poincaré notes to be published in [Mumford and Desolneux 09]) via the first variation of energy $E(\psi)$ of the path $\psi(x,t) \in \mathbf{Diff}(\mathbb{R}^n)$:

$$E(\psi) = \int_0^1 \left\| \frac{\partial \psi}{\partial t}(\psi^{-1}(x,t),t) \right\|_L^2 dt$$

In our case, we have the homogeneous space $\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$, not a group. Fortunately Arnold's formula for geodesics on groups extend with only small changes to equations for geodesics on homogeneous spaces $H\backslash G$:

Theorem 1. *Let G be any lie group and H a subgroup. Let $\langle g_1, g_2 \rangle_L = (L(g_1), g_2)$ be a non-negative symmetric inner product on $\mathfrak{g}$ with null space $\mathfrak{h}$, defined by a non-negative self-adjoint linear map $L : \mathfrak{g} \to \mathfrak{g}^*$. Assume this inner product is invariant under $Ad_h, h \in H$. As above, this defines a G-invariant metric on $H\backslash G$. Let $g(t)$ be any path in G and define $u(t) = g_t \cdot g^{-1}$ to be its tangent path in $\mathfrak{g}$. Then:*

$$\{H \cdot g(t)\} \subset H\backslash G \text{ is a geodesic } \iff Lu_t = ad_u^*(Lu) \text{ in } \mathfrak{g}^*$$

where $ad_u^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is the adjoint of $ad_u, u \in \mathfrak{g}$.

The proof essentially the same as Arnold's theorem for the case $H = \{e\}$. Note that it is convenient computationally to choose a *splitting* $\mathfrak{g} = \mathfrak{h} \oplus W$. Then L defines an isomorphism $W \to W^*$ whose inverse we may call L^i. Translating W around the group G, we get a splitting of the tangent bundle to G into the 'vertical' vectors tangent to the cosets of H and a 'horizontal' bundle $\mathcal{W}$ given by W. Then the above theorem can be applied to paths for which $u(t) \in W$ for all t and these correspond to the paths in G which are 'horizontal' lifts of paths in $H\backslash G$. Then geodesics in $H\backslash G$ are defined by solutions of Arnold's equation with $u(t) \in W$.

This applies to our case of $\mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$. On the lie algebra level, $\mathrm{PSL}_2(\mathbb{R})$ becomes the vector fields spanned by 1, cos and sin, i.e. those whose only non-zero Fourier coefficients are $-1, 0, +1$. The simplest complement W are the vector fields with these coefficients zero. In fact, the Green's function G has this property, so convolution with this G is the above operator L^i in this case. The geodesic equation is therefore simply another special case of EPDiff. Given a path $\psi(\theta, t)$ in $\mathbf{Diff}(S^1)$, let $v(\theta, t) = \frac{\partial \psi}{\partial t}(\psi^{-1}(\theta, t), t)$ be the scalar vector field it defines on a circle and let L be the Weil-Petersson differential operator $L = -\mathcal{H}(\partial_\theta^3 + \partial_\theta)$. Then EPDiff takes the form

$$(Lv)_t + v.(Lv)_\theta + 2v_\theta.Lv = 0. \tag{3}$$

We call $v(\theta, t)$ the velocity of the path and $m(\theta, t) = Lv(\theta, t)$ the momentum. Then inversely $v(\theta, t) = G * m(\theta, t)$. Note that momenta can be distributions. Both u and v will have vanishing $-1, 0, +1$ Fourier coefficients. In this article we consider an ansatz, a special form of momentum which we call a Teichon (or an N-Teichon), given by a sum of N delta-functions, i.e.

$$\begin{aligned} m(\theta, t) &= \sum_{j=1}^{N} a_j \delta(\theta - b_j), \\ v(\theta, t) &= \sum_{j=1}^{N} a_j G(\theta - b_j). \end{aligned}$$

Plugging this expressions into EPDiff (3) we get a system of ODEs describing the evolution of a_k's and b_k's:

$$\begin{cases} \dot{a}_k = -a_k \sum_{j=1}^{N} a_j G'(b_k - b_j), \\ \\ \dot{b}_k = \sum_{j=1}^{N} a_j G(b_k - b_j). \end{cases} \tag{4}$$

where a_j, b_j are functions of time t. For well-posedness of the problem we need to require that $m(\theta, t)$ should have its 0th, ±1st Fourier coefficients zero, i.e. we have the following constraints:

$$\sum_{j=1}^{N} a_j = \sum_{j=1}^{N} a_j e^{ib_j} = \sum_{j=1}^{N} a_j e^{-ib_j} = 0. \tag{5}$$

If they are satisfied at time $t = 0$ they will be satisfied for all t. The minimum number N of distinct delta functions (all b_k's are different) to satisfy this condition is four. Note that EPDiff is invariant under the action $\mathrm{Ad}_A, A \in \mathrm{PSL}_2$, so we may always normalize Teichons by the Möbius group.

