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FLUX RECOVERY AND A POSTERIORI ERROR ESTIMATORS: CONFORMING ELEMENTS FOR SCALAR ELLIPTIC EQUATIONS*

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Abstract. In this paper, we first study two flux recovery procedures for the conforming finite element approximation to general second-order elliptic partial differential equations. One is accurate in a weighted L^2 norm studied in [Z. Cai and S. Zhang, *SIAM J. Numer. Anal.*, 47 (2009), pp. 2132–2156] for linear elements, and the other is accurate in a weighted $H(\mathrm{div})$ norm, up to the accuracy of the current finite element approximation. For the L^2 recovered flux, we introduce and analyze an a posteriori error estimator that is more accurate than the explicit residual-based estimator. Based on the $H(\mathrm{div})$ recovered flux, we introduce two a posteriori error estimators. One estimator may be regarded as an extension of the recovery-based estimator studied in [Z. Cai and S. Zhang, *SIAM J. Numer. Anal.*, 47 (2009), pp. 2132–2156] to higher-order conforming elements. The global reliability and the local efficiency bounds for this estimator are established provided that the underlying problem is neither convection- nor reaction-dominant. The other is proved to be exact locally and globally on any given mesh with no regularity assumptions with respect to a norm depending on the underlying problem. Numerical results on test problems for these estimators are also presented.

Key words. a posteriori error estimator, flux recovery

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1. Introduction.

Since Babǔska's pioneering work [3] in 1976, the *a posteriori* error estimation and adaptive methods have been extensively studied. Impressive progress has been made during the past three decades, and there is now a vast literature in this research area. For references up to 2003 and historic remarks, for example, see the survey articles of Eriksson et al. [24], Bank [6], Becker and Rannacher [10], the books of Verfürth [37], Ainsworth and Oden [1], Babuška and Strouboulis [4], Bangerth and Rannacher [5], and the references therein.

Existing error estimators can be categorized as three classes: the residual, the gradient recovery, and the hierarchical bases. Obviously, the residual is the only quantity directly related to the true error and, hence, a natural means for developing estimators. There are three kinds of residual-based estimators: explicit, implicit, and equilibrated. For simple model problems, the energy norm of the true error is equal to the dual norm of the residual (element residuals and jumps on interior edges). Unfortunately, the dual norm is not computationally feasible. So the explicit residual estimators are basically *estimations* of the dual norm of the residual and are not accurate for error control in general. For details on implicit and equilibrated residual methods and bibliographical remarks, see the book by Ainsworth and Oden [1].

Recently, for simple model problems and linear elements, estimators with guaranteed reliability bounds are studied through the equilibrated residual method combining with the introduction of a dual mesh [29] or the method of hypercircle. Estimators resulting from the method of hypercircle had been studied by Ladeveze and D. Leguillon [28], Vejchodsky [36], Braess and Schöberl [14], Verfürth [39], etc. More recently,

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it is further extended to higher-order elements for Poisson equations in [13]. Estimators of this type are based on the so-called Prager–Synge identity, which holds for only positive definite problems, and requires recovery of the flux in the $H(\mathrm{div})$ conforming finite element spaces satisfying the equilibrium equation exactly. Estimators developed in [28, 36, 14] differ in local recovery procedures. For applications of this type of methods to the reaction-dominant diffusion and the interface problems, see [39] and [21], respectively.

The existing recovery-based estimators (the Zienkiewicz–Zhu (ZZ) estimator and its variations) are simply the L^2 norm of the difference between the direct and post-processed approximations of the gradient/flux. The recovered gradient/flux is a projection of the direct approximation onto vector-valued continuous finite element space with respect to either a discrete or L^2 inner product. When the underlying problem is smooth and the finite element approximation has a superconvergence property, this type of estimator is *asymptotically* exact. This property guarantees accurate error control for sufficiently small mesh size. However, for nonsmooth problems, in particular, those with discontinuous gradient/flux, it is well known [7, 31, 19] that these estimators are not efficient on relatively coarse meshes. That is, they might overrefine regions where there are no errors and, hence, they fail to reduce the global error. One could overcome this difficulty by applying the method on each subdomain separately. For reasons why this local approach is not favorable, see detailed discussions in [31]. By simply projecting the direct approximation of the flux onto conforming finite element spaces of $H(\mathrm{div})$, we developed a global approach in [19] for the conforming linear finite element approximation to the interface problem. It was shown in [19] that this global approach is robust with respect to the diffusion coefficients. The approach was further extended to the mixed and nonconforming elements in [20]. Other drawbacks of the recovery-based estimators include the limitation to linear elements and the unreliability on coarse meshes (see [1] for a one-dimensional example). For recent development on higher-order finite element approximations, see [9, 8, 41].

The purpose of this paper is twofold: (1) constructing accurate approximations to the flux based on the current Galerkin finite element approximation, and (2) using the recovered flux to design a posteriori error estimators that overcome the drawbacks of existing estimators mentioned above. Given the Galerkin finite element approximation, we consider two flux recovery procedures with recovered fluxes in $H(\mathrm{div})$ conforming finite element spaces such as Raviart–Thomas and Brezzi–Douglas–Martini elements [16]. The reason for using these finite element spaces is to accommodate possible discontinuities of the flux and, hence, to eliminate possible overrefinements on regions where there are no errors. The first one is simply a weighted L^2 projection of the direct approximation of the flux, which was studied in [19] for linear elements. For higher-order elements, we show that this L^2 recovered flux is again accurate in the weighted L^2 norm up to the accuracy of the finite element approximation in the energy norm. Essentially, this L^2 recovery procedure guarantees that the recovered flux approximately satisfies the constitutive equation. To recover a more accurate flux, we introduce a new procedure that approximately satisfies both the constitutive and the equilibrium equations. We show that this procedure, referred to as $H(\mathrm{div})$ recovery, is accurate in a weighted $H(\mathrm{div})$ norm. It is important to point out that the $H(\mathrm{div})$ recovery always results in a linear, symmetric, and positive definite problem that can be solved very efficiently by fast multigrid iterative methods, even if the underlying problem is highly nonlinear or convection-dominant.

With the L^2 recovered flux, we introduce and analyze an a posteriori error estimator that is the recovery-based estimator plus weighted element residuals. This estimator is comparable to the explicit residual-based estimator (see, e.g., [1, 37]), but it is more accurate than the latter. It is important to point out that the element residual is necessary for higher-order elements and guarantees the reliability on coarse meshes. We analyze this estimator by establishing a global reliability bound, provided that the underlying problem is coercive in the energy norm, and a local efficiency bound. A similar idea, adding two additional terms to the ZZ estimator, was studied by Fierro and Veeser [25] and the resulting estimator is rather sophisticated.

With the $H(\mathrm{div})$ recovered flux, we introduce and analyze two a posteriori error estimators. The first one is the recovery-based estimator, defined as the weighted L^2 norm of the difference between recovered and direct fluxes. This estimator may be regarded as an extension of the estimator developed in [19] for linear elements to higher-order elements. Based on the discussion in section 6.2, the new recovery procedure is necessary for guaranteeing reliability. Under the assumption that the underlying problem is neither convection- nor reaction-dominant, we prove global reliability and the local efficiency without regularity assumptions for higher-order finite element approximations. Finally, it is also important to point out that straightforward extensions of existing recovery-based estimators to higher-order elements usually fail, and developing a viable estimator is nontrivial. For example, Bank, Xu, and Zheng in [8] recently studied the recovery-based estimator for Lagrange triangular elements of degree p, and their scheme requires recovery of all partial derivatives of pth order instead of the gradient with only 2 (or 3) partial derivatives of first-order in two (or three) dimensions.

The second estimator is defined by adding the L^2 norm of the element residuals to the recovery-based estimator. Apparently, the element residual is natural for designing a posteriori error estimators and is inexpensive to compute. More importantly, it is essential for guaranteeing the reliability on coarse meshes. By using the L^2 norm on the element residual, we are able to show that this estimator is exact locally and globally on any given mesh, including an arbitrary initial mesh, with no regularity assumptions. Exactness on any given mesh implies that the estimator is ideal for error control (or so-called solution verification) on coarse meshes. Error control on coarse meshes is of paramount importance for simulating physical phenomena in engineering applications and scientific predictions with limited computer resources. No regularity assumptions in this paper means that only assumptions on the existence of the underlying problem are required. This is weaker than those required for approximation theory and much weaker than those required by the current theory of the recovery-based estimators. Therefore, the estimators can be applied to problems of practical interest, such as interface singularities, discontinuities in the form of shock-like fronts, and of interior or boundary layers.

The paper is organized as follows. Elliptic problems and their finite element approximation are described in sections 2 and 3, respectively. Section 4 introduces a flux recovery procedure. Two a posteriori error estimators are defined in section 5 and analyzed in section 6. Finally, numerical experiments for some test problems are presented in section 7.

1.1. Notation. We use standard notations and definitions for the Sobolev spaces $H^s(\Omega)^d$ and $H^s(\partial\Omega)^d$ for $s \geq 0$. The standard associated inner products are denoted by $(\cdot,\, \cdot)_{s,\Omega}$ and $(\cdot,\, \cdot)_{s,\partial\Omega}$, and their respective norms are denoted by $\|\cdot\|_{s,\Omega}$ and $\|\cdot\|_{s,\partial\Omega}$. (We suppress the superscript d because the dependence on dimension will be clear by context. We also omit the subscript Ω from the inner product and norm designation when there is no risk of confusion.) For $s=0$, $H^s(\Omega)^d$ coincides with $L^2(\Omega)^d$. In this case, the inner product and norm are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. Set

$$H^1_D(\Omega) := \{v \in H^1(\Omega) \,:\, v = 0 \text{ on } \Gamma_D\}.$$

We denote the duals of $H^1_D(\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$ by $H^{-1}_D(\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$ with norms defined by

$$\|\phi\|_{-1,\,D} = \sup_{0\neq\psi\in H^1_D(\Omega)} \frac{(\phi,\,\psi)}{\|\psi\|_1} \quad \text{and} \quad \|\phi\|_{-1/2,\,\partial\Omega} = \sup_{0\neq\psi\in H^{\frac{1}{2}}(\partial\Omega)} \frac{(\phi,\,\psi)}{\|\psi\|_{1/2,\partial\Omega}}.$$

When $\Gamma = \partial\Omega$, denote $H^1_D(\Omega)$ by $H^1_0(\Omega)$. Finally, set

$$H(\mathrm{div};\Omega) = \{\boldsymbol{\tau} \in L^2(\Omega)^d \,:\, \nabla\cdot\boldsymbol{\tau} \in L^2(\Omega)\},$$

which is a Hilbert space under the norm

$$\|\boldsymbol{\tau}\|_{H(\mathrm{div};\,\Omega)} = \left(\|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\nabla\cdot\boldsymbol{\tau}\|^2_{0,\Omega}\right)^{\frac{1}{2}},$$

and define the subspace

$$H_N(\mathrm{div};\Omega) = \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) \,:\, \mathbf{n}\cdot\boldsymbol{\tau} = 0 \text{ on } \Gamma_N\}.$$

2. Elliptic problem. Let Ω be a bounded, open, connected subset of $\Re^d$ ($d=2$ or 3) with a Lipschitz continuous boundary $\partial\Omega$. Denote $\mathbf{n} = (n_1,\ldots,n_d)$, the outward unit vector normal to the boundary. We partition the boundary of the domain Ω into two open subsets Γ_D and Γ_N such that $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. For simplicity, we assume that Γ_D is not empty (i.e., $\mathrm{mes}\,(\Gamma_D) \neq 0$). (Otherwise, solutions of the partial differential equations considered in this paper are unique up to an additive constant.)

