Review Problems.

1. Assume $u = \Theta(\theta) R(r) \Rightarrow R'' \Theta + \frac{1}{r} R' \Theta + \frac{1}{r^2} R \Theta'' = 0$

$\Rightarrow (R'' + \frac{1}{r} R') \Theta + \frac{1}{r^2} R \Theta'' = 0 \Rightarrow (R'' + \frac{1}{r} R') \Theta = -\frac{1}{r^2} R \Theta''$

$\Rightarrow \dfrac{r^2 R'' + r R'}{R} = -\dfrac{\Theta''}{\Theta} = \lambda$

we have 2 odes.

$\Theta'' + \lambda \Theta = 0 \qquad r^2 R'' + r R' - \lambda R = 0$

and. $\Theta(\theta + 2\pi) = \Theta(\theta) \qquad R$ is bounded.

Consider $\Theta'' + \lambda \Theta = 0$ and $\Theta(\theta + 2\pi) = \Theta(\theta)$

① $\lambda < 0$. $\lambda = -\mu^2 \quad \Theta(\theta) = c_1 e^{\mu\theta} + c_2 e^{-\mu\theta}$

periodic $\Rightarrow c_1 = c_2 = 0 \Rightarrow$ trival

② $\lambda = 0 \quad \Theta = c_1 + c_2\theta$, periodic $\Rightarrow c_2 = 0$

$\Rightarrow \Theta = C$

$r^2 R'' + r R' = 0 \quad \Rightarrow R = k_1 + k_2 \ln r$. $r \to 0$. $\ln r \to -\infty \Rightarrow k_2 = 0$

So $R = k_1$

$\Rightarrow$ for $\lambda = 1$. $u = \Theta R = c \cdot k =$ constant is a solution.

③ $\lambda > 0$. $\lambda = \mu^2 \Rightarrow \Theta(\theta) = c_1 \cos\mu\theta + c_2 \sin\mu\theta$

periodic with period $2\pi \Rightarrow n \dfrac{2\pi}{\mu} = 2\pi \Rightarrow \mu = n$

$r^2 R'' + r R' - n^2 R = 0 \Rightarrow R(r) = c_1 r^{-n} + c_2 r^{n}$

as $r \to 0$, $r^{-n} \to +\infty \quad \Rightarrow c_1 = 0$

so $R = r^n$

So, $r^n \cos n\theta$. $r^n \sin n\theta$ are solution of p.d.e with boundry condition

$\Rightarrow u(r,\theta) = \dfrac{c_0}{2} + \sum_{n=1}^{\infty} r^n (c_n \cos n\theta + b_n \sin n\theta)$

$\sin\theta = u(1,\theta) = \dfrac{c_0}{2} + \sum_{n=1}^{\infty} (c_n \cos n\theta + b_n \sin n\theta)$

$\Rightarrow$ all $c_i = 0$. $i = 0, 1, 2 \cdots$

$b_1 = 1$. $\quad b_2, b_3 \cdots = 0$

So. $u(r,\theta) = r \sin\theta$.

2. Assume $u = X(x)Y(y) \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y}$.

$u(x,0) \Rightarrow X(x)Y(0) = 0 \Rightarrow Y(0) = 0$

$u(0,y) = 0 \Rightarrow X(0) = 0$, $\quad u(1,y) = 0 \Rightarrow X(1) = 0$

so X has 2 boundary conditions $\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$

$\Rightarrow \begin{cases} X'' + \lambda X = 0 \\ X(0) = X(1) = 0 \end{cases}$ and $Y'' - \lambda Y = 0$.

$\Rightarrow \lambda_n = \left(\frac{n\pi}{1}\right)^2 = n^2\pi^2$. $X_n = \sin(n\pi x)$

$Y'' - n^2\pi^2 Y = 0 \Rightarrow Y_n = C_1 \cosh(n\pi y) + C_2 \sinh(n\pi y)$

$Y_n(0) = 0 \Rightarrow C_1 = 0$

$\Rightarrow Y_n = \sinh(n\pi y)$

so $U_n(x,y) = \sin(n\pi x)\sinh(n\pi y)$

$\Rightarrow u(x,y) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)\sinh(n\pi y)$

$u_y(x,y) = \sum_{n=1}^{\infty} C_n n\pi \sin(n\pi x)\cosh(n\pi y)$

$u_y(x,1) = \sum_{n=1}^{\infty} C_n n\pi \sin(n\pi x)\cosh(n\pi) = \sin(2\pi x)$

$\Rightarrow n = 2$. $\quad C_2 \cdot 2\pi \cosh(2\pi) = 1$

$C_2 = \frac{1}{2\pi\cosh(2\pi)}$

all other $C_n = 0$

$\Rightarrow u(x,y) = \frac{1}{2\pi\cosh(2\pi)} \sin(2\pi x)\sinh(2\pi y)$.