5. Estimates

5.1. **4-Teichon ODE.**

We consider the case of $N = 4$ deltas. Due to the Möbius invariance, we can normalize the initial conditions giving us the following initial configuration for momentum $m(\theta)$:

$$\begin{aligned}
&m(\theta, t=0) = \textstyle\sum_{j=1}^4 a_j \delta(\theta - b_j), \text{ where} \\
&(a_1, a_2, a_3, a_4) = (1, -1, 1, -1), \\
&(b_1, b_2, b_3, b_4) = (2\pi - d_0/2, d_0/2, \pi - d_0/2, \pi + d_0/2), \\
&a_0 = 1,\ d_0 \in (0, \epsilon).
\end{aligned}$$

If this holds for time 0, then it is easy to check that for all times t

$$\begin{aligned}
(a_1, a_2, a_3, a_4) &= (a(t), -a(t), a(t), -a(t)), \\
(b_1, b_2, b_3, b_4) &= (2\pi - d(t)/2,\ d(t)/2,\ \pi - d(t)/2,\ \pi + d(t)/2).
\end{aligned}$$

for some functions $a(t), d(t)$. Then the system (4) becomes a system of just two variables $a(t)$ and $d(t)$

$$\begin{aligned}
\dot{a} &= a^2[G'(-d) + G'(\pi - d)], \\
\dot{d} &= -2a[G(0) - G(-d) + G(\pi) - G(\pi - d)].
\end{aligned} \tag{6}$$

Equivalently, if $E(d)$ denotes $G(0) - G(-d) + G(\pi) - G(\pi - d)$ and prime is the differentiation in d, then the system is:

$$\begin{aligned}
\dot{a} &= a^2 E'(d), \\
\dot{d} &= -2aE(d).
\end{aligned} \tag{7}$$

Notice that (7) is a Hamiltonian system with the conserved energy $H = a^2 E(d) = k^2$, for some constant k. Therefore we can have the equation just in terms of d

$$\dot{d} = -2k\sqrt{E(d)}, \text{ where } k = a_0\sqrt{E(d_0)}. \tag{8}$$

The other equation is a conserved quantity, which relates $a(t)$ and $d(t)$:

$$a(t)\sqrt{E(d(t))} = k.$$

Geodesic in the universal Teichmüller space with the above initial momentum $m(\theta)$ could be seen on fig.2 (evolution of $\log|a_k|$'s and b_k's is shown on fig.3).

5.2. **Estimates of $a(t)$, $d(t)$.**

For small $d \in (0, \epsilon)$ we have an expansion of $E(d) = G(0) - G(-d) + G(\pi) - G(\pi - d)$:

$$E(d) = d^2\left(\frac{1}{2} - \log\frac{d}{2}\right) - d^4\left(\frac{3}{48} - \frac{1}{12}\log\frac{d}{2}\right) + O(d^6 \log d).$$

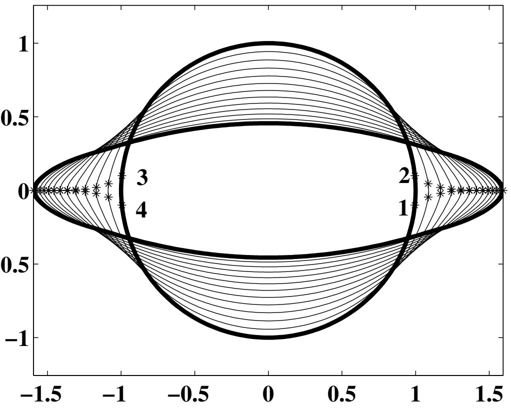

FIGURE 2. Geodesic with initial momentum $m(\theta) = \sum_{k=1}^{4} a_k \delta(\theta - b_k)$ upto time T=6, $(a_{k=1}^4)_{t=0} = (1, -1, 1, -1), (b_{k=1}^4)_{t=0} = (2\pi - 0.1, 0.1, \pi - 0.1, \pi + 0.1)$. Position of delta functions (i.e. b_k's) is marked by asterisks.

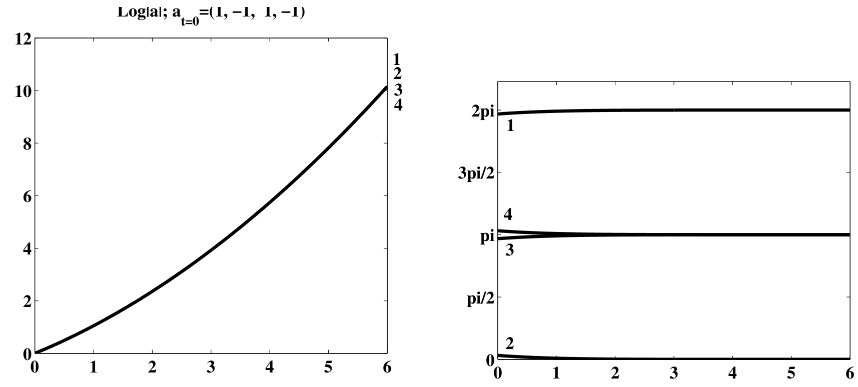

FIGURE 3. Time evolution of $\log|a_k|$'s (left) and b_k's (right) for the 4-Teichon: $(a_{k=1}^4)_{t=0} = (1, -1, 1, -1), (b_{k=1}^4)_{t=0} = (2\pi - 0.1, 0.1, \pi - 0.1, \pi + 0.1)$.

And the basic inequality

$$d^2\left(\frac{1}{2} - \log\frac{d}{2} - \gamma\right) \leq E(d) \leq d^2\left(\frac{1}{2} - \log\frac{d}{2}\right). \tag{9}$$

Here γ is chosen to guarantee that $-\gamma d^2 \leq E(d) - d^2\left(\frac{1}{2} - \log\frac{d}{2}\right)$, or in other words $\gamma \geq -E(d)/d^2 + \left(\frac{1}{2} - \log\frac{d}{2}\right)$. Since the RHS is an increasing function of d, and $d(t)$ will be decreasing with t it suffices to take any $\gamma \geq -E(d_0)/d_0^2 + \left(\frac{1}{2} - \log\frac{d_0}{2}\right)$ (in simulations the value used was $d_0 = .2$, hence we may take $\gamma = .0102$).