Consider the second-order elliptic boundary value problem

$$-\nabla\cdot\,(A\nabla\, u) + \mathbf{b}\cdot\nabla u + b_0 u = f \quad \text{in } \Omega \tag{2.1}$$

with boundary conditions

$$u = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n}\cdot A\nabla\, u = 0 \quad \text{on } \Gamma_N, \tag{2.2}$$

where the symbols $\nabla\cdot$ and ∇ stand for the divergence and gradient operators, respectively; and $A = (a_{ij})_{d\times d}$, $\mathbf{b} = (b_i)_{d\times 1}$, and b_0 and f are given matrix-, vector-, and scalar-valued functions, respectively. Assume that the diffusion tensor A is uniformly symmetric positive definite: there exist positive constants $0 < \Lambda_0 \leq \Lambda_1$ such that

$$\Lambda_0 \boldsymbol{\xi}^T\boldsymbol{\xi} \leq \boldsymbol{\xi}^T A\boldsymbol{\xi} \leq \Lambda_1 \boldsymbol{\xi}^T\boldsymbol{\xi} \tag{2.3}$$

for all $\boldsymbol{\xi} \in \Re^d$ and almost all $x \in \bar{\Omega}$. We assume homogeneous boundary conditions for simplicity and assume that a_{ij} and b_i are in $L^\infty(\Omega)$.

For any $v \in H^1(\Omega)$, denote

$$|\!|\!| v |\!|\!|_\Omega = \left(\|v\|^2_{0,\Omega} + \|A^{1/2}\nabla v\|^2_{0,\Omega}\right)^{1/2},$$

the H^1 norm weighted by the diffusion tensor. Let

$$X\,v = \mathbf{b}\cdot\nabla v + b_0 v.$$

It is known that

$$\|X\,v\|_{0,\Omega} \le C_x\,\|\!|v|\!\|_{\Omega} \quad \forall\, v\in H^1(\Omega), \tag{2.4}$$

where C_x apparently depends upon bounds of the coefficients A, $\mathbf{b}$, and b_0. Here and hereafter, we use C with or without subscripts to denote a generic positive constant, possibly different at different occurrences, that is independent of the mesh parameter h_K introduced in subsequent sections, but may depend upon the domain Ω.

Let $U = H^1_D(\Omega)$. The corresponding variational form of system (2.1) is to find $u\in U$ such that

$$a(u,\,v) = (f,\,v) \quad \forall\, v\in U, \tag{2.5}$$

where the bilinear form is defined by

$$a(u,\,v) = (A\nabla u,\,\nabla v) + (Xu,\,v).$$

Assume that (2.5) has a unique solution in U for any given $f\in H^{-1}(\Omega)$. It is then well known that problem (2.5) satisfies the following H^{1+r} regularity estimate:

$$\|u\|_{1+r} \le C\,\|f\|_{-1+r} \tag{2.6}$$

with $r>0$.

In the remainder of this section, we describe two special cases of the boundary value problem in (2.1)–(2.2).

Elliptic interface problem. Let $\{\Omega_i\}_{i=1}^n$ be a partition of the domain Ω with Ω_i being an open polygonal domain. Let $\alpha(x)$ be positive and piecewise constant on subdomains, $\{\Omega_i\}$, of Ω with possible large jumps across subdomain boundaries (interfaces):

$$\alpha(x) = \alpha_i > 0 \quad \text{in } \Omega_i$$

for $i=1,\dots,n$. The elliptic interface problem is the boundary value problem in (2.1–2.2) with

$$A = \alpha\, I \quad \text{and} \quad \mathbf{b} = \mathbf{0},$$

where $I = I_{d\times d}$ is the identity matrix; that is,

$$\left\{\begin{aligned} -\nabla\cdot(\alpha\nabla\, u) + b_0 u &= f \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \Gamma_D,\\ \mathbf{n}\cdot(\alpha\nabla\, u) &= 0 \quad \text{on } \Gamma_N. \end{aligned}\right. \tag{2.7}$$

The corresponding energy norm for this problem is

$$\|\!|v|\!\|_{\Omega} = \left(\|\alpha^{1/2}\nabla v\|^2_{0,\Omega} + \|b_0^{1/2} v\|^2_{0,\Omega}\right)^{1/2}.$$

It is well known that the smoothness of the solution depends upon the jumps of the diffusion coefficient α and that r could be very small.

Singularly perturbed reaction-diffusion equation. Let ε be a very small perturbation parameter $0 < \varepsilon \ll 1$. The singularly perturbed reaction-diffusion equation is the boundary value problem in (2.1)–(2.2) with

$$A = \varepsilon I, \quad \mathbf{b} = \mathbf{0}, \quad \text{and} \quad b_0 = 1;$$

that is,

$$\left\{\begin{aligned} -\varepsilon\,\Delta u + u &= f \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \Gamma_D,\\ \mathbf{n}\cdot(\varepsilon\nabla u) &= 0 \quad \text{on } \Gamma_N. \end{aligned}\right. \tag{2.8}$$

The corresponding energy norm for this problem is

$$|||v|||_\Omega = \left(\varepsilon\|\nabla v\|_{0,\Omega}^2 + \|v\|_{0,\Omega}^2\right)^{1/2}.$$

It is well known that the solution of the problem satisfies the regularity estimate

$$|||u|||_\Omega \le C\|f\|_{0,\Omega},$$

which follows easily from

$$|||v|||_\Omega^2 = (f,\, u) \le \|f\|_{0,\Omega}\|u\|_{0,\Omega} \le \frac{1}{2}\left(\|f\|_{0,\Omega}^2 + \|u\|_{0,\Omega}^2\right).$$

Moreover, if $u \in H^2(\Omega)$, then

$$\varepsilon\|u\|_{2,\Omega} + |||u|||_\Omega \le C\|f\|_{0,\Omega},$$

which is a direct consequence of the H^2 regularity estimate of the Poisson operator: $\|u\|_{2,\Omega} \le C\left(\|\Delta u\|_{0,\Omega} + \|u\|_{0,\Omega}\right)$, the first equation in (2.8), the triangle inequality, and the energy estimate above. These estimates indicate that the L^2, H^1, and H^2 norms of the solution have the respective scales: 1, $\varepsilon^{-1/2}$, and ε^{-1}.

3. Finite element approximation.

For simplicity of presentation, we consider only triangular and tetrahedra elements for the respective two and three dimensions. Extension to rectangular and standard isoparametric elements is straightforward. Assuming that the domain Ω is polygonal, let $\mathcal{T} = \{K\}$ be a finite element partition that is regular (see [15]); i.e., for all $K \in \mathcal{T}$, there exists a positive constant κ such that

$$h_K \le \kappa\,\rho_K,$$

where h_K denotes the diameter of the element K and ρ_K denotes the diameter of the largest circle that may be inscribed in K. Note that the assumption of regularity does not exclude highly, locally refined meshes.

Let $P_k(K)$ be the space of polynomials of degree k on element K. Denote the finite element space of order k associated with the triangulation $\mathcal{T}$ by

$$\mathcal{U}^k = \mathcal{U}^k(\mathcal{T}) = \{v \in U : v|_K \in P_k(K) \quad \forall\, K \in \mathcal{T}\} \subset U.$$

It has the following approximation property: if $k \ge 1$ is an integer and $l \in [1,\, k+1]$, then

$$\inf_{\varphi\in\mathcal{U}^k}\left(\|v-\varphi\|_{0,\Omega} + \|h\,(v-\varphi)\|_{1,\Omega}\right) \le C\,\|h^l\,v\|_{l,\Omega} \quad \forall\, v \in H^l(\Omega) \tag{3.1}$$

with the weighted norm defined by

$$\|h^l\, v\|_{m,\Omega} = \left(\sum_{K\in\mathcal{T}} h_K^{2l}\|v\|_{m,K}^2\right)^{1/2}.$$

The finite element approximation of (2.5) is then to find $u_{\mathcal{T}} \in \mathcal{U}^k$ such that

$$a(u_{\mathcal{T}},\, v) = (f,\, v) \quad \forall\, v \in \mathcal{U}^k. \tag{3.2}$$

To the best of our knowledge, there are no a priori error estimates for the Galerkin finite element approximation to the general elliptic equation without the assumption that the largest mesh size is sufficiently small. Below, we state the a priori error estimate for the singularly perturbed reaction-diffusion problem, which is straightforward from the standard error analysis. Based on the following theorem and the regularity estimates in the previous section, it is easy to see that the mesh size has to be small enough to guarantee the accuracy of the finite element approximation.

THEOREM 3.1. *For the singularly perturbed reaction-diffusion problem, assume that $u_{\mathcal{T}} \in \mathcal{U}^1$ is the piecewise linear finite element approximation defined in* (3.2), *then there exists a positive constant C such that*

$$|\!|\!| u - u_{\mathcal{T}} |\!|\!|_\Omega \le C\varepsilon^{-1/2}\left(\sum_{K\in\mathcal{T}} h_K^2\left(\varepsilon^2\,|u|_{2,K}^2 + \varepsilon\,|u|_{1,K}^2\right)\right)^{1/2}.$$

Proof. Let u_I be the interpolant of u in $\mathcal{U}^1$, then a standard argument shows that

$$|\!|\!| u - u_{\mathcal{T}} |\!|\!|_\Omega \le |\!|\!| u - u_I |\!|\!|_\Omega = \left(\sum_{K\in\mathcal{T}}\left(\varepsilon\|\nabla(u-u_I)\|_{0,K}^2 + \|u-u_I\|_{0,K}^2\right)\right)^{1/2},$$

which, together with the approximation property, implies the theorem. ∎

4. Flux recovery. The flux, $\boldsymbol{\sigma} = -A\nabla u$, is an important physical quantity, often the primary concern in practice. Hence, it is desirable to compute an accurate approximation to the flux based on the current finite element approximation, $u_{\mathcal{T}}$. In this section, we study two flux recovery procedures. One is referred to as the L^2 recovery studied in [19] for linear finite elements. The other is new and referred to as $H(\mathrm{div})$ recovery.

In both recovery procedures, the flux is approximated using $H(\mathrm{div})$ conforming finite elements. Of the several families of the $H(\mathrm{div};\Omega)$ conforming finite element spaces (see, e.g., [16]), we consider only Raviart–Thomas elements for simplicity and remark on Brezzi–Douglas–Marini elements. Denote the local Raviart–Thomas space of index k on element $K \in \mathcal{T}$ by

$$RT_k(K) = P_k(K)^d + \mathbf{x}\, P_k(K)$$

with $\mathbf{x} = (x_1, \ldots, x_d)$. Denoting

$$\Sigma = H_N(\mathrm{div};\Omega),$$

the standard $H(\mathrm{div};\,\Omega)$ conforming Raviart–Thomas space of index k is then defined by

$$\mathcal{V}^k = \mathcal{V}^k(\mathcal{T}) = \{\boldsymbol{\tau} \in \Sigma \,:\, \boldsymbol{\tau}|_K \in RT_k(K) \;\; \forall\, K \in \mathcal{T}\}. \tag{4.1}$$

It is well known (see [16]) that $\mathcal{V}^k$ has the following approximation property: let $k \geq 0$ be an integer, and let $l \in [1,\, k+1]$

$$\inf_{\boldsymbol{\tau} \in \mathcal{V}^k} \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_{H(\mathrm{div};\,\Omega)} \leq C \left(\|h^l\, \boldsymbol{\sigma}\|^2_{l,\Omega} + \|h^l\, \nabla \cdot \boldsymbol{\sigma}\|^2_{l,\Omega} \right)^{1/2} \tag{4.2}$$

for $\boldsymbol{\sigma} \in H^l(\Omega)^d \cap \Sigma$ with $\nabla \cdot \boldsymbol{\sigma} \in H^l(\Omega)$.

