Robust Recovery Based a posteriori Error Estimators for Finite Element Methods

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A Posteriori Error Estimations

- Finite element solution u_h of a PDE, on mesh $\mathcal{T}$.
- Indicator $\eta_K(u_h)$ - a computable quantity for each $K \in \mathcal{T}$ based on the solution u_h and other known information
- Estimator $\eta(u_h) = \left(\sum_{K\in\mathcal{T}} \eta_K^2\right)^{1/2}$
- $$C_e^{-1}\eta \le \|u - u_h\| \le C_r\eta$$

  and

  $$c_e^{-1}\eta_K \le \|u - u_h\|_K$$

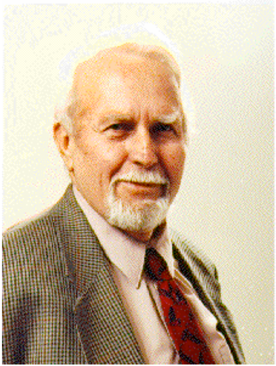
Recovery-Based Estimators

- Recovery-based Estimators:
  $\sigma(u_h)$: Quantity to be recovered: gradient (∇u_h) or flux ($-A\nabla u_h$) (or others)
  $G(\sigma(u_h))$ Recovered Quantity in a FE space V,

$$\eta_G = \|G(\sigma(u_h)) - \sigma(u_h)\|$$

- Several Whats and How:
  - Measure by what norm (measure error in what norm)?
  - What quantity?
  - Recover it in what finite element space?
  - How to recover?

Olgierd C. Zienkiewicz (May 18, 1921-Jan. 2, 2009)



A simple error estimator and adaptive procedure for practical engineerng analysis
OC **Zienkiewicz**, JZ **Zhu** - International Journal for Numerical Methods in Engineering, 1987 - interscience.wiley.com
... 0. C. **ZIENKIEWICZ** AND J. Z. **ZHU** Department of C id Engineering, L'niuersity of Swunseu,
Swunsea, SA2 8PP UK ... Page 6. 342 0. C. **ZIENKIEWICZ** AND JZ **ZHU** ...
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Zienkiewicz & Zhu error estimator

- Poisson equation, $-\Delta u = f$, solved by linear conforming FEM,
-
$$S_1 = \{v \in C^0(\Omega) | v|_K \in P_1(K), \forall\, K \in \mathcal{T}\}$$
  numerical solution $u_h \in S_1$
- ∇u_h is a piecewise constant vector, recover it in the standard piecewise linear continuous FE space S_1^2, and measure the error in H^1 semi norm

Zienkiewicz & Zhu error estimator

- Gradient Recovery by Zienkiewicz & Zhu: Find $G(\nabla u_h) \in S_1^2$,

$$G(\nabla u_h)(z) = \frac{1}{|\omega_z|} \int_{\omega_z} \nabla\, u_h \, dx \quad \forall\, z \in \mathcal{N}$$

  z is a vertex of $\mathcal{T}$, ω_z is the union of elements that shares the vertex z

- Error Estimator:

$$\eta_K = \|\nabla u_h - G(\nabla u_h)\|_{0,K}, \quad \eta = \|\nabla u_h - G(\nabla u_h)\|_{0,\Omega},$$

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A Test Problem for Zienkiewicz & Zhu error estimator

$-\Delta u = 1$ in $\Omega = (-1, 1)^2/[0, 1] \times [-1, 0]$, $u = 0$ on $\partial\Omega$.

L Shape domain, singularity at the origin

Solved by linear conforming FEM and adaptive refined by ZZ error estimator.

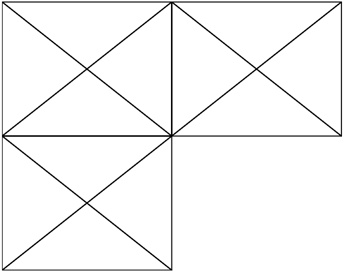

Figure: the initial mesh

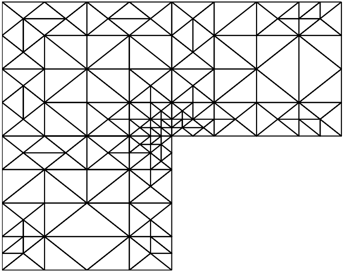

Figure: mesh after several refinements

A Test Problem

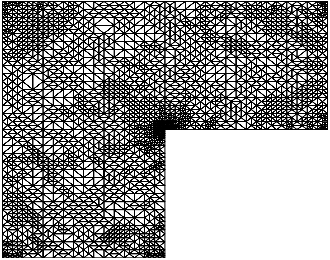

Figure: the mesh

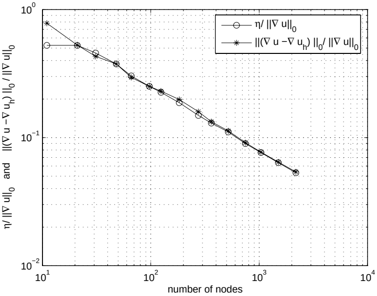

Figure: error and ZZ estimator η

Explanations

- Explanations from Superconvergence by Many Authors
  - $$\|\nabla u - G(\nabla u_h)\| \text{ is much smaller than } \|\nabla u - \nabla u_h\|$$
  - Unstructured meshes need special treating
  - Super-convergence needs high regularity of the solution (not satisfied in most of the problems)
  - Against the philosophy of Adaptive methods
- Reliability and efficiency bounds proof by Carstensen Group, 2002

$$C\|\nabla u_h - G(\nabla u_h)\| + h.o.t \leq \|\nabla u - \nabla u_h\| \leq C\|\nabla u_h - G(\nabla u_h)\| + h.o.t$$