Based on this we get the following inequalities for the RHS of the equation (8):

$$-2kd\sqrt{\frac{1}{2}-\log\frac{d}{2}} \le -2k\sqrt{E(d)} \le -2kd\sqrt{\frac{1}{2}-\log\frac{d}{2}-\gamma}. \tag{10}$$

From the theory of ODEs we can conclude that for all $t \in [0,+\infty)$ if $d_0 \in (0,\epsilon)$ ($d(t)$ will stay in $(0,\epsilon)$ for all t), then the following estimate for $d(t)$ is valid:

$$\boxed{\begin{array}{l} 2\exp\left\{\frac{1}{2}-(kt+c_1)^2\right\} \le d(t) \le 2\exp\left\{\frac{1}{2}-(kt+c_2)^2-\gamma\right\}, \\ \\ \text{where } k = a_0\sqrt{E(d_0)} > 0, \\ c_1 = \sqrt{\frac{1}{2}-\log\frac{d_0}{2}}, \\ c_2 = \sqrt{\frac{1}{2}-\log\frac{d_0}{2}-\gamma};\ c_1 > c_2. \end{array}} \tag{11}$$

Denote the bounds

$$\begin{aligned} d_1(t) &:= 2\exp\left\{\frac{1}{2}-(kt+c_1)^2\right\}, \\ d_2(t) &:= 2\exp\left\{\frac{1}{2}-(kt+c_2)^2-\gamma\right\}. \end{aligned} \tag{12}$$

Using inequalities (9) we get the following estimates:

$$\frac{1}{\sqrt{\frac{1}{2}-\log\frac{d}{2}}} \le \frac{d}{\sqrt{E(d)}} \le \frac{1}{\sqrt{\frac{1}{2}-\log\frac{d}{2}-\gamma}}.$$

Combining it with the estimates on $d(t)$ and the fact that both bounds are increasing functions of d we conclude that the product ad decreases as $1/t$, or specifically

$$\begin{gathered} \frac{k}{\sqrt{\frac{1}{2}-\log\frac{d_1}{2}}} \ \le\ a(t)d(t) = k\frac{d}{\sqrt{E(d)}} \ \le\ \frac{k}{\sqrt{\frac{1}{2}-\log\frac{d_2}{2}-\gamma}}, \\ \boxed{\frac{k}{kt+c_1} \ \le\ a(t)d(t) \ \le\ \frac{k}{kt+c_2}.} \end{gathered} \tag{13}$$

Finally an estimate for $a(t)$:

$$\begin{gathered} \frac{k}{d_2\sqrt{\frac{1}{2}-\log\frac{d_2}{2}}} \ \le\ a(t) = \frac{k}{\sqrt{E(d)}} \ \le\ \frac{k}{d_1\sqrt{\frac{1}{2}-\log\frac{d_1}{2}-\gamma}}, \\ \boxed{\frac{k\exp\left\{(kt+c_2)^2-\frac{1}{2}+\gamma\right\}}{2\sqrt{(kt+c_2)^2+\gamma}} \ \le\ a(t) \ \le\ \frac{k\exp\left\{(kt+c_1)^2-1/2\right\}}{2\sqrt{(kt+c_1)^2-\gamma}}.} \end{gathered} \tag{14}$$

5.3. **Estimating the velocity field $v(\theta,t)$.** For $\theta \in I = [\delta,\pi-\delta]\cup[\pi+\delta,2\pi-\delta]$ (away from points $0,\pi,2\pi$, where $G(\theta)$ is only C^1) we have the following Taylor expansion:

$$G(\theta+\Delta\theta) = G(\theta) + \Delta\theta\, G'(\theta) + \frac{\Delta\theta^2}{2}G''(\theta) + \frac{\Delta\theta^3}{6}G'''(\theta) + \frac{\Delta\theta^4}{24}G^{(iv)}(\theta) + O(\Delta\theta^5).$$

Therefore using the above we have the following expansion of the velocity field $v(\theta, t)$

$$\begin{aligned} v(\theta,t) &= a[G(\theta+d/2)-G(\theta-d/2)+G(\theta+\pi+d/2)-G(\theta+\pi-d/2)] \\ &= ad[G'(\theta)+G'(\theta+\pi)]+ad^3/24[G'''(\theta)+G'''(\theta+\pi)]+O(ad^5). \end{aligned}$$

For derivatives of the Green's function we get

$$\begin{aligned} G'(\theta)+G'(\theta+\pi) &= 2\sin\theta\log|\tan\theta/2|, \\ G'''(\theta)+G'''(\theta+\pi) &= -2\sin\theta\log|\tan\theta/2|+2\cot\theta. \end{aligned}$$

And applying estimates (11), (13) for ad, ad^3 we finally arrive at

$$\boxed{\begin{array}{l} \textbf{First order approximation:} \\ v(\theta,t)\approx 2\frac{k}{kt+c}\sin\theta\log|\tan\theta/2|+O\left(\frac{1}{te^{2kt^2}}\right),\ \text{for some } c. \\ \\ \textbf{Third order approximation:} \\ v(\theta,t)\approx 2\frac{k}{kt+c}\sin\theta\log|\tan\theta/2| \\ +r(t)(\cot\theta-\sin\theta\log|\tan\theta/2|)+O\left(\frac{1}{t\exp(4kt^2)}\right), \\ \text{where } r(t) \text{ s.t. } \frac{k}{3}\frac{\exp(1-2(kt+c_1)^2)}{kt+c_1} \ \le\ r(t)\ \le\ \frac{k}{3}\frac{\exp(1-2(kt+c_2)^2-2\gamma)}{kt+c_2}. \end{array}} \tag{15}$$

Here the constant k is as before $k = a_0\sqrt{E(d_0)}$, and we can choose c somewhere between c_2 and c_1 ($c_2 = \sqrt{1/2-\log\frac{d_0}{2}-\gamma} < c_1 = \sqrt{1/2-\log\frac{d_0}{2}}$).