With the definition of the flux, the general second-order elliptic boundary value problem in (2.1–2.2) may be rewritten as the first-order system

$$\begin{cases} \boldsymbol{\sigma} + A\nabla u \quad = \mathbf{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{\sigma} + Xu = f & \text{in } \Omega, \end{cases} \tag{4.3}$$

with boundary conditions

$$u = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot \boldsymbol{\sigma} = 0 \quad \text{on } \Gamma_N. \tag{4.4}$$

Let $\bar{u}_{\mathcal{T}} \in \mathcal{U}^k$ be the current approximation of the exact solution $u \in U$ of (2.1–2.2). Define the bilinear forms

$$\hat{b}(\boldsymbol{\sigma},\, \boldsymbol{\tau}) = (A^{-1}\boldsymbol{\sigma},\, \boldsymbol{\tau}) \quad \text{and} \quad b(\boldsymbol{\sigma},\, \boldsymbol{\tau}) = \hat{b}(\boldsymbol{\sigma},\, \boldsymbol{\tau}) + (\nabla \cdot \boldsymbol{\sigma},\, \nabla \cdot \boldsymbol{\tau}) \tag{4.5}$$

for any $(\boldsymbol{\sigma},\, \boldsymbol{\tau}) \in \Sigma \times \Sigma$. The L^2 recovery procedure is to find $\hat{\boldsymbol{\sigma}}_{\mathcal{T}} \in \mathcal{V}^m$ such that

$$\hat{b}(\hat{\boldsymbol{\sigma}}_{\mathcal{T}},\, \boldsymbol{\tau}) = -(\nabla\, \bar{u}_{\mathcal{T}},\, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{V}^m. \tag{4.6}$$

The $H(\mathrm{div})$ recovery procedure is to find $\boldsymbol{\sigma}_{\mathcal{T}} \in \mathcal{V}^m$ such that

$$b(\boldsymbol{\sigma}_{\mathcal{T}},\, \boldsymbol{\tau}) = -(\nabla\, \bar{u}_{\mathcal{T}},\, \boldsymbol{\tau}) + (f - X\bar{u}_{\mathcal{T}},\, \nabla \cdot \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{V}^m. \tag{4.7}$$

Note that we assume $f \in L^2(\Omega)$ in the $H(\mathrm{div})$ recovery.

Denote the true errors by

$$e = u - \bar{u}_{\mathcal{T}}, \quad \hat{\mathbf{E}} = \boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_{\mathcal{T}}, \quad \text{and} \quad \mathbf{E} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathcal{T}},$$

and denote the norms induced by the bilinear forms by

$$\|\boldsymbol{\tau}\|_{\hat{B},\Omega} = \sqrt{\hat{b}(\boldsymbol{\tau},\, \boldsymbol{\tau})} \quad \text{and} \quad \|\boldsymbol{\tau}\|_{B,\Omega} = \sqrt{b(\boldsymbol{\tau},\, \boldsymbol{\tau})},$$

which are weighted L^2 and $H(\mathrm{div})$ norms, respectively.

THEOREM 4.1. *The following a priori error bounds for the recovered fluxes hold:*

$$\|\hat{\mathbf{E}}\|_{\hat{B},\Omega} \leq C \left(\inf_{\boldsymbol{\tau} \in \mathcal{V}^m} \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_{\hat{B},\Omega} + \|e\|_{\Omega} \right) \tag{4.8}$$

and

$$\|\mathbf{E}\|_{B,\Omega} \leq C \left(\inf_{\boldsymbol{\tau} \in \mathcal{V}^m} \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_{B,\Omega} + \|e\|_{\Omega} \right). \tag{4.9}$$

Proof. Since proofs on (4.8) and (4.9) are similar, we show only the validity of (4.9). For all $\boldsymbol{\tau} \in \mathcal{V}^m$, using both equations in (4.3) gives the error equation

$$b(\mathbf{E},\, \boldsymbol{\tau}) = -(\nabla\, e,\, \boldsymbol{\tau}) - (X\, e,\, \nabla \cdot \boldsymbol{\tau}). \tag{4.10}$$

Using (4.10) with $\boldsymbol{\tau} = \boldsymbol{\tau} - \boldsymbol{\sigma}_{\mathcal{T}}$ and the Cauchy–Schwarz inequality yield

$$\begin{aligned}\|\mathbf{E}\|_{B,\Omega}^2 &= b(\mathbf{E}, \mathbf{E}) = b(\mathbf{E}, \boldsymbol{\sigma} - \boldsymbol{\tau}) + b(\mathbf{E}, \boldsymbol{\tau} - \boldsymbol{\sigma}_{\mathcal{T}})\\ &= b(\mathbf{E}, \boldsymbol{\sigma} - \boldsymbol{\tau}) - (\nabla e, \boldsymbol{\tau} - \boldsymbol{\sigma}_{\mathcal{T}}) - (X e, \nabla\cdot(\boldsymbol{\tau} - \boldsymbol{\sigma}_{\mathcal{T}}))\\ &\le \|\mathbf{E}\|_{B,\Omega}\|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_{B,\Omega} + \left(\|A^{1/2}\nabla e\|_{0,\Omega}^2 + \|X e\|_{0,\Omega}^2\right)^{\frac{1}{2}} \|\boldsymbol{\tau} - \boldsymbol{\sigma}_{\mathcal{T}}\|_{B,\Omega},\end{aligned}$$

which, combining with the triangle inequality and (2.4), gives

$$\begin{aligned}\|\mathbf{E}\|_{B,\Omega}^2 &\le \|\mathbf{E}\|_{B,\Omega}\|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_{B,\Omega} + C\,|\!|\!| e|\!|\!|_{\Omega}\left(\|\boldsymbol{\tau} - \boldsymbol{\sigma}\|_{B,\Omega} + \|\mathbf{E}\|_{B,\Omega}\right)\\ &\le \|\mathbf{E}\|_{B,\Omega}\left(\|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_{B,\Omega} + C\,|\!|\!| e|\!|\!|_{\Omega}\right) + C\,|\!|\!| e|\!|\!|_{\Omega}\,\|\boldsymbol{\tau} - \boldsymbol{\sigma}\|_{B,\Omega}\end{aligned}$$

for all $\boldsymbol{\tau} \in \mathcal{V}^m$. Now, the error bound in (4.9) follows from the above inequality and the ϵ inequality $(ab \le \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2)$. ◻

Remark 4.1. Theorem 4.1 and the approximation property in (4.2) indicate that we should use the Raviart–Thomas elements of index $k-1$ for approximating $\boldsymbol{\sigma}$ in (4.6) and (4.7) in order to be accurate up to that of the current finite element approximation. We may also use other families, e.g., Brezzi–Douglas–Marini elements of index k [16], of $H(\mathrm{div};\,\Omega)$ conforming finite element spaces of appropriate order to approximate the flux in (4.6) and (4.7).

Remark 4.2. The resulting system of linear equations from the L^2 recovery is a mass matrix and, hence, it can be very efficiently solved with several sweeps of the Jacobi iteration or, better, the preconditioned conjugate gradients with the Jacobi preconditioner. Due to the hierarchical structure of the meshes in adaptive finite element method and the availability of the finite element approximations on previous meshes, problem (4.7) can be solved efficiently by a fast $H(\mathrm{div})$-type full multigrid method on a composite grid. For efficient full multigrid methods, see, e.g., [33, 34]. For fast $H(\mathrm{div})$-type multigrid methods, see [2, 35, 26, 34].

Remark 4.3. For the purpose of a posteriori error estimators, problem (4.7) may be approximated roughly. As demonstrated numerically in section 7.2, one iteration of an $H(\mathrm{div})$ V(1,1)-cycle multigrid method is sufficient to produce a reliable and efficient estimator. Estimators based on the localization of problem (4.7) are the subject of a forthcoming paper.

5. Error estimator. Based on the L^2 recovered flux, $\hat{\boldsymbol{\sigma}}_{\mathcal{T}}$, defined in (4.6), we introduce a new a posteriori error estimator:

$$\zeta_K = \left(\|A^{-1/2}\hat{\boldsymbol{\sigma}}_{\mathcal{T}} + A^{1/2}\nabla\bar{u}_{\mathcal{T}}\|_{0,K}^2 + \beta_K^2\|\nabla\cdot\hat{\boldsymbol{\sigma}}_{\mathcal{T}} + X\bar{u}_{\mathcal{T}} - f\|_{0,K}^2\right)^{1/2} \forall\, K \in \mathcal{T} \tag{5.1}$$

and

$$\zeta = \left(\sum_{K\in\mathcal{T}} \zeta_K^2\right)^{1/2} \tag{5.2}$$

with $\beta_K > 0$ depending upon coefficients of the underlying problem. For example, for the interface problem described in section 2, we choose

$$\beta_K = \alpha_K^{-1/2} h_K,$$

and for the singularly perturbed reaction-diffusion equation, we choose

$$\beta_K = \min\{\varepsilon^{-1/2} h_K,\, 1\}.$$

Unlike existing recovery-based estimators, including those in [19], the estimator ζ has an extra term, which is a weighted element residual. This term guarantees reliability of the estimator on coarse meshes, which we believe is necessary for higher-order elements (see Figure 1). This is because, in general, the element residual is not higher-order compared to the first term. This estimator is comparable to the explicit residual-based estimator (see, e.g., [1, 37]), but it is more accurate than the latter (see, e.g., Figures 10 and 12 for a singularly perturbed reaction-diffusion test problem). A similar idea, adding two additional terms to the ZZ estimator, was studied in [25] and the resulting estimator is sophisticated.

With the $H(\mathrm{div})$ recovered flux, $\boldsymbol{\sigma}_{\mathcal{T}}$, defined in (4.7), we introduce two *a posteriori* error estimators. The first one is defined as follows:

$$\xi_K = \|A^{-1/2}\boldsymbol{\sigma}_{\mathcal{T}} + A^{1/2}\nabla \bar{u}_{\mathcal{T}}\|_{0,K} \quad \forall\, K \in \mathcal{T} \tag{5.3}$$

and

$$\xi = \left(\sum_{K\in\mathcal{T}} \xi_K^2\right)^{1/2} = \|A^{-1/2}\boldsymbol{\sigma}_{\mathcal{T}} + A^{1/2}\nabla \bar{u}_{\mathcal{T}}\|_{0,\,\Omega}. \tag{5.4}$$

This estimator may be considered as an extension of the recovery-based estimator studied in [19] to both higher-order conforming elements and general scalar elliptic partial differential equations. As shown in [19], estimators of this type are robust with respect to the diffusion tensor and are possibly asymptotically exact. By adding "element" residuals, we have the second estimator defined by

$$\eta_K = \left(\xi_K^2 + \|\nabla\cdot\boldsymbol{\sigma}_{\mathcal{T}} + X\bar{u}_{\mathcal{T}} - f\|_{0,\,K}^2\right)^{1/2} \quad \forall\, K \in \mathcal{T} \tag{5.5}$$

and

$$\eta = \left(\sum_{K\in\mathcal{T}} \eta_K^2\right)^{1/2} = \left(\xi^2 + \|\nabla\cdot\boldsymbol{\sigma}_{\mathcal{T}} + X\bar{u}_{\mathcal{T}} - f\|_{0,\,\Omega}^2\right)^{1/2}. \tag{5.6}$$

We show that the estimator η is locally and globally exact on any given mesh with respect to a norm depending upon the underlying problem.