III Let $u = X(x)T(t)$, $XT'' - 4X''T = 0$

$$\Rightarrow \frac{X''}{X} = \frac{1}{4}\frac{T''}{T} = -\lambda$$

$$\Rightarrow X'' + \lambda X = 0, \quad T'' + 4\lambda T = 0$$

$u_x(0,t) = 0 \Rightarrow X'(0)T(t) = 0 \Rightarrow X'(0) = 0$

$u_x(2,t) = 0 \Rightarrow X'(2) = 0$

$u_t(x,0) \overset{=0}{\Rightarrow} T'(0) = 0$

for $X'' + \lambda X = 0$, $X'(0) = X'(2) = 0$, $\lambda_n = (\frac{n\pi}{2})^2$

we know $n = 0, 1, 2, \dots$, $X_n = \cos(\frac{n\pi}{2}x)$.

(you need to do this eigenvalue/function problem by yourself in the exam)

$\Rightarrow T'' + 4(\frac{n\pi}{2})^2 T = 0 \Rightarrow T'' + n^2\pi^2 T = 0$

$\cancel{T = T(0)e^{\dots}}$ $\cancel{T_n = T_n(0)e^{-n^2\pi^2 t}}$ $\cancel{T_n^0 = c_1\cosh(n\pi t) + c_2\sinh(n\pi t)}$

$\cancel{T_n' = T_n(0)(-n^2\pi^2)e^{-n^2\pi^2 t}}$

$\cancel{T_n'(0) = 0 \Rightarrow T_n(0)}$ $\cancel{T_n' = \dots}$

$T_n = c_1\cosh(n\pi t) + c_2\sinh(n\pi t)$

$T_n' = c_1 n\pi\sinh(n\pi t) + c_2 n\pi\cosh(n\pi t)$

$T_n'(0) = 0 = c_2 \cdot n\pi \cdot 1 \Rightarrow c_2 = 0$

$\Rightarrow T_n = c_1\cosh(n\pi t)$

$$\Rightarrow u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n\cosh(n\pi t)\cos(\frac{n\pi}{2}x)$$

$$u(x,0) = f \Rightarrow f = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n\cos(\frac{n\pi}{2}x).$$

$$\Rightarrow c_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 f(x)\,dx$$

$$c_n = \int_0^2 f(x)\cos(\frac{n\pi}{2}x)\,dx.$$

IV. $u = X(x)T(t)$.

$$XT' = 4X''T \Rightarrow \frac{X''}{X} = \frac{1}{4}\frac{T'}{T} = -\lambda$$

$u_x(0,t) = 0 \Rightarrow X'(0) = 0$.

$u(2,t) = 0 \Rightarrow X(2) = 0$

$\Rightarrow X'' + \lambda X = 0. \quad X'(0) = X(2) = 0.$

① $\lambda < 0$, $\lambda = -\mu^2$. $X = c_1\cosh(\mu x) + c_2\sinh(\mu x)$, $X' = c_1\mu\sinh(\mu x) + c_2\mu\cosh(\mu x)$

$X'(0) = 0 \Rightarrow c_2 = 0 \Rightarrow X = c_1\cosh\mu x$. $X(2) = 0 \Rightarrow c_1\cosh 2\mu = 0$

$\Rightarrow c_1 = 0 \Rightarrow$ trival.

② $\lambda = 0$. $X = c_1 + c_2 x$. $X'(0) = c_2$. $X'(0) = 0 \Rightarrow c_2 = 0$.

$X(2) = 0 \Rightarrow c_1 = 0$

So trival.

③ $\lambda > 0$. $\lambda = \mu^2$. $X = c_1\cos(\mu x) + c_2\sin(\mu x)$

$X' = -c_1\mu\sin(\mu x) + c_2\mu\cos(\mu x)$

$0 = X'(0) = c_2\mu \Rightarrow c_2 = 0$.

$\Rightarrow X = c_1\cos(\mu x)$, $0 = X(2) = c_1\cos(2\mu) = 0$.

So $2\mu = (n - \frac{1}{2})\pi$, $n = 1, 2, \ldots$

$\mu = (\frac{n}{2} - \frac{1}{4})\pi$. $\lambda_n = \left(\frac{2n-1}{4}\right)^2\pi^2$. $X_n = \cos\left(\frac{2n-1}{4}\pi x\right)$.

$T' + 4\lambda T = 0 \Rightarrow T_n = T_n(0)\, e^{-4\left(\frac{2n-1}{4}\right)^2\pi^2 t}$.

So $u = \sum_{n=1}^{\infty} T_n(0)\, e^{-\frac{(2n-1)^2}{4}\pi^2 t}\cos\left(\frac{2n-1}{4}\pi x\right)$.

$u(x,0) = f(x) \Rightarrow f(x) = \sum_{n=1}^{\infty} T_n(0)\cos\left(\frac{2n-1}{4}\pi x\right)$.

multiply $\cos\left(\frac{2n-1}{4}\pi x\right)$ both sides, integrate from -4 to 4

$$\int_{-4}^{4} f(x)\cos\left(\frac{2n-1}{4}\pi x\right)dx = T_n(0)\int_{-4}^{4}\cos^2\left(\frac{2n-1}{4}\pi x\right)dx = 4T_n(0)$$

$\Rightarrow T_n(0) = \frac{1}{4}\int_{-4}^{4} f(x)\cos\left(\frac{2n-1}{4}\pi x\right)dx$. here we extend $f(x)$ evenly to $(0,-4)$, and o extension $f(x)$ to $(2,4)$.