Remark. Notice that $\|v(\theta,t)\|_{WP}$ should be constant, but in the first order expansion there is a t-term which is monotonically decreasing. It is actually the case that $\sin\theta\log|\tan\theta/2|$ has infinite WP-norm.

5.4. **Estimating the fingerprint $\psi(\theta, t)$.** The fingerprint $\psi(\theta,t)$ evolves under the action of the velocity field $v(\theta,t)$ according to

$$\psi_t(\psi^{-1}(\theta,t),t) = v(\theta,t).$$

Denote $\psi^i(\theta,t)$ the inverse of $\psi(\theta,t)$ in θ-variable for each fixed t-variable. It is easy to check that $\psi^i(\theta,t)$ satisfies the transport equation:

$$\psi^i_t(\theta,t) = -v(\theta,t)\psi^i_\theta(\theta,t). \tag{16}$$

Remark. In this case $\psi^i(\theta,t)$ is in the left coset space $\mathbf{Diff}(S^1)/\mathrm{PSL}_2(\mathbb{R})$ and we have the same setup as in Mumford, Sharon [Sharon and Mumford 06].

We are going to solve the transport equation (16) for the inverse fingerprint ψ^i with *first order approximation* of v (15).

Note that $G(\theta)$ is smooth, except for the logarithmic pole in $G''(\theta)$ at $\theta = 0$. Let us write out the first order Taylor's series with the remainder term in the integral form:

$$\begin{aligned} G(\theta+\delta) &= G(\theta)+G'(\theta)\delta+R_1(\theta), \\ &\quad \text{where } R_1(\theta) = \int_\theta^{\theta+\delta} G''(t)(\theta+\delta-t)dt. \end{aligned}$$

The remainder term $R_1(\theta)$ for all δ's small will be uniformly bounded for all θ away from zero. Employing the fact that $G''(\theta) = \cos\theta \log[2(1-\cos\theta)] + 1/2\cos\theta + 1$ one can easily verify that the remainder term $R_1(\theta)$ at $\theta = 0$:

$$R_1(0) = \int_0^\delta G''(t)(\delta - t)dt = \int_0^\delta ((2-t^2)\log t + 3/2 + O(t^2))(\delta - t)dt = O(\delta^2 \log\delta).$$

Therefore the first order estimate of $v(\theta, t)$ in (15) is valid uniformly for all $\theta \in [0, 2\pi]$ and t, while the third order estimate is valid uniformly away from $\theta = 0, \pi$.

From PDE theory it is known that the solution remains constant along the characteristics of the equation (16). The characteristic equations are:

$$\begin{cases} \dfrac{\partial\theta}{\partial s} = 2\dfrac{k}{kt+c}\sin\theta\log|\tan\theta/2|, \\[2ex] \dfrac{\partial t}{\partial s} = 1. \end{cases}$$

Using the fact that $\int \frac{d\theta}{\sin\theta} = \log|\tan\theta/2|$ we get

$$\frac{\log|\tan(\theta-\pi)/2|}{\log|\tan(\theta_0-\pi)/2|} = \left(\frac{kt+c}{c}\right)^2.$$

Given point (θ, t) a characteristic passes through the point $(\theta_0, 0)$, where θ_0 is expressed using the previous formula, i.e.

$$\theta_0 = 2\arctan\left[(\tan(\theta-\pi)/2)^{\left(\frac{c}{kt+c}\right)^2}\right] + \pi.$$

Since ψ^i stays constant along characteristics and since $\psi^i(\theta, t = 0) = \theta$:

$$\psi^i(\theta, t) = \psi^i(\theta_0, 0) = \theta_0 = 2\arctan\left[(\tan(\theta-\pi)/2)^{\left(\dfrac{c}{kt+c}\right)^2}\right] + \pi, \tag{17}$$
$$\text{where } \tan^\beta = \text{sign}(\tan)|\tan|^\beta.$$

If we denote $\left(\dfrac{c}{kt+c}\right)^2 = \beta$, then the above inverse fingerprint is the fingerprint of the 'eye' shape with angles at its corners $\alpha\pi$, $\alpha = 2\beta/(\beta+1)$ (example of the fingerprint on fig.4, see also [Sharon and Mumford 06]). In other words the curve starts as a circle (angles at the corners of the 'eye' are π) and angle gets smaller with time as $\frac{2\pi}{1+(kt/c+1)^2}$.

Invert expression (17) to get the approximate evolution of the fingerprint $\psi \in \mathrm{PSL}_2(\mathbb{R})\backslash\mathbf{Diff}(S^1)$:

$$\boxed{\begin{array}{l} \psi(\theta, t) = 2\arctan\left[(\tan(\theta-\pi)/2)^{\left(\dfrac{kt+c}{c}\right)^2}\right] + \pi, \\ \text{where } \tan^\beta = \text{sign}(\tan)|\tan|^\beta. \end{array}} \tag{18}$$

This fingerprint still defines an eye-shaped figure (see fig.4), one just needs to perform welding using the fact that $\psi = \phi_{\text{int}}^{-1} \circ \phi_{\text{ext}}$.

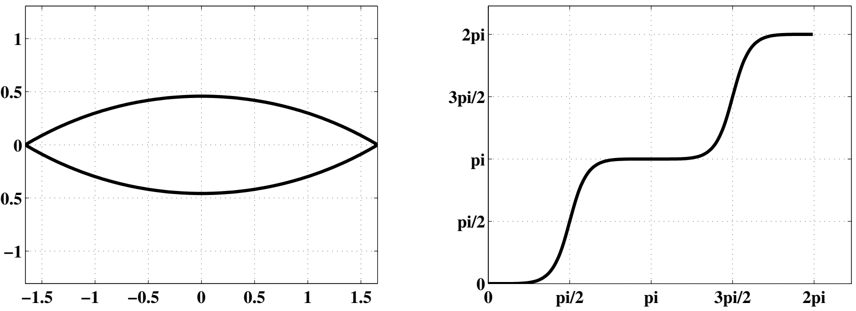

Figure 4. Example of the eye-shape and its fingerprint ψ as given by equation (17).