Remark 5.1. For diffusion and diffusion-reaction problems, the estimators based on the L^2 recovery, the ζ defined in (5.2) and those developed in [19], are sufficient. The ξ and η defined in the respective (5.4) and (5.6) are designed for the convection-diffusion-reaction and the Helmholtz problems, i.e., when $\mathbf{b} \neq 0$ or b_0 is nonpositive in (2.1). The ξ and η differ in norms measuring the error: the "energy" norm and a stronger norm. When the right-hand side is only in L^2, then the stronger norm measured by the η is too strong for the underlying problem. Hence, one can either use the ξ, which is not reliable on relatively coarse meshes, or modify the η as follows:

$$\hat{\eta} = \left(\sum_{K\in\mathcal{T}} \hat{\eta}_K^2\right)^{1/2} \qquad \text{with}$$

$$\hat{\eta}_K = \left(\|A^{-1/2}\boldsymbol{\sigma}_{\mathcal{T}} + A^{1/2}\nabla \bar{u}_{\mathcal{T}}\|_{0,K}^2 + \beta_K^2 \|\nabla\cdot\boldsymbol{\sigma}_{\mathcal{T}} + X\bar{u}_{\mathcal{T}} - f\|_{0,\,K}^2\right)^{1/2}.$$

It is easy to show that the $\hat{\eta}$ is robust with proper choices of β_K in a similar fashion as the proofs for the ζ.

6. Analysis. In this section, we establish reliability and efficiency bounds for the estimators ζ and ξ and local and global exactness for the estimator η.

Let U^* denote the dual space of U equipped with the dual norm

$$\|f\|_{U^*} = \sup_{v\in U} \frac{(f,\, v)}{\|v\|_\Omega}$$

for any $f \in U^*$. For any element $K \in \mathcal{T}$, denote

$$\|(\boldsymbol{\tau},\, v)\|_{1,K} = \left(\|A^{-1/2}\boldsymbol{\tau}\|^2_{0,\,K} + \|v\|^2_{0,\,K} + \|A^{1/2}\nabla v\|^2_{0,\,K}\right)^{\frac{1}{2}}$$

and

$$\|(\boldsymbol{\tau},\, v)\|_{2,K} = \left(\|A^{-1/2}\boldsymbol{\tau} + A^{1/2}\nabla v\|^2_{0,\,K} + \|\nabla\cdot\boldsymbol{\tau} + X\, v\|^2_{0,\,K}\right)^{\frac{1}{2}}.$$

Let

$$\|(\boldsymbol{\tau},\, v)\|_{1,\Omega} = \left(\sum_{K\in\mathcal{T}} \|(\boldsymbol{\tau},\, v)\|^2_{1,K}\right)^{\frac{1}{2}} = \left(\|v\|^2_\Omega + \|A^{-1/2}\boldsymbol{\tau}\|^2_{0,\Omega}\right)^{\frac{1}{2}}$$

and

$$\begin{aligned}\|(\boldsymbol{\tau},\, v)\|_{2,\Omega} &= \left(\sum_{K\in\mathcal{T}} \|(\boldsymbol{\tau},\, v)\|^2_{2,K}\right)^{\frac{1}{2}} \\ &= \left(\|A^{-1/2}\boldsymbol{\tau} + A^{1/2}\nabla v\|^2_{0,\Omega} + \|\nabla\cdot\boldsymbol{\tau} + Xv\|^2_{0,\Omega}\right)^{\frac{1}{2}}.\end{aligned}$$

Obviously, $\|(\boldsymbol{\tau},\, v)\|_{1,\Omega}$ defines a norm for $(\boldsymbol{\tau},\, v) \in L^2(\Omega)^{d\times d} \times H^1_D(\Omega)$. If $\boldsymbol{\tau} = -A\nabla v$, then

$$\|(\boldsymbol{\tau},\, v)\|_{1,\Omega} = \left(\|v\|^2_{0,\Omega} + 2\,\|A^{1/2}\nabla v\|_{0,\Omega}\right)^{1/2}$$

defines an "energy" norm for $v \in H^1_D(\Omega)$. We show that $\|(\boldsymbol{\tau},\, v)\|_{2,\Omega}$ also defines a norm on $\Sigma \times U$ in section 6.2.

THEOREM 6.1. *For any $(\boldsymbol{\tau},\, v) \in \Sigma \times U$, there exists positive constants C_1 and C_2 such that*

$$C_1\,\|(\boldsymbol{\tau},\, v)\|^2_{1,\Omega} \le \|A^{-1/2}\boldsymbol{\tau} + A^{1/2}\nabla v\|^2_{0,\Omega} + \|\nabla\cdot\boldsymbol{\tau} + Xv\|^2_{U^*} \le \bar{C}_1\,\|(\boldsymbol{\tau},\, v)\|^2_{1,\Omega} \tag{6.1}$$

with $\bar{C}_1 = 2\max\{2,\, 1 + C_X^2\}$ and that

$$C_2\,\left(\|v\|^2_\Omega + \|\boldsymbol{\tau}\|^2_{B,\Omega}\right) \le \|(\boldsymbol{\tau},\, v)\|^2_{2,\Omega} \le 2C_X\left(\|v\|^2_\Omega + \|\boldsymbol{\tau}\|^2_{B,\Omega}\right). \tag{6.2}$$

Similar results as those in (6.1) and (6.2) were proved in [12] and [18], respectively. The upper bounds in (6.1) and (6.2) are direct consequences of the triangle inequality, (2.4), and the fact that

$$\|\nabla\cdot\boldsymbol{\tau}\|_{U^*} \le \|A^{-1/2}\boldsymbol{\tau}\|_{0,\Omega} \quad \text{and} \quad \|Xv\|_{U^*} \le \|Xv\|_{0,\Omega}.$$

The proof of the lower bound in (6.2) is quite technical in [18]. This can be proved alternatively by first establishing

$$\|v\|^2_\Omega + \|\boldsymbol{\tau}\|^2_{B,\Omega} \le C\,\left(\|(\boldsymbol{\tau},\, v)\|^2_{2,\Omega} + \|v\|^2_{0,\Omega}\right) \tag{6.3}$$

and then using the standard compactness argument to remove $\|v\|^2_{0,\Omega}$ in the above inequality (see [17]). For the reader's convenience, we provide the proof in section 6.4.

6.1. Analysis for estimator $\boldsymbol{\zeta}$. To establish the local efficiency, we assume that there exist a positive constant C_a independent of β_K such that

$$\beta_K \|f + \nabla \cdot (A \nabla \bar{u}_{\mathcal{T}}) - X \bar{u}_{\mathcal{T}}\|_{0,K} \le C_a |\!|\!| e |\!|\!|_K + \beta_K \|f - v\|_{0,K}, \tag{6.4}$$

for all $v \in \mathcal{P}_{\mathcal{T}} = \{v \in L^2(\Omega) : v|_K \in P_k(K) \text{ for all } K \in \mathcal{T}\}$. For the choices of β_K in section 5, this was proved for the elliptic interface problem in [11] and for the singularly perturbed reaction-diffusion equations in [38].

THEOREM 6.2. *Let $f_{\mathcal{T}}$ be the L^2 projection of f onto $\mathcal{P}_{\mathcal{T}}$. Assume that* (6.4) *holds. Then the error estimator ζ defined in* (5.2) *satisfies the local efficiency bound: there exists a positive constant C_e such that*

$$\zeta_K \le C_e \|(\hat{\mathbf{E}}, e)\|_{1,K} + \beta_K \|f - f_{\mathcal{T}}\|_{0,K}, \tag{6.5}$$

and the global efficiency bound: there exists a positive constant $\hat{C}_e$ such that

$$\zeta \le \hat{C}_e \left(|\!|\!| e |\!|\!|_\Omega + \left(\sum_{K \in \mathcal{T}} \beta_K^2 \|f - f_{\mathcal{T}}\|_{0,K}^2 \right)^{1/2} \right). \tag{6.6}$$

Moreover, both C_e and $\hat{C}_e$ are independent of the jumps and ε for the interface and singularly perturbed reaction-diffusion problems, respectively.

Proof. For any $K \in \mathcal{T}$, (2.3) implies there exists a positive constant Λ_K such that

$$\boldsymbol{\xi}^T A \boldsymbol{\xi} \le \Lambda_K \boldsymbol{\xi}^T \boldsymbol{\xi}$$

for all $\boldsymbol{\xi} \in \Re^d$ and almost all $x \in \bar{K}$. It then follows from the triangle inequality, (6.4), and the inverse inequality with constant C_i that

$$\begin{aligned}
&\beta_K \|f - \nabla \cdot \hat{\boldsymbol{\sigma}}_{\mathcal{T}} - X \bar{u}_{\mathcal{T}}\|_{0,K} \\
&\le \beta_K \|f + \nabla \cdot (A \nabla \bar{u}_{\mathcal{T}}) - X \bar{u}_{\mathcal{T}}\|_{0,K} + \beta_K \|\nabla \cdot (\hat{\boldsymbol{\sigma}}_{\mathcal{T}} + A \nabla \bar{u}_{\mathcal{T}})\|_{0,K} \\
&\le C_a |\!|\!| e |\!|\!|_K + \beta_K \|f - f_{\mathcal{T}}\|_{0,K} + C_i \, \beta_K h_K^{-1} \Lambda_K^{1/2} \|A^{-1/2} \hat{\boldsymbol{\sigma}}_{\mathcal{T}} + A^{1/2} \nabla \bar{u}_{\mathcal{T}}\|_{0,K} \\
&\le C_a |\!|\!| e |\!|\!|_K + \beta_K \|f - f_{\mathcal{T}}\|_{0,K} + C_i C \, \|A^{-1/2} \hat{\boldsymbol{\sigma}}_{\mathcal{T}} + A^{1/2} \nabla \bar{u}_{\mathcal{T}}\|_{0,K},
\end{aligned}$$

where C in the above inequality equals one for the interface problems and the singularly perturbed reaction-diffusion equations. Using the first equation in (4.3) and the triangle inequality gives

$$\|A^{-1/2} \hat{\boldsymbol{\sigma}}_{\mathcal{T}} + A^{1/2} \nabla \bar{u}_{\mathcal{T}}\|_{0,K} = \|A^{-1/2} \hat{\mathbf{E}} + A^{1/2} \nabla e\|_{0,K} \le \|(\hat{\mathbf{E}}, e)\|_{1,K}.$$

Now, combining the above two inequalities yields the local efficiency bound in (6.5). The global efficiency bound in (6.6) is a direct consequence of (6.5) and (4.8). ∎

To establish the global reliability bound, we assume that there exist an $e_I \in \mathcal{U}^1$ and a positive constant C_I independent of β_K such that

$$\left(\sum_{K \in \mathcal{T}} \beta_K^{-2} \|e - e_I\|_{0,K}^2 + \|A^{1/2} \nabla (e - e_I)\|_\Omega^2 \right)^{1/2} \le C_I \, |\!|\!| e |\!|\!|_\Omega. \tag{6.7}$$

For both interface problems and singularly perturbed reaction-diffusion equations, (6.7) holds for the choices of β_K in section 5 with $e_I = Ie$, where I is a Clément-type interpolation operator (see, e.g., [11, 19, 32, 38]).