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Zienkiewicz & Zhu error estimator

- Norm: H^1 semi norm
- Quantity: Gradient
- Space: Piecewise linear continuous FE
- How: Local patch averaging

Question ?

Are these choices still suitable for more general and complicated problems?

A Benchmark Test Ptoblem

interface problem

$$\begin{cases} -\nabla \cdot (a\nabla u) & = & 0 \;\text{ in } \Omega = (-1, 1)^2 \\ u & = & g \;\text{ on } \partial\Omega \end{cases}$$

with $a = R$ in Quadrants 1 and 3 and 1 in Quadrants 2 and 4

A Benchmark Test Ptoblem

- **exact solution**
  $u(r,\theta) = r^\alpha \mu(\theta) \in H^{1+\alpha-\epsilon}(\Omega)$ with

$$\mu(\theta) = \begin{cases} \cos((\frac{\pi}{2}-\sigma)\alpha)\cdot\cos((\theta-\frac{\pi}{2}+\rho)\alpha) & \text{if } 0 \le \theta \le \frac{\pi}{2}, \\ \cos(\rho\alpha)\cdot\cos((\theta-\pi+\sigma)\alpha) & \text{if } \frac{\pi}{2} \le \theta \le \pi, \\ \cos(\sigma\alpha)\cdot\cos((\theta-\pi-\rho)\alpha) & \text{if } \pi \le \theta \le \frac{3\pi}{2}, \\ \cos((\frac{\pi}{2}-\rho)\alpha)\cdot\cos((\theta-\frac{3\pi}{2}+\sigma)\alpha) & \text{if } \frac{3\pi}{2} \le \theta \le 2\pi. \end{cases}$$

- **example** $\alpha = 0.1 \Rightarrow u \in H^{1.1-\epsilon}(\Omega)$
  $R \approx 161.448$, $\rho = \pi/4$, and $\sigma \approx -14.923$.
- **Singularity only at the origin, not on the interfaces. The function u is equally good/bad in the 4 quadrants.**

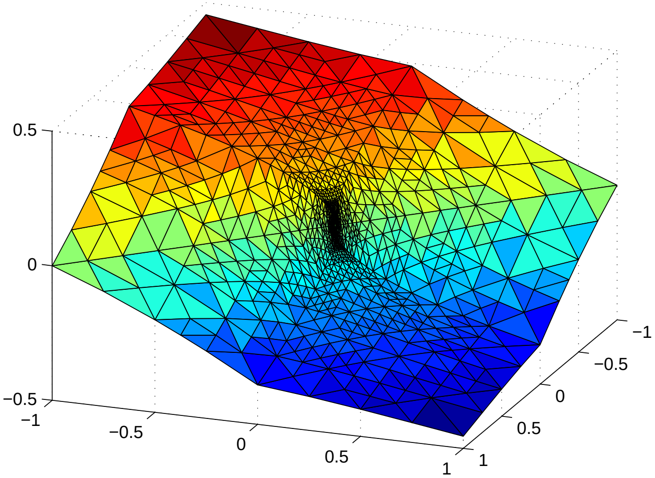

Figure: solution u

Energy norm, not H^1-semi norm

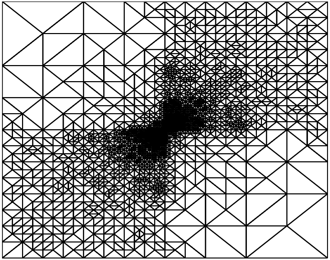

Figure: the mesh generated by Babuska-Miller error estimator corresponding to the H1 semi norm of the error

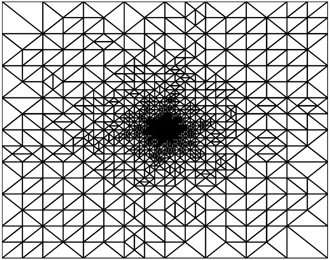

Figure: the mesh generated by Bernadi-Verfurth error estimator corresponding to the energy norm of the error

Energy norm, not H^1 semi norm

Quantity and Space

- ZZ (Zienkiewicz-Zhu) gradient recovery-based estimator: recover the gradient in continuous linear FE spaces
- Modified Carstensen flux recovery-based error estimator
  $\eta_C = \min_{\boldsymbol{\tau} \in S_1^d} \|a^{-1/2}(a\nabla u_h + \boldsymbol{\tau})\|_{0,\Omega}$.
  recover the flux in continuous linear FE spaces