The limiting fingerprint is

$$\psi_\infty(\theta) = \begin{cases} \pi/2, \text{ for } \theta \in (0,\pi); \\ 3\pi/2, \text{ for } \theta \in (\pi, 2\pi). \end{cases}$$

We have proved the existence of the solution for infinite time and have found bounds on its longterm behavior. In the limit the shape represented by the fingerprint (18) will become an infinite slit on the plane. Note, that our soliton trajectory is asymptotic to the following one parameter subgroup in $\mathbf{Diff}(S^1)$:

$$\begin{aligned} \psi_\beta(\theta) &= 2\arctan(\tan(\theta/2)^\beta), \\ & \text{ where } \tan^\beta = \operatorname{sign}(\tan)|\tan|^\beta, \\ \psi_{\beta_1\beta_2} &= \psi_{\beta_1} \circ \psi_{\beta_2}. \end{aligned}$$

6. Numerical Experiments

6.1. **Numerical methods.** System (4) is in fact a Hamiltonian system with Hamiltonian $H_{ab} = \sum_{i,j=1}^N a_i a_j G(b_i - b_j)$ and it could be rewritten as

$$\begin{cases} \dot{a}_k = -\dfrac{\partial H_{ab}}{\partial b_k}, \\ \dot{b}_k = \dfrac{\partial H_{ab}}{\partial a_k}. \end{cases}$$

That is why it is reasonable to use symplectic methods (i.e. preserving the Hamiltonian H_{ab}) of integration of system (4). Also notice on fig.5 how conventional Runge-Kutta method (`ode45` solver in Matlab) fails to conserve the energy.

We are going to describe shortly the main definitions behind Euler-A and Lobatto methods (mainly from [Hairer et al. 02]).

We treat non-autonomous systems of first-order ordinary differential equations

$$\dot{y} = f(t,y), \quad y(t_0) = y_0.$$

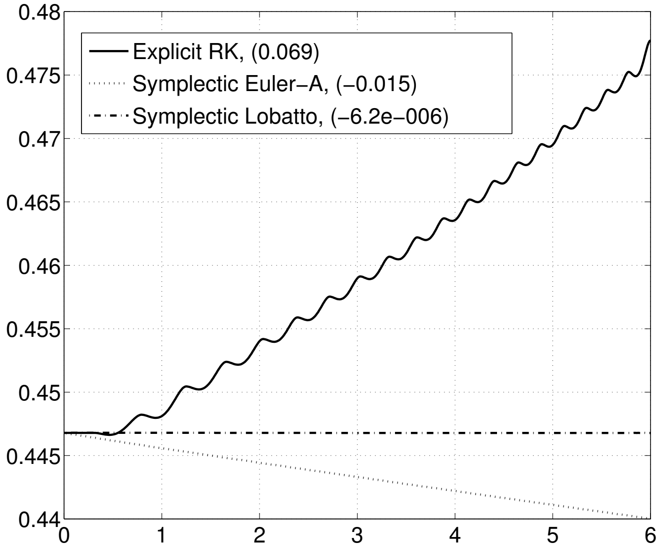

FIGURE 5. Evolution of conserved Hamiltonian H_{ab} in numerical simulations. Case of a 4-Teichon, $\Delta t = .01$, $(a^4_{k=1})_{t=0} = (1,-1,1,-1)$, $(b^4_{k=1})_{t=0} = (2\pi - 0.1, 0.1, \pi - 0.1, \pi + 0.1)$. Method used, relative change of H_{ab}.

Definition. *Let β_i, α_{ij} $(i, j = 1, \ldots, s)$ be real numbers, h is a constant step size and let $\sigma_i = \sum_{j=1}^{s} \alpha_{ij}$. An* **s-stage Runge-Kutta method** *is given by*

$$\begin{aligned} k_i &= f(t_0 + \sigma_i h, y_0 + h \textstyle\sum_{j=1}^{s} \alpha_{ij} k_j), \quad i = 1, \ldots, s \\ y_1 &= y_0 + h \textstyle\sum_{i=1}^{s} \beta_i k_i. \end{aligned} \tag{19}$$

Here we allow a full matrix (α_{ij}) of non-zero coefficients, thus making this an *implicit* integrator. In case when $\alpha_{ij} = 0$ for $i \le j$ the method becomes an *explicit* Runge-Kutta method.

It is customary to display coefficients in the following way:

$$\begin{array}{c|ccc} \sigma_1 & \alpha_{11} & \ldots & \alpha_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_s & \alpha_{s1} & \ldots & \alpha_{ss} \\ \hline & \beta_1 & \ldots & \beta_s \end{array}$$

Definition. *We consider differential equations in the partitioned form*

$$\dot{y} = f(y, z), \quad \dot{z} = g(y, z), \tag{20}$$

where y, z may be vectors of different dimensions.