THEOREM 6.3. *Assume that $\bar{u}_{\mathcal{T}} = u_{\mathcal{T}}$ and that the bilinear form $a(\cdot,\cdot)$ is coercive, i.e., there exists a positive constant γ such that*

$$\gamma \,|\!|\!| v |\!|\!|^2 \le a(v,\,v) \quad \forall\, v \in U.$$

If (6.7) *holds, then the error estimator ζ defined in* (5.2) *satisfies the global reliability bound: there exists a positive constant C_r such that*

$$\|(\hat{\mathbf{E}},\,e)\|_{1,\Omega} \le C_r \zeta. \tag{6.8}$$

Proof. It follows from the coercivity of $a(\cdot,\cdot)$, the orthogonality property of the finite element approximation, integration by parts, and the Cauchy–Schwarz inequality that for all $v \in \mathcal{U}^k$

$$\begin{aligned}
\gamma \,|\!|\!| e |\!|\!|_\Omega^2 &\le a(e,\,e) = a(e,\,e-v) = (A\nabla(u-u_{\mathcal{T}}),\,\nabla(e-v)) + (X(u-u_{\mathcal{T}}),\,e-v)\\
&= (A\nabla u + \hat{\boldsymbol{\sigma}}_{\mathcal{T}},\,\nabla(e-v)) - (\hat{\boldsymbol{\sigma}}_{\mathcal{T}} + A\nabla u_{\mathcal{T}},\,\nabla(e-v)) + (X(u-u_{\mathcal{T}}),\,e-v)\\
&= (f - \nabla\cdot\hat{\boldsymbol{\sigma}}_{\mathcal{T}} - Xu_{\mathcal{T}},\,e-v) - (\hat{\boldsymbol{\sigma}}_{\mathcal{T}} + A\nabla u_{\mathcal{T}},\,\nabla(e-v))\\
&\le \sum_{K\in\mathcal{T}} \|f - \nabla\cdot\hat{\boldsymbol{\sigma}}_{\mathcal{T}} - Xu_{\mathcal{T}}\|_{0,\,K}\,\|e-v\|_{0,\,K} + \|A^{\frac12}\nabla u_{\mathcal{T}} + A^{-\frac12}\hat{\boldsymbol{\sigma}}_{\mathcal{T}}\|_{0,\,\Omega}\,|\!|\!| e-v |\!|\!|_\Omega\\
&\le \zeta\left(\sum_{K\in\mathcal{T}} \beta_K^{-2}\|e-v\|_{0,\,K}^2 + |\!|\!| e-v |\!|\!|_\Omega^2\right)^{1/2} \le C_I\,\zeta\,|\!|\!| e |\!|\!|_\Omega.
\end{aligned}$$

Hence, choosing $v = e_I$ and using (6.7), we have

$$|\!|\!| e |\!|\!|_\Omega \le C_I \gamma^{-1}\zeta,$$

which, together with the triangle inequality, yields

$$\|A^{-1/2}\hat{\mathbf{E}}\|_{0,\,\Omega} \le \|A^{-\frac12}\hat{\mathbf{E}} + A^{\frac12}\nabla e\|_{0,\,\Omega} + \|A^{\frac12}\nabla e\|_{0,\,\Omega} \le \left(1 + C_I\gamma^{-1}\right)\zeta.$$

Combining the above two inequalities implies (6.8). This completes the proof of the theorem. ∎

6.2. Analysis for estimator $\boldsymbol{\xi}$. By the constitutive equation in (4.3), the estimator ξ has the following *local* representation in terms of the true error $(\mathbf{E},\,e) = (\boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathcal{T}},\,u - \bar{u}_{\mathcal{T}})$:

$$\xi_K = \xi_K(\mathbf{E},\,e) = \|A^{-1/2}\mathbf{E} + A^{1/2}\nabla e\|_{0,K} \tag{6.9}$$

for any element $K \in \mathcal{T}$.

THEOREM 6.4. *Assume that $\Lambda_0 \ge 1$. Then the estimator ξ defined in* (5.4) *satisfies the global reliability bound*

$$\|(\mathbf{E},\,e)\|_{1,\Omega} \le C\,\left(\xi + \|h\,(f - \nabla\cdot\boldsymbol{\sigma}_{\mathcal{T}} - X\bar{u}_{\mathcal{T}})\|_{0,\Omega}\right). \tag{6.10}$$

Proof. It follows from (6.1) and (6.9) with $(\boldsymbol{\tau},\,v) = (\mathbf{E},\,e)$ that

$$\|(\mathbf{E},\,e)\|_{1,\Omega} \le C_1\left(\xi^2 + \|\nabla\cdot\mathbf{E} + Xe\|_{U^*}^2\right)^{1/2}. \tag{6.11}$$

For any given $v \in U$, there exists a $\boldsymbol{\tau} \in \Sigma$ such that (see [15])

$$\nabla \cdot \boldsymbol{\tau} = v \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{\tau}\|_{1,\Omega} \le C\,\|v\|_{0,\Omega}. \tag{6.12}$$

Let $\Pi_0 : H(\text{div};\,\Omega) \cap L^t(\Omega)^d \mapsto \mathcal{V}^0$ for fixed $t > 2$ be the well-known RT_0 interpolation operator (see [16]), then it satisfies the following approximation and stability properties:

$$\|\nabla \cdot (\boldsymbol{\tau} - \Pi_0\boldsymbol{\tau})\|_{0,K} \le C\,h_K \|\nabla \cdot \boldsymbol{\tau}\|_{1,K} \quad \text{and} \quad \|\Pi_0\boldsymbol{\tau}\|_{0,\Omega} \le C\,\|\boldsymbol{\tau}\|_{1,\Omega},$$

which, together with (6.12), the error equation in (4.10), the Cauchy–Schwarz inequality, and (2.3), yields

$$\begin{aligned}
&(\nabla \cdot \mathbf{E} + Xe,\, v) = (\nabla \cdot \mathbf{E} + Xe,\, \nabla \cdot \boldsymbol{\tau}) \\
&= (\nabla \cdot \mathbf{E} + Xe,\, \nabla \cdot (\boldsymbol{\tau} - \Pi_0\boldsymbol{\tau})) + (\nabla \cdot \mathbf{E} + Xe,\, \nabla \cdot \Pi_0\boldsymbol{\tau}) \\
&= \sum_{K\in\mathcal{T}} (\nabla \cdot \mathbf{E} + Xe,\, \nabla \cdot (\boldsymbol{\tau} - \Pi_0\boldsymbol{\tau}))_K + (A^{-1/2}\mathbf{E} + A^{1/2}\nabla e,\, A^{-1/2}\Pi_0\boldsymbol{\tau}) \\
&\le C \sum_{K\in\mathcal{T}} h_K \|\nabla \cdot \mathbf{E} + Xe\|_{0,K}\, \|\nabla \cdot \boldsymbol{\tau}\|_{1,K} + \|A^{-1/2}\mathbf{E} + A^{1/2}\nabla e\|_{0,\Omega}\, \|A^{-1/2}\Pi_0\boldsymbol{\tau}\|_{0,\Omega} \\
&\le C\,\|h(\nabla \cdot \mathbf{E} + Xe)\|_{0,\Omega}\, \|v\|_{1,\Omega} + C\Lambda_0^{-1/2}\xi\, \|v\|_{0,\Omega}.
\end{aligned}$$

Hence,

$$\|\nabla\cdot\mathbf{E}+Xe\|_{U^*} \le C\,\left(\|h(\nabla \cdot \mathbf{E} + Xe)\|_{0,\Omega} + \xi\right) = C\,\left(\|h\,(f - \nabla \cdot \boldsymbol{\sigma}_{\mathcal{T}} - X\bar{u}_{\mathcal{T}})\|_{0,\Omega} + \xi\right).$$

Combining with (6.11) proves the validity of (6.10) and, hence, the theorem. ◻

Let $m = k - 1$ in (4.7), then the second term in (6.10),

$$\|h\,(f - \nabla \cdot \boldsymbol{\sigma}_{\mathcal{T}} - X\bar{u}_{\mathcal{T}})\|_{0,\Omega} = \|h\,(\nabla \cdot \mathbf{E} + Xe)\|_{0,\Omega},$$

is a higher-order term comparing to the estimator ξ. This can be seen clearly from the triangle inequality and (4.9). Alternatively, this term can be bounded by the so-called oscillation of the element residual and the estimator. To do so, let

$$R = f - \nabla \cdot \boldsymbol{\sigma}_{\mathcal{T}} - X\bar{u}_{\mathcal{T}} = \nabla \cdot \mathbf{E} + Xe,$$

and let $\mathbb{P}_{k-1}$ be the L^2 projection operator onto the discontinuous piecewise polynomial space of degree $k - 1$ with respect to the triangulation $\mathcal{T}$,

$$\{v \in L^2(\Omega) \,:\, v|_K \in P_{k-1}(K)\ \forall\ K \in \mathcal{T}\}.$$

LEMMA 6.5. *Assume that $\Lambda_0 \ge 1$. Then there exists a positive constant C such that*

$$\|f - \nabla \cdot \boldsymbol{\sigma}_{\mathcal{T}} - X\bar{u}_{\mathcal{T}}\|_{0,\Omega} = \|R\|_{0,\Omega} \le C\,\xi + \|R - \mathbb{P}_{k-1}R\|_{0,\Omega}. \tag{6.13}$$

Proof. There exists a $\boldsymbol{\tau} \in \Sigma$ such that (see [15])

$$\nabla \cdot \boldsymbol{\tau} = R \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{\tau}\|_{1,\Omega} \le C\,\|R\|_{0,\Omega}. \tag{6.14}$$

Let $\Pi_{k-1} : H(\mathrm{div};\,\Omega) \cap L^t(\Omega)^d \mapsto \mathcal{V}^{k-1}$ for fixed $t > 2$ be the well-known RT_{k-1} interpolation operator (see [16]), then we have

$$\nabla \cdot \Pi_{k-1}\boldsymbol{\tau} = \mathbb{P}_{k-1}\nabla \cdot \boldsymbol{\tau} = \mathbb{P}_{k-1}R \quad \text{and} \quad \|\Pi_{k-1}\boldsymbol{\tau}\|_{0,\Omega} \le C\,\|\boldsymbol{\tau}\|_{1,\Omega}.$$

Now, it follows from the error equation in (4.10), the Cauchy–Schwarz inequality, and (2.3) that

$$\begin{aligned}
\|R\|_{0,\Omega}^2 &= (R,\,R) = (R,\,R - \nabla\cdot\Pi_{k-1}\boldsymbol{\tau}) + (A^{-1/2}\mathbf{E} + A^{1/2}\nabla e,\,A^{-1/2}\Pi_{k-1}\boldsymbol{\tau})\\
&= (R,\,R - \mathbb{P}_{k-1}R) + (A^{-1/2}\boldsymbol{\sigma}_{\mathcal{T}} + A^{1/2}\nabla\bar{u}_{\mathcal{T}},\,A^{-1/2}\Pi_{k-1}\boldsymbol{\tau})\\
&\le \|R - \mathbb{P}_{k-1}R\|_{0,\Omega}\|R\|_{0,\Omega} + C\Lambda_0^{-1/2}\xi\,\|R\|_{0,\Omega},
\end{aligned}$$

which implies (6.13) and, hence, the lemma. ☐

Remark 6.6. The assumption, $\Lambda_0 \ge 1$, in Theorem 6.4 and Lemma 6.5 excludes both convection- and reaction-dominant problems but not the interface problems. For the singularly perturbed reaction-diffusion problems, the estimator ξ would fail when the mesh size is not small enough, more precisely, if $h_K \ge C\,\varepsilon^{1/2}$.

THEOREM 6.7. *For any element $K \in \mathcal{T}$, the estimator ξ satisfies the local efficiency bound*

$$\xi_K \le \|A^{-1/2}\mathbf{E}\|_{0,K} + \|A^{1/2}\nabla e\|_{0,K} \le \sqrt{2}\,\|(\mathbf{E},\,e)\|_{1,K}. \tag{6.15}$$

Proof. The local efficiency bound in (6.15) is a simple consequence of the local representation of the estimator ξ in (6.9) and the triangle inequality. ☐

6.3. Analysis for estimator $\boldsymbol{\eta}$. By the constitutive and equilibrium equations in (4.3), we have the following *local* representation of the estimator η:

$$\eta_{_K} = \eta_{_K}(\mathbf{E},\,e) = \left(\|A^{-1/2}\mathbf{E} + A^{1/2}\nabla e\|_{0,K}^2 + \|\nabla\cdot\mathbf{E} + Xe\|_{0,\,K}^2\right)^{1/2} \tag{6.16}$$

for any element $K \in \mathcal{T}$.