4. $x' = 3 - t + 4x.$ $\quad f(t,x) = 3 - t + 4x$

Suppose the time step size is h.

Forward Euler: $x^{n+1} = x^n + h(3 - t^n + 4x^n)$

Improved Euler:
$$\begin{cases} f^n = 3 - t^n + 4x^n \\ \cancel{x^{n+1} = x^n + \frac{1}{2} f_n h + \frac{1}{2}(3 - t - h + 4x^n} \\ x^{n+1} = x^n + \frac{h}{2}\cdot\left(f_n + f(t+h,\, x^n + h f_n)\right). \end{cases}$$

Backward Euler. $\cancel{x^{n+1} = 3 - t^{n+1} + 4x^{n+1}}$

$$x^{n+1} = x^n + h\left(3 - t^{n+1} + 4x^{n+1}\right)$$
$$= x^n + h(3 - t^{n+1}) + 4h\, x^{n+1}$$
$$\Rightarrow (1-4h)\, x^{n+1} = x^n + h(3 - t^{n+1})$$
$$x^{n+1} = \frac{1}{1-4h}\left[x^n + h(3 - t^{n+1})\right]$$

VI: 1. $x' = 14x - 2x^2 - xy$, $y' = 16y - 2y^2 - xy$

$= x(14 - 2x - y)$ $= y(16 - 2y - x)$

$x = 0$, $14 - 2x - y = 0$ or $y = 0$. $16 - 2y - x = 0$

$(0,0)$ $(0,8)$ $(7,0)$ $(4,6)$

$$J = \begin{pmatrix} 14 - 4x - y & -x \\ -y & 16 - 4y - x \end{pmatrix}$$

$\vec{X_0} = (0,0)$ $\frac{d\vec{x}}{dt} = \begin{pmatrix} 14 & 0 \\ 0 & 16 \end{pmatrix} \vec{x}$. $\lambda = 14$. $\lambda = 16$.

unstable, hyperbolic

$\vec{X_1} = (0,8)$ $\frac{d\vec{x}}{dt} = \begin{pmatrix} 6 & 0 \\ -8 & -16 \end{pmatrix} (\vec{x} - \vec{x_1})$. $\lambda_1 = 6$. $\lambda_2 = -16$.

unstable. saddle point

$\vec{X_2} = (7,0)$ $\frac{d\vec{x}}{dt} = \begin{pmatrix} -14 & -7 \\ 0 & 9 \end{pmatrix}$ $\lambda_1 = -14$. $\lambda_2 = 9$.

unstable, saddle point

$\vec{X_3} = (4,6)$ $\frac{d\vec{x}}{dt} = \begin{pmatrix} -8 & -4 \\ -6 & -12 \end{pmatrix}$. $\lambda = -10 + \sqrt{28}$. $\lambda = -10 - \sqrt{28}$

< 0 < 0

unstable, hyperbolic

draw the phase portrait by yourself.

2. $x' = x + \sin y$, $y' = 2y$

$\Rightarrow y = 0$, $x = 0$. only critical pt.

$J = \begin{bmatrix} -1 & \cos y \\ 0 & 2 \end{bmatrix}$ $A\Big|_{(0,0)} = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$

$\frac{d\vec{x}}{dt} = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} \vec{x}$. $\lambda_1 = -1$. $\lambda_2 = 2$

saddle point. unstable!

3. $x' = x(3-y)$. $y' = y(x-2)$

$x=0,\ y=3$ $\Rightarrow$ ~~0~~ $(0,0)$. $(3,2)$. 2 critical points.

$x=2\ y=0$

$$J = \begin{pmatrix} 3-y & -x \\ y & x-2 \end{pmatrix}.$$

for $(0,0)$. $\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\vec{x}$. $\lambda_1 = 3$. $\lambda_2 = -2$

unstable. hyperbolic!

for $(2,3)$. $\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & -2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} x-2 \\ y-3 \end{pmatrix}$.

$\lambda^2 + 6 = 0$, $\lambda = \pm\sqrt{6}\, i$

undermined! no conclusion can be drawn from the information.

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4. $x'' + 4x - x^3 = 0$

$$\begin{cases} x' = y \\ y' = -4x + x^3 = x(-4+x^2) \end{cases}$$

$y=0$. $x=0$ or $x=\pm 2$.

so. $(0,0)$. $(2,0)$. $(-2,0)$

$$J = \begin{pmatrix} 0 & 1 \\ -4+3x^2 & 0 \end{pmatrix}$$

$(0,0)$. $\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\vec{x}$. $\lambda = \pm 2i$. undermined!

$(2,0)$ $\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} x-2 \\ y \end{pmatrix}$ $\lambda^2 - 8 = 0$ $\lambda = \pm 2\sqrt{2}$. unstable saddle point

$(-2,0)$ $\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} x+2 \\ y \end{pmatrix}$ the same.