The idea is to take two different Runge-Kutta methods, and to treat the y-variables with the first method (α_{ij}, β_i)*, and the z-variables with the second method* $(\hat{\alpha}_{ij}, \hat{\beta}_i)$*.*

Let α_{ij}, β_i *and* $\hat{\alpha}_{ij}, \hat{\beta}_i$ *be the coefficients of two Runge-Kutta methods. A* **partitioned Runge-Kutta method** *for the solution of (20) is given by*

$$\begin{aligned} k_i &= f\left(y_0 + h\sum_{j=1}^{s}\alpha_{ij}k_j, z_0 + h\sum_{j=1}^{s}\hat{\alpha}_{ij}l_j\right), \\ l_i &= g\left(y_0 + h\sum_{j=1}^{s}\alpha_{ij}k_j, z_0 + h\sum_{j=1}^{s}\hat{\alpha}_{ij}l_j\right), \\ y_1 &= y_0 + h\sum_{i=1}^{s}\beta_i k_i, \quad z_1 = z_0 + h\sum_{i=1}^{s}\hat{\beta}_i l_i. \end{aligned} \tag{21}$$

A particular example of this method is a *symplectic Euler-A (or Euler-B)* method:

$$\begin{cases} y_{n+1} = y_n + hf(y_n, z_{n+1}) \\ z_{n+1} = z_n + hg(y_n, z_{n+1}), \end{cases} \text{or} \quad \begin{cases} y_{n+1} = y_n + hf(y_{n+1}, z_n) \\ z_{n+1} = z_n + hg(y_{n+1}, z_n). \end{cases}$$

Here implicit Euler method with $\beta_1 = 1, \alpha_{11} = 1$ is combined with the explicit Euler method with $\hat{\beta}_1 = 1, \hat{\alpha}_{11} = 0$ (or vice versa).

Another example of the 3-stage partitioned RK method is Lobatto IIIA-B method. The coefficients of the method are given in Table 1

TABLE 1. Coefficients of the 3-stage Lobatto IIIA-B pair

0	0	0	0
1/2	5/24	1/3	−1/24
1	1/6	2/3	1/6
	1/6	2/3	1/6

0	1/6	−1/6	0
1/2	1/6	1/3	0
1	1/6	5/6	0
	1/6	2/3	1/6

In the figure 5 you can see the performance of three algorithms in preserving the Hamiltonian H_{ab}. Clearly, Lobatto IIIA-B method does the best job. This is due to the fact that Lobatto IIIA-B is a fourth order method, while Euler-A is just a first order method. In table 2 the comparison data is provided.

TABLE 2. Comparison of numerical methods used

Method	Explicit RK	Euler-A			Lobatto IIIA-B
Δt	10^{-2}	10^{-2}	10^{-3}	10^{-4}	10^{-2}
Time spent	.3 sec	2 sec	22 sec	734 sec	13 sec
Change in H_{ab}	7.0e10^{-2}	-1.5e10^{-2}	-1.5e10^{-3}	-1.5e10^{-4}	-6.0e10^{-6}

6.2. **Numerical experiments with N-Teichons.** The conjecture is that we can approximate any shape with a relatively small number of Teichons (around 20). In other words setting an initial momentum on the circle as a sum of 20 delta functions $a_i\delta(\theta - b_i)$ scattered at positions b_i on the circle and satisfying conditions (5). Then solve the EPDiff forward in time until time $T = 1$ (using Lobatto IIIA-B scheme) and see what kind of shapes are appearing. One of the ideas is to learn the 'syntax' of Teichon interaction: how they repel or attract each other and what kind of shapes are produced as a result of these interactions. From the experiments we can say a few certain rules of this 'syntax'.

To simplify the discussion we can say that shapes basically have two sets of features: some number of extremities (or limbs) and some number of concavities (or dents). We are going to state the conjecture on the formation of these two features and provide figures of randomly generated shapes to support it.

Extremities are obtained via 'pinching', i.e. two (or more) close Teichons are attracted towards each other. The faster they are heading (large values of a_j's) the pointier is the extremity.

Concavities, or the dents, are obtained via 'ripping': two (or more) close Teichons running away from each other. The faster they are running the more concave that part of the shape will be.

Parts of the shape with no Teichons, or with Teichons moving slowly (small values of a_j) remain circular.

Let us see how these rules apply to a Donald-Duck-like shape on fig.6 (evolution of a_k, b_k is shown on fig.7). It is comprised of the pointier end formed by 'large' Teichons $1, 2, 3$ and the dull end formed by solitons $4, 5, 6, 7$. The sharper end is formed by 'fast running' solitons $1, 2, 3$. The 'ripping' by Teichons $1, 8$ is somewhat bigger than the 'ripping' done by solitons $3, 4$, since Teichons $1, 8$ run faster away from each other than $3, 4$. The upper part of the Donald's head remains circular due to the absence of any solitons.

One can see how these general rules hold true most of the time on figs.8,9,10. Here there are five more figures with random shapes generated by 16-Teichons. To the right from the shape you can see the 1-Teichon number j and the corresponding value of a_j at the initial time.

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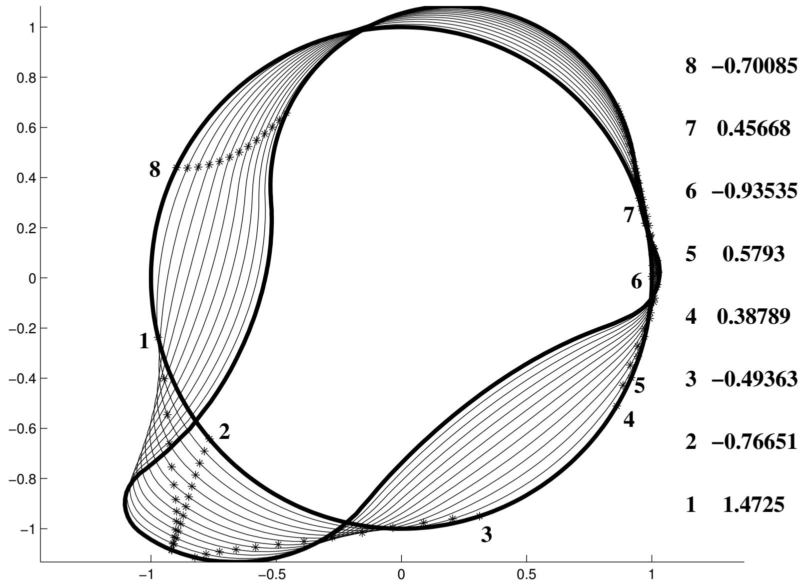

FIGURE 6. Evolution of an 8-Teichon from the circle to a Donald-Duck-like shape. Positions of individual 1-Teichons are marked by asterisks. The initial values of $(a_k)_{k=1}^{8}$ are given on the right.