LEMMA 6.8. *The $\|(\boldsymbol{\tau}\,,\,v)\|_{2,\,\Omega}$ defines a norm in the product space $\Sigma \times U$, which is equivalent to the norm $\left(\|\!|v|\!\|_\Omega^2 + \|\boldsymbol{\tau}\|_{B,\Omega}^2\right)^{1/2}$.*

Proof. With the equivalence in (6.2), it suffices to show that $\|(\cdot\,,\,\cdot)\|_{2,\,\Omega}$ satisfies the triangle inequality. For any $(\boldsymbol{\tau}_i,\,v_i) \in \Sigma \times U$ $(i = 1,\,2)$, let

$$a_i = \|A^{-1/2}\boldsymbol{\tau}_i + A^{1/2}\nabla v_i\|_{0,\,\Omega} \quad \text{and} \quad b_i = \|\nabla\cdot\boldsymbol{\tau}_i + X\,v_i\|_{0,\,\Omega};$$

then $\|(\boldsymbol{\tau}_i,\,v_i)\|_{2,\,\Omega}^2 = a_i^2 + b_i^2$. It follows from the Cauchy–Schwarz inequality that

$$\begin{aligned}
\|(\boldsymbol{\tau}_1,\,v_1) + (\boldsymbol{\tau}_2,\,v_2)\|_{2,\,\Omega}^2 &\le \sum_{i=1}^{2}(a_i^2 + b_i^2) + 2\,(a_1a_2 + b_1b_2)\\
&\le \sum_{i=1}^{2}(a_i^2 + b_i^2) + 2\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}\\
&= \left(\|(\boldsymbol{\tau}_1,\,v_1)\|_{2,\,\Omega} + \|(\boldsymbol{\tau}_2,\,v_2)\|_{2,\,\Omega}\right)^2.
\end{aligned}$$

Taking the square root on both sides of the above inequality proves the triangle inequality and, hence, the lemma. ☐

Remark 6.9. When $X\,v = v$, it follows from integration by parts that

$$\|(\boldsymbol{\tau}\,,\,v)\|_{2,\,\Omega} = \Big(\|v\|^2_{0,\Omega} + \|A^{1/2}\nabla v\|^2_{0,\Omega} + \|A^{-1/2}\boldsymbol{\tau}\|^2_{0,\Omega} + \|\nabla\cdot\boldsymbol{\tau}\|^2_{0,\Omega}\Big)^{1/2}$$
$$= \big(\|\!|v|\!\|^2_{\Omega} + \|\boldsymbol{\tau}\|^2_{B,\Omega}\big)^{1/2}$$

for any $(\boldsymbol{\tau}\,,\,v)\in\Sigma\times U$. Moreover, when $\boldsymbol{\tau} = -A\nabla v$, we have

$$\|(-A\nabla v\,,\,v)\|_{2,\,\Omega} = \Big(\|v\|^2_{0,\Omega} + 2\|A^{1/2}\nabla v\|^2_{0,\Omega} + \|\nabla\cdot(A\nabla v)\|^2_{0,\Omega}\Big)^{1/2},$$

which is stronger than the "energy" norm, $\|\!|v|\!\|_{\Omega}$, of v.

THEOREM 6.10. *The a posteriori error estimator η defined in* (5.6) *is exact locally and globally with respect to the seminorm $\|(\cdot\,,\,\cdot)\|_{2,K}$ and the norm $\|(\cdot\,,\,\cdot)\|_{2,\Omega}$*

$$\eta_K = \|(\mathbf{E},\,e)\|_{2,K} \quad \textit{and} \quad \eta = \|(\mathbf{E},\,e)\|_{2,\Omega}, \tag{6.17}$$

respectively.

Proof. Equation (6.17) is a direct consequence of the local representation of the estimator η in (6.16). ∎

Remark 6.11. Obviously, Theorem 6.10 indicates that the estimator η satisfies both the (local) efficiency and (global) reliability bounds with both constants being one.

Remark 6.12. When $Xv = v$, Remark 6.9 and Theorem 6.10 imply that the estimator η is exact globally with respect to the norm $\big(\|\!|v|\!\|^2_{\Omega} + \|\boldsymbol{\tau}\|^2_{B,\Omega}\big)^{1/2}$. Similarly, if $Xv = b_0(x)v$ with $b_0(x) > 0$ for almost all $x\in\bar{\Omega}$, then the modified estimator

$$\tilde{\eta} = \Big(\|A^{-1/2}\boldsymbol{\sigma}_{\mathcal{T}} + A^{1/2}\nabla\bar{u}_{\mathcal{T}}\|^2_{0,\,\Omega} + \|b_0^{-1/2}(\nabla\cdot\boldsymbol{\sigma}_{\mathcal{T}} + b_0\bar{u}_{\mathcal{T}} - f)\|^2_{0,\,\Omega}\Big)^{1/2} \tag{6.18}$$

is exact globally with respect to the norm

$$\Big(\|b_0^{1/2}v\|^2_{0,\Omega} + \|A^{1/2}\nabla v\|^2_{0,\Omega} + \|A^{-1/2}\boldsymbol{\tau}\|^2_{0,\Omega} + \|b_0^{-1/2}\nabla\cdot\boldsymbol{\tau}\|^2_{0,\Omega}\Big)^{1/2}.$$

Here the recovered flux, $\boldsymbol{\sigma}_{\mathcal{T}}\in\mathcal{V}^m$, in (6.18) is the solution of the following problem (a modification of problem (4.7)):

$$\hat{b}(\boldsymbol{\sigma}_{\mathcal{T}},\,\boldsymbol{\tau}) + (b_0^{-1}\nabla\cdot\boldsymbol{\sigma},\,\nabla\cdot\boldsymbol{\tau}) = -(\nabla\,\bar{u}_{\mathcal{T}},\,\boldsymbol{\tau}) + \big(b_0^{-1}(f - X\bar{u}_{\mathcal{T}}),\,\nabla\cdot\boldsymbol{\tau}\big) \quad \forall\,\boldsymbol{\tau}\in\mathcal{V}^m.$$

6.4. Proof of the lower bound in (6.2).

Proof. To show the validity of the lower bound in (6.2), we first establish (6.3). For any $(\boldsymbol{\tau},\,v)\in\Sigma\times U$, it follows from integration by parts, the Cauchy–Schwarz inequality, (2.4), and the Poincaré inequality that

$$\begin{aligned}
&\|A^{\frac12}\nabla\,v\|^2_{0,\,\Omega} = (A^{\frac12}\nabla\,v + A^{-\frac12}\boldsymbol{\tau},\,A^{\frac12}\nabla\,v) - (\boldsymbol{\tau},\,\nabla\,v)\\
&= (A^{\frac12}\nabla\,v + A^{-\frac12}\boldsymbol{\tau},\,A^{\frac12}\nabla\,v) + (\nabla\cdot\boldsymbol{\tau},\,v)\\
&= (A^{\frac12}\nabla\,v + A^{-\frac12}\boldsymbol{\tau},\,A^{\frac12}\nabla\,v) + (\nabla\cdot\boldsymbol{\tau} + Xv,\,v) - (Xv,\,v)\\
&\le \|A^{\frac12}\nabla\,v + A^{-\frac12}\boldsymbol{\tau}\|_{0,\,\Omega}\,\|A^{\frac12}\nabla\,v\|_{0,\,\Omega} + \|\nabla\cdot\boldsymbol{\tau} + Xv\|_{0,\,\Omega}\,\|v\|_{0,\,\Omega} + \|Xv\|_{0,\,\Omega}\,\|v\|_{0,\,\Omega}\\
&\le \Big(\|A^{\frac12}\nabla\,v + A^{-\frac12}\boldsymbol{\tau}\|_{0,\,\Omega} + C\,\|v\|_{0,\,\Omega}\Big)\,\|A^{\frac12}\nabla\,v\|_{0,\,\Omega} + \|\nabla\cdot\boldsymbol{\tau} + Xv\|_{0,\,\Omega}\|v\|_{0,\,\Omega}.
\end{aligned}$$

Hence,

$$\|A^{\frac{1}{2}}\nabla v\|_{0,\,\Omega}^2 \le C\,\left(\|(\boldsymbol{\tau},\,v)\|_{2,\Omega} + \|v\|_{0,\,\Omega}^2\right), \tag{6.19}$$

which, together with the triangle inequality, gives

$$\begin{aligned}\|A^{-\frac{1}{2}}\boldsymbol{\tau}\|_{0,\,\Omega}^2 &\le 2\,\left(\|A^{-\frac{1}{2}}\boldsymbol{\tau} + A^{\frac{1}{2}}\nabla v\|_{0,\,\Omega}^2 + \|A^{\frac{1}{2}}\nabla v\|_{0,\,\Omega}^2\right)\\ &\le C\,\left(\|(\boldsymbol{\tau},\,v)\|_{2,\Omega} + \|v\|_{0,\,\Omega}^2\right).\end{aligned}$$

By the triangle inequality, (2.4), and (6.19), we have

$$\begin{aligned}\|\nabla\cdot\boldsymbol{\tau}\|_{0,\,\Omega}^2 &\le 2\left(\|\nabla\cdot\boldsymbol{\tau} + Xv\|_{0,\,\Omega}^2 + \|Xv\|_{0,\,\Omega}^2\right) \le 2\left(\|\nabla\cdot\boldsymbol{\tau} + Xv\|_{0,\,\Omega}^2 + C\,|\!|\!|v|\!|\!|_{\Omega}^2\right)\\ &\le C\,\left(\|(\boldsymbol{\tau},\,v)\|_{2,\Omega} + \|v\|_{0,\,\Omega}^2\right).\end{aligned}$$

Combining the above three inequalities yields (6.3).

With (6.3), we show the validity of the lower bound in (6.2) by the compactness argument. To this end, assume that the lower bound in (6.2) is not true. This implies that there exists a sequence $\{(\boldsymbol{\tau}_n,\,v_n)\} \in \Sigma\times U$ such that

$$\|\boldsymbol{\tau}_n\|_{B,\Omega}^2 + |\!|\!|v_n|\!|\!|_{\Omega}^2 = 1 \quad \text{and} \quad \|(\boldsymbol{\tau}_n,\,v_n)\|_{2,\Omega} \le \frac{1}{n}. \tag{6.20}$$

Since U is compactly contained in $L^2(\Omega)$, there exists a subsequence $\{v_{n_k}\} \in U$ which converges in $L^2(\Omega)$. For any $k,\ l$ and $(\boldsymbol{\tau}_{n_k},\,v_{n_k})$, $(\boldsymbol{\tau}_{n_l},\,v_{n_l}) \in \Sigma\times U$, it follows from (6.3) and the triangle inequality that

$$\begin{aligned}&\|\boldsymbol{\tau}_{n_k} - \boldsymbol{\tau}_{n_l}\|_{B,\Omega}^2 + |\!|\!|v_{n_k} - v_{n_l}|\!|\!|_{\Omega}^2\\ &\le C\left(\|(\boldsymbol{\tau}_{n_k} - \boldsymbol{\tau}_{n_l},\,v_{n_k} - v_{n_l})\|_{2,\Omega}^2 + \|v_{n_k} - v_{n_l}\|_{0,\,\Omega}^2\right)\\ &\le C\left(\|(\boldsymbol{\tau}_{n_k},\,v_{n_k})\|_{2,\Omega} + \|(\boldsymbol{\tau}_{n_l},\,v_{n_l})\|_{2,\Omega} + \|v_{n_k} - v_{n_l}\|_{0,\,\Omega}^2\right) \to 0 \quad \text{as } k,\,l\to\infty.\end{aligned}$$

This implies that $\{(\boldsymbol{\tau}_{n_k},\,v_{n_k})\}$ is a Cauchy sequence in the complete space $\Sigma\times U$. Hence, there exists $(\boldsymbol{\tau},\,v)\in\Sigma\times U$ such that

$$\lim_{k\to\infty}\left(\|\boldsymbol{\tau}_{n_k} - \boldsymbol{\tau}\|_{B,\Omega} + |\!|\!|v_{n_k} - v|\!|\!|_{\Omega}\right) = 0.$$

Next, we show that

$$v = 0 \quad \text{and} \quad \boldsymbol{\tau} = \mathbf{0}, \tag{6.21}$$

which contradict with (6.20) that

$$\|\boldsymbol{\tau}\|_{B,\Omega}^2 + |\!|\!|v|\!|\!|_{\Omega}^2 = \lim_{k\to\infty}\|\boldsymbol{\tau}_{n_k}\|_{B,\Omega}^2 + |\!|\!|v_{n_k}|\!|\!|_{\Omega}^2 = 1.$$

To this end, for any $\phi\in U$, integration by parts and the Cauchy–Schwarz inequality give

$$\begin{aligned}a(v_{n_k},\,\phi) &= (A\nabla v_{n_k},\,\nabla\phi) + (Xv_{n_k},\,\phi) = (A\nabla v_{n_k} + \boldsymbol{\tau}_{n_k},\,\nabla\phi) + (Xv_{n_k} + \nabla\cdot\boldsymbol{\tau}_{n_k},\,\phi)\\ &\le \|(\boldsymbol{\tau}_{n_k},\,v_{n_k})\|_{2,\Omega}\,|\!|\!|\phi|\!|\!|_{\Omega} \le \frac{1}{n_k}\,|\!|\!|\phi|\!|\!|_{\Omega}.\end{aligned}$$

Since $\lim_{k\to\infty} v_{n_k} = v$ in U, we then have

$$|a(v,\,\phi)| = \lim_{k\to\infty} |a(v_{n_k},\,\phi)| \le 0,$$

which, together with the uniqueness of the variational problem (2.5), implies

$$v = 0.$$

Now, $\boldsymbol{\tau} = \mathbf{0}$ follows from (6.3):

$$\|\boldsymbol{\tau}\|^2_{B,\Omega} = \lim_{k\to\infty} \|\boldsymbol{\tau}_{n_k}\|^2_{B,\Omega} \le C \lim_{k\to\infty} \left(\|(\boldsymbol{\tau}_{n_k},\, v_{n_k})\|^2_{2,\Omega} + \|v_{n_k}\|^2_{0,\,\Omega} \right) = 0.$$

This completes the proof of (6.21) and, hence, the lower bound in (6.2). ∎

Remark 6.13. When $Xv = b_0 v$ with $b_0 \ge 0$, the compactness argument in the above proof is not needed. That is, (6.2) may be proved in the same fashion as that of (6.3).

7. Numerical examples. In this section, we report numerical results on several one- and two-dimensional test problems. Starting with a coarse triangulation $\mathcal{T}_0$, a sequence of meshes $\{\mathcal{T}_k\}$ is generated by using a standard adaptive meshing algorithm that adopts the Dörfler's bulk marking strategy [23]; i.e., at each refinement step, elements $K \in \mathcal{M}_k$ satisfying

$$\left(\sum_{K\in\mathcal{M}_k} \eta_K^2 \right)^{1/2} \ge \Theta \left(\sum_{K\in\mathcal{T}_k} \eta_K^2 \right)^{1/2} \qquad (\Theta = 1/2)$$

are marked for refinement. In two dimensions, marked triangles are refined regularly by dividing each into four congruent triangles. Additionally, irregularly refined triangles are needed in order to make the triangulation admissible.

7.1. One-dimensional Poisson equation. Consider a one-dimensional Poisson equation

$$\left\{ \begin{array}{l} -u'' = f \quad \text{in} \;\; I = (0,1), \\ u(0) = u(1) = 0. \end{array} \right. \tag{7.1}$$

For $i = 1,\, 2$, denote by

$$\mathcal{U}^i = \{ v \in C^0([0,1]) \,:\, v|_K \in P_i(K) \;\forall\; K \in \mathcal{T} \}$$

the finite element space of order i, where $\mathcal{T}$ is a one-dimensional mesh of $(0,1)$ with element $K = (x_k,\, x_{k+1})$, and $P_i(K)$ is the collection of all polynomials of degree i. Let u_i be the finite element approximation; i.e., $u_i \in \mathcal{U}_0^i \equiv \mathcal{U}^i \cap H_0^1(I)$ satisfies

$$(u_i',\, v') = (f,\, v) \quad \forall\; v \in \mathcal{U}_0^i.$$

Let $\hat{\sigma}_i \in \mathcal{U}^i$ and $\sigma_i \in \mathcal{U}^i$ be the L^2 and $H(\mathrm{div})$ recovered fluxes satisfying the equations

$$(\hat{\sigma}_i,\, \tau) = -(u_i',\, \tau) \quad \forall\; \tau \in \mathcal{U}^i$$

and

$$(\sigma_i,\, \tau) + (\sigma_i',\, \tau') = -(u_i',\, \tau) + (f,\, \tau') \quad \forall\; \tau \in \mathcal{U}^i,$$

respectively. For two test problems, we will present numerical results for the four estimators

$$\chi_i = \|\hat{\sigma}_i + u_i'\|_{0,I}, \quad \zeta_i = \left(\chi_i^2 + \|h(\sigma_i' - f)\|_{0,I}^2\right)^{1/2},$$

$$\xi_i = \|u_i' + \sigma_i\|_{0,I}, \quad \text{and} \quad \eta_i = \left(\|u_i' + \sigma_i\|_{0,I}^2 + \|\sigma_i' - f\|_{0,I}^2\right)^{1/2}.$$

To visually illustrate that $\|h(\sigma_i' - f)\|_{0,I}$ is a higher-order term, we introduce

$$\varrho_i = \left(\|u_i' + \sigma_i\|_{0,I}^2 + \|h(\sigma_i' - f)\|_{0,I}^2\right)^{1/2}.$$

The relative error estimator is calculated as the ratio of the estimator and $\|u_i'\|_{0,I}$.

The first test problem is (7.1) with the right-hand side function $f = 30x^4 - 20x^3$ and the exact solution $u = x^5(1-x)$. The initial mesh is a uniform grid with the mesh size 0.1. For the estimator χ_2, adaptive calculation is stopped after 10 refinements because local mesh refinements do not improve the accuracy of the approximation (see Figure 1), which indicates the failure of the estimator χ_2 on coarse meshes. For the estimator ξ_2, the stopping criterion is that the relative error is less than 10^{-4}. Figure 2 shows that the recovery-based estimator ξ_2 is very accurate and that the term $\|h(\sigma_i - f)\|_{0,I}$ is higher-order. Moreover, the slope of the log(dof)-log(relative error) for both ξ_2 and ϱ_2 is -2, which implies the optimal decay of the error with respect to the number of unknowns. This test problem provides an example showing that a straightforward extension of recovery-based estimators to the quadratic element fails on coarse meshes.

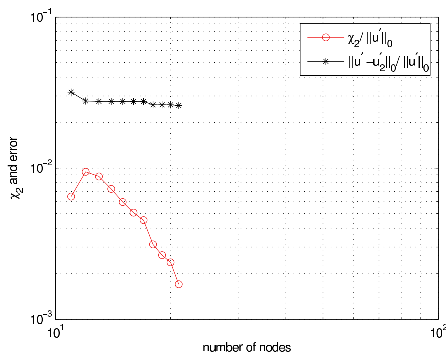

FIG. 1. χ_2 *vs. errors.*

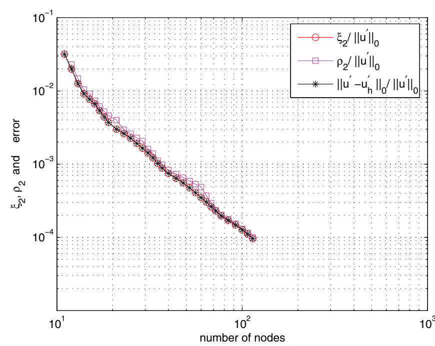

FIG. 2. ξ_2 *and* ϱ_2 *vs. errors.*

The second test problem is (7.1) with the right-hand side function $f = \mu \sin (2^m \pi x)$, where m is a fixed integer and μ is an arbitrary constant. The exact solution of this problem is

$$u = \frac{1}{4^m \pi^2} \mu \sin(2^m \pi x),$$

which is oscillatory. The problem was constructed by Ainsworth and Oden in [1] in order to show that existing recovery-based estimators could be unreliable on coarse meshes. More specifically, consider a uniform mesh of $(0, 1)$ with element nodes located at the points

$$x_k = k/2^n \quad \text{for } k = 0, \ldots, 2^n.$$

When $m \geq n$, it is easy to see that the error estimator χ_1 defined above and any recovery-based estimators solely based on u_1 is zero, but the true error is proportional to $|\mu|$ and could be arbitrarily large. (For more details, see page 83 of [1].) When $m > n$, a simple calculation shows that $(f, \tau') = 0$ for all $\tau \in \mathcal{U}^1$. This implies that σ_1 is identical zero and, hence, the estimator ξ_1 is also unreliable on coarse meshes.

On the other hand, for the test problem with $\mu = 1$, $m = 5$ and $n = 1$, Figure 3 clearly shows that the estimator η_1 is reliable on coarse meshes even though the true error does not decrease until the mesh is fine enough. It also shows that η_1 is accurate with respect to the norm

$$\left(\|(u-u_1)'\|^2_{0,\Omega} + \|(\sigma-\sigma_1)'\|^2_{0,\Omega}\right)^{1/2}.$$

Moreover, when meshes are fine enough, the slope of the log(dof)-log(relative error) is -1, which indicates the optimal (quadratic) decay of the error with respect to the number of unknowns. Here, we do not present numerical results of the estimator ζ_1 that should also be reliable on coarse meshes but is less accurate than η_1.

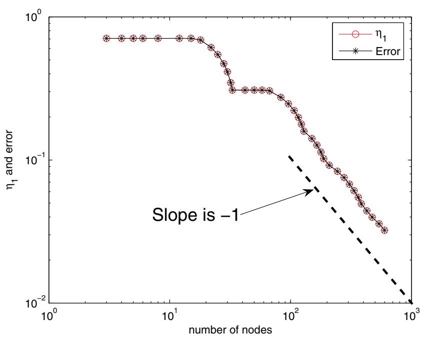

FIG. 3. η_1 *vs. error.*

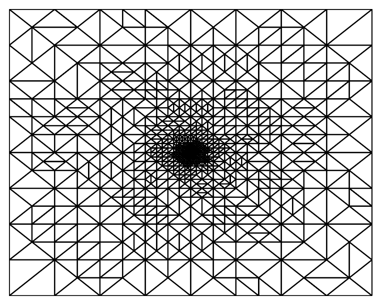

FIG. 4. *Mesh generated by* ξ.

7.2. Two-dimensional interface problem. This subsection presents two sets of numerical results for a benchmark interface problem with intersecting interfaces [30]. Let $\Omega = (-1,1)^2$, $\Gamma_N = \emptyset$, $f = 0$, $b_0 = 0$, $\alpha(x) = R$ in $(0,1)^2 \cup (-1,0)^2$, and $\alpha(x) = 1$ in $\Omega \setminus ([0,1]^2 \cup [-1,0]^2)$, then the exact solution of (2.7) in the polar coordinates at the origin is $u(r,\theta) = r^\beta \mu(\theta)$, where $\mu(\theta)$ is a given smooth function of θ. For $\beta = 0.5$ and 0.25, $R \approx 5.8$ and 25.3, respectively. For this test problem including $\beta = 0.1$, numerical results on various error estimators were reported in [19] for the conforming linear element and in [20] for the mixed and nonconforming elements of the lowest order. In particular, the recovery estimators in [19] based on the L^2 recovery are robust, accurate, and computationally efficient. The purpose of the first test problem with $\beta = 0.5$ is to show that the estimators ξ and η, defined respectively in (5.4) and (5.6), work well for quadratic elements.