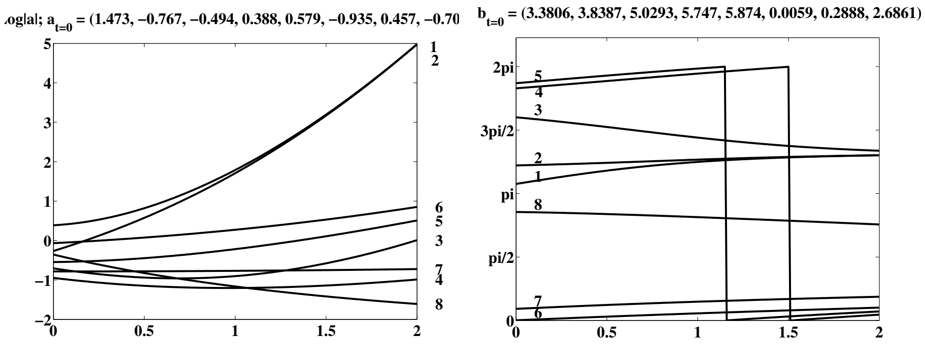

FIGURE 7. Time evolution of $\log|a_k|$'s (left) and b_k's (right) for the solitons representing Donald-Duck-like shape (fig.6)

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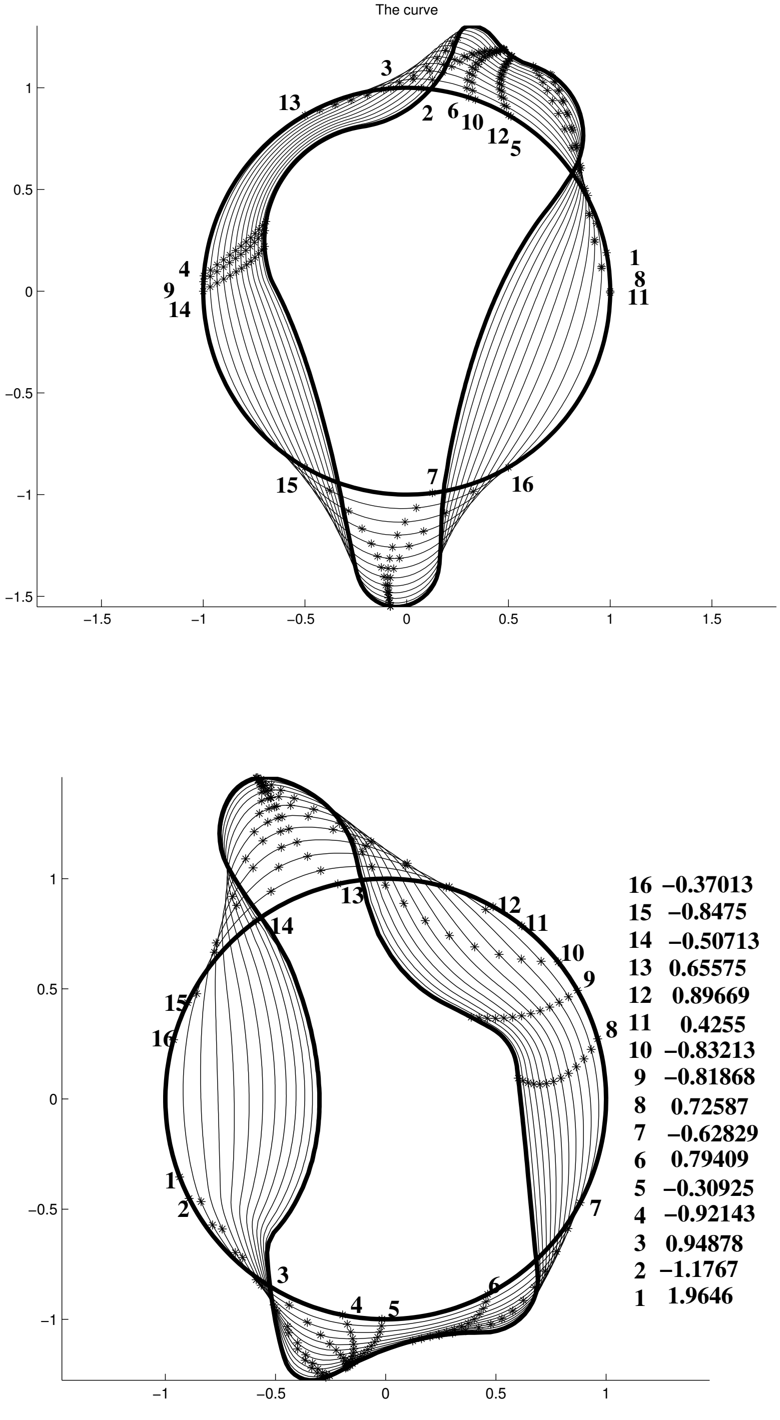
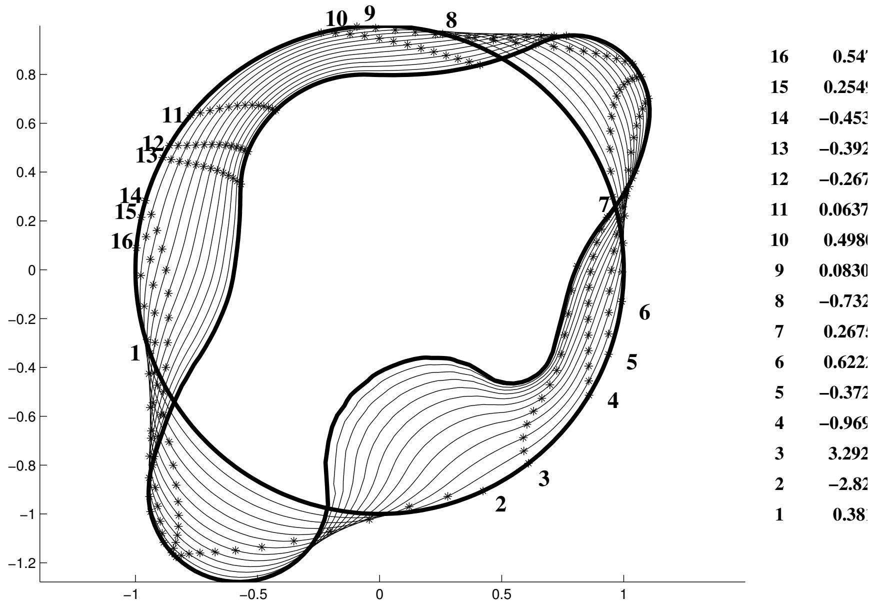