To visually illustrate that $\|h(\nabla \cdot \boldsymbol{\sigma}_\mathcal{T} - f)\|^2_{0,\Omega}$ is a higher-order term, we also introduce

$$\varrho = \left(\xi^2 + \|h(\nabla \cdot \boldsymbol{\sigma}_\mathcal{T} - f)\|^2_{0,\Omega}\right)^{1/2}.$$

Meshes generated by ξ and η are similar, and the mesh generated by ξ is depicted in Figure 4. As expected, refinements are centered around the origin. The estimator

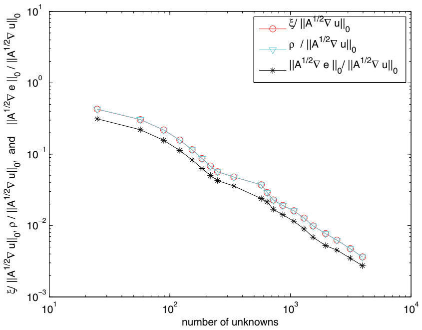

FIG. 5. *ξ and ρ vs. error.*

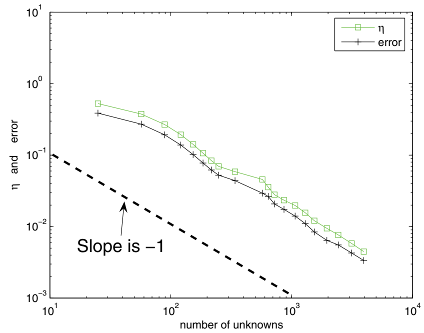

FIG. 6. *η vs. error.*

ξ, the quantity ϱ, and the relative error $\|A^{1/2}\nabla e\|_{0,\Omega}/\|A^{1/2}\nabla u\|_{0,\Omega}$ as functions of the number of unknowns are drawn in Figure 5. The effectivity index for ξ is about 1.35. As shown in Theorem 6.6, the estimator η is exact with respect to the norm $\left(\|A^{-1/2}\mathbf{E} + A^{1/2}\nabla e\|_{0,\Omega}^2 + \|\nabla\cdot\mathbf{E}\|_{0,\Omega}^2\right)^{1/2}$. But in Figure 6, we illustrate the estimator η and the error in the norm $\left(\|A^{1/2}\nabla e\|_{0,\Omega}^2 + \|\nabla\cdot\mathbf{E}\|_{0,\Omega}^2\right)^{1/2}$. Since the slopes of the log(dof)-log(error) in Figures 7 and 8 are -1, this means that the error decays optimally with respect to the number of unknowns.

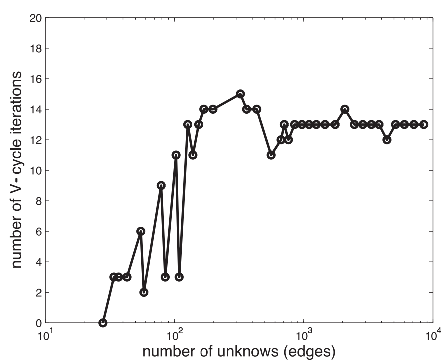

FIG. 7. *Number of V-cycle iterations and number of unknowns (edges).*

The second set of numerical results is for $\beta = 0.25$ and for linear element. The vertex bisection is used for generating adaptive meshes. The purpose here is to show numerically that the $H(\mathrm{div})$ recovery defined in (4.7) may be computed very efficiently by multigrid methods, and that the one step multigrid iteration on problem (4.7) is sufficient to generate a robust and accurate estimator.

A $V(1,1)$-cycle multigrid (MG) method developed in [35, 26] is employed for numerically solving problem (4.7) on adaptive meshes. (For a detailed matrix presentation of the algorithm and a recent review, see Algorithm F.1.1 in [34] and [40], respectively.) The smoothing consists of one Gauss–Seidel (GS) iteration of the underlying $H(\mathrm{div})$ system and one GS iteration of a curl-curl system related to the null space of the divergence operator. On each level, the smoothing is only performed on the new unknowns and their neighbors. Hence, the cost of one smoothing step is of

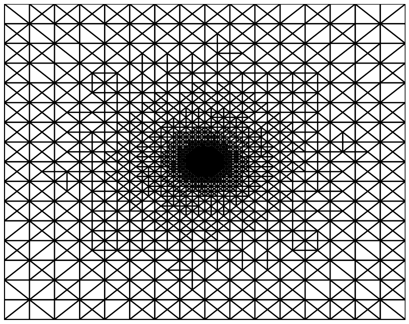

FIG. 8. *Mesh generated by ξ with H(div) MG solver.*

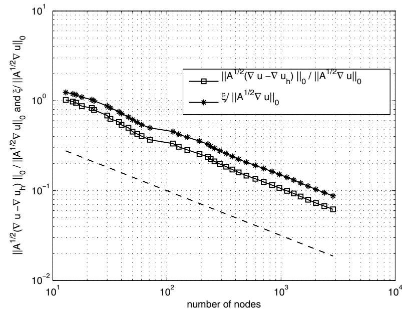

FIG. 9. *ξ vs. error.*

$O(N)$ with a relatively small constant, where N is the number of unknowns on the finest mesh.

For an initial guess, it is natural to use prolongation of the solution at the previous adaptive step. The stopping criterion is that the ratio of the ℓ^2 norms of the current and initial residuals is less than 0.001. The number of MG iterations versus the number of unknowns (edges) is depicted in Figure 7, and it shows that the number of MG iterations is constant once the underlying discretization is relatively large enough (several hundred unknowns). This is different from the observation of section 6.3 in [22].

The mesh generated by the ξ and the ξ versus the true error as functions of the number of nodes is depicted in Figures 8 and 9, respectively. The decay of the error is optimal and the effectivity constant is 1.4. The estimator ξ using a direct solver for problem (4.7) is also tested and its performance is similar to that using the MG solver, and, hence, not reported here. Finally, the estimator ξ using one MG iteration is tested and, again, its performance (see Figures 10 and 11) is very similar to that using the MG solver. Note that the computational cost in this case is comparable to explicit error estimators.

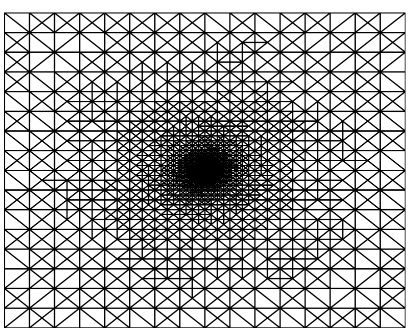

FIG. 10. *Mesh generated by ξ with one step V cycle H(div) MG solver.*

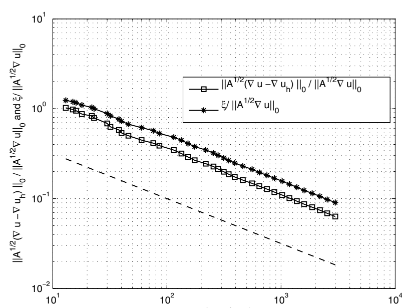

FIG. 11. *ξ with one step V cycle H(div) MG solver and error.*

7.3. Singularly perturbed reaction-diffusion problem. In this section, we report numerical results on one- and two-dimensional singularly perturbed reaction-diffusion problems in (2.8). A robust explicit residual-based estimator for this problem was studied by Verfürth in [38] (see also [27]) and defined as

$$\eta_V = \left(\varepsilon^{-1/2} \sum_{e\in\mathcal{E}} \alpha_K \|\varepsilon[\mathbf{n}\cdot\nabla u_\mathcal{T}]_e\|_{0,e}^2 + \sum_{K\in\mathcal{T}} \alpha_K^2 \|f+\varepsilon\Delta u_\mathcal{T} - u_\mathcal{T}\|_{0,K}^2 \right)^{1/2}, \tag{7.2}$$

where $\alpha_K = \min\{\varepsilon^{-1/2}h_K, 1\}$. This is the only existing robust estimator to the best of our knowledge.

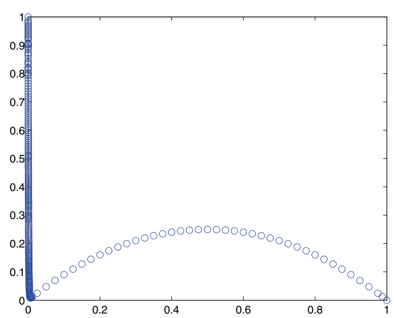

FIG. 12. *Numerical solution.*

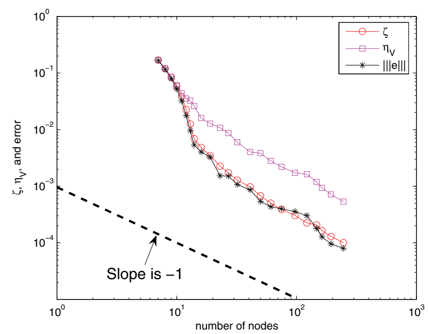

FIG. 13. *ζ, η_V, and error.*

The first test problem is a one-dimension singularly perturbed reaction-diffusion problem, i.e., (2.8) with

$$\Omega = (0,\,1), \quad \varepsilon = 10^{-6}, \quad f = 2\varepsilon + x(1-x), \quad \text{and} \quad \Gamma_N = \emptyset.$$

This problem has an exact solution $u = e^{-x/\sqrt{\varepsilon}} + x(1-x)$. Using linear element, $\Theta = 0.7$ in the Dörfler marking, and the stopping criterion when the error is less than 10^{-4}, a numerical solution on the finest mesh generated by ζ, is depicted in Figure 12. In Figure 13, we plot the estimators ζ and η_V defined in (5.2) and (7.2), respectively, and the true error on the adaptive meshes generated by ζ. It is clear that ζ is more accurate than η_V.

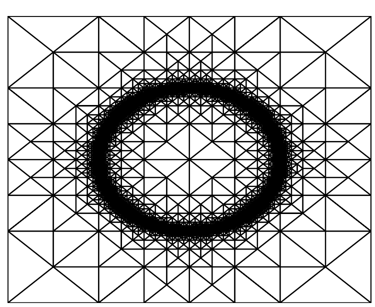

FIG. 14. *Mesh generated by ζ.*

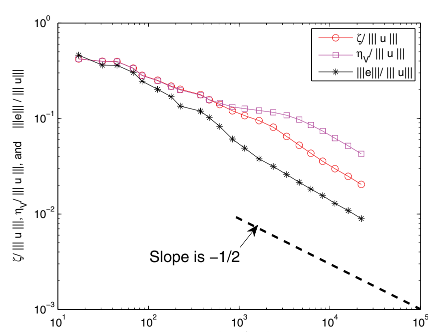

FIG. 15. *ζ, η_V, and error.*

The second test problem for a singularly perturbed reaction-diffusion equation is (2.8) in two dimensions with

$$\Omega = (-1,1)^2, \quad \varepsilon = 10^{-4}, \quad \text{and} \quad \Gamma_N = \emptyset.$$

The right-hand side f is chosen so that the exact solution is

$$u = \tanh(\varepsilon^{-1/2}\,(x^2 + y^2 - 1/4)) - 1.$$

This problem exhibits an interior layer along the circle of radius 1/2 centered at the origin. The coarsest triangulation is obtained by first partitioning the domain Ω into 4 squares with sides of length 1 and then dividing each square into two triangles by connecting the top-left and right-bottom corners of the square. Using linear element, $\Theta = 0.5$ in the Dörfler marking, and the stopping criterion when the error is less than 10^{-4}, the finest mesh generated by ζ is depicted in Figure 14, and the estimators ζ and η_V and the true error in the energy norm is reported in Figure 15. (Meshes generated by η are similar to those by ζ.) Again, it is clear that the ζ is more accurate than the η_V and that the mesh generated by ζ is optimal. Finally, the effectivity index $\zeta/\|\!|e|\!\|$ is about 2.3.

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