16	−0.37013
15	−0.8475
14	−0.50713
13	0.65575
12	0.89669
11	0.4255
10	−0.83213
9	−0.81868
8	0.72587
7	−0.62829
6	0.79409
5	−0.30925
4	−0.92143
3	0.94878
2	−1.1767
1	1.9646

FIGURE 8. Evolution of a 16-Teichon from the circle to the bird-like shape (top) and to the bottle-like shape (bottom). Positions of individual 1-Teichons are marked by asterisks. Initial values of $(a_k)_{k=1}^{16}$ are given on the right. (Data for the bird-like shape is not available.)



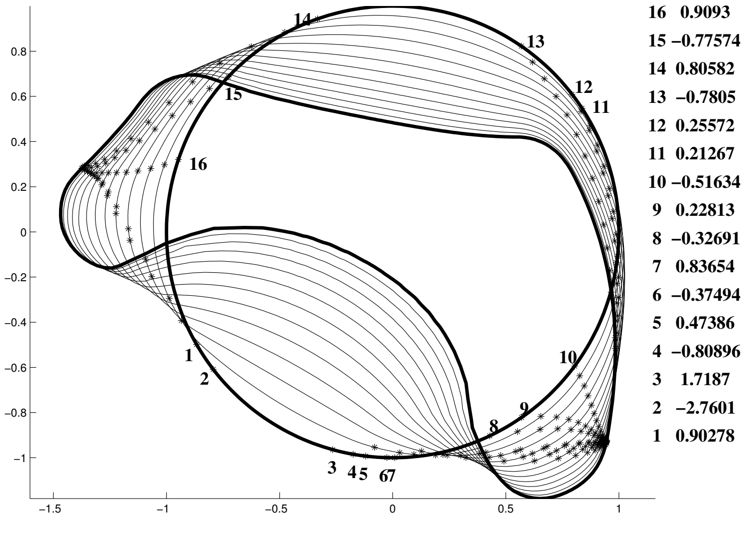


Figure 9. Evolution of a 16-Teichon from the circle to the amoeba-like shape (top) and the boomerang-like shape (bottom), represented by a 16-Teichon. Positions of individual 1-Teichons are marked by asterisks. Initial values of $(a_k)_{k=1}^{16}$ are given on the right.

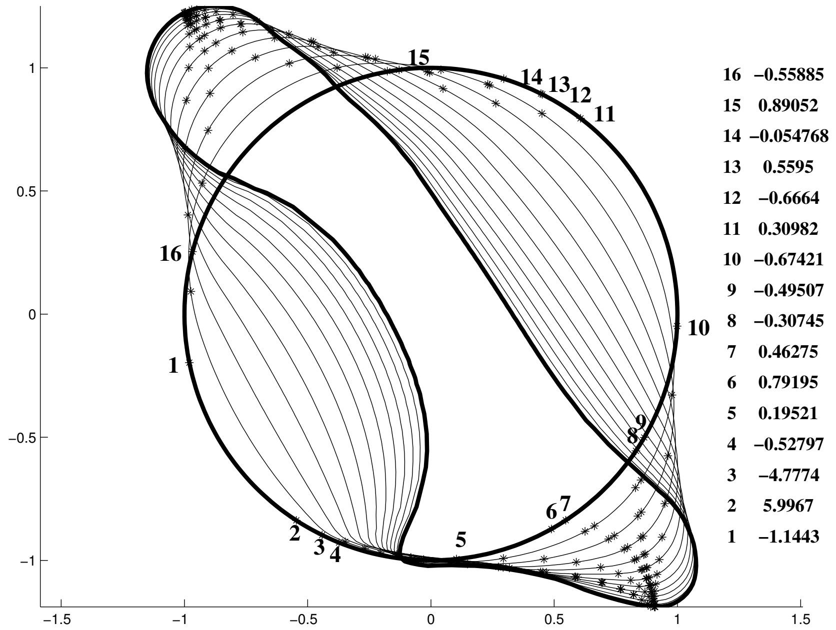


FIGURE 10. Evolution of a 16-Teichon from the circle to the sock-like shape, represented by a 16-Teichon. Positions of individual 1-Teichons are marked by asterisks. Initial values of $(a_k)_{k=1}^{16}$ are given on the right.