Numerical Results by Gradient/Flux Recovery Error Estimators in Continuous Function Spaces S_1^2

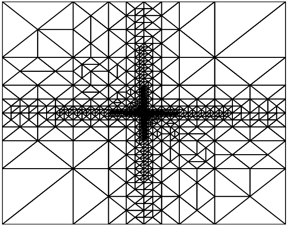

Figure: mesh by η_{ZZ}

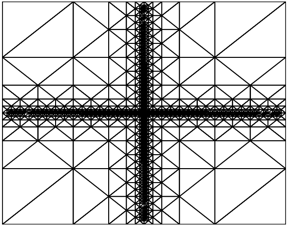

Figure: mesh by η_c

Lots of over refinements along the interface because of recovering in a wrong space!

Space and Quatity need to be reconsidered.

An Analysis of a Model Problem

- Diffusion Equations

$$-\nabla \cdot (A\nabla u) = f \in \Omega \quad f \in L^2(\Omega)$$

- Intrinsic Continuities of the Problem
  - Solution:

    $$u \in H^1(\Omega)$$

    "Continuous"
  - Gradient:

    $$u \in H^1(\Omega) \Rightarrow \nabla u \in H(\text{curl}; \Omega)$$

    Tangential Component of the Gradient is "Continuous"
  - Flux:

    $$\sigma = -A\nabla u \in H(\text{div}; \Omega)$$

    Normal Component of the Flux is "Continuous"
  - Both the flux σ and gradient ∇u are discontinuous, if A is discontinuous

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The reason ZZ error estimator fails for problem with discontinuous coefficients

- Recovered quantity: in a global continuous space
- True quantities: Gradient/Flux are not global continuous, only tangential or normal components are continuous!

Ask too much, thus, artificial, unnecessary over-refinements!

Guidelines of Recovery-Based Error Estimators

- FEM: violates the physical continuities of the one or more quantities.
- Recover a quantity whose continuity is *violated* by the method.
- in the conforming FE space where the true quantity lives in,

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Interface Problems

$$\begin{aligned} -\nabla \cdot (a(x)\nabla u) &= f \text{ in } \Omega \subset \mathcal{R}^d \\ u &= 0 \text{ on } \Gamma_D \\ \mathbf{n} \cdot (a(x)\nabla u) &= 0 \text{ on } \Gamma_N \end{aligned}$$

$a(x)$ is positive piecewise constant w.r.t $\bar{\Omega} = \cup_{i=1}^{n} \bar{\Omega}_i$,
$a(x) = a_i > 0$ in Ω_i

Linear Conforming FEM for Interface Problems

- Variational Problem: Find $u \in H^1_D(\Omega)$, such that

$$(a(x)\nabla u, \nabla v) = (f, v) \quad \forall\, v \in H^1_D(\Omega)$$

- FE space Piecewise linear continuous finite element space

$$S_1 := \{v \in C^0(\Omega) : v|_K \in P_1(K), \forall\, K \in \mathcal{T}\}$$

- Discrete Problem: Find $u_h \in S_{1,0} := S_1 \bigcap H^1_D(\Omega)$, such that

$$(a(x)\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall\, v_h \in S_{1,0}$$

Linear Conforming FEM for Interface Problems

- Comparison of Continuous and Discrete Solutions:

Gradient	$\nabla u \in H(\mathrm{curl};\Omega)$	$\nabla u_h \in \nabla S_{1,0} \subset H(\mathrm{curl};\Omega)$
Flux	$\sigma = -a\nabla u \in H(\mathrm{div};\Omega)$	$-a\nabla u_h \in \nabla S_{1,0} \not\subset H(\mathrm{div};\Omega)$

- Quantity to recover: the Flux (not the Gradient).
- In what space? $H(div;\Omega)$-conforming FE space RT_0 or BMD_1.
- What if in S_1^2? Over-refinements

Robust Flux Recovery Error Estimators for Interface Problems: Conforming FEs

- L^2-Projection Flux Recovery: Find $\sigma_h \in RT_0$, $\sigma_h \cdot \mathbf{n} = 0$ on Γ_N such that
$$(a(x)^{-1}\sigma_h, \boldsymbol{\tau}) = (-\nabla u_h, \boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in RT_0$$
(Preconditioned by diagonal matrix of the mass matrix, condition number is independent of h)
- Explicit Recovery: weighted averages of $-a\nabla u_h$
- Error Estimator:

$$\eta_K = \|a^{1/2}\nabla u_h + a^{-1/2}\sigma_h\|_{0,K}, \quad \eta = \|a^{1/2}\nabla u_h + a^{-1/2}\sigma_h\|_{0,\Omega},$$

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Robust Flux Recovery Error Estimators for Interface Problems: Conforming FEs

- Robustness: C_e and C_r is independent of the jumps of the coefficients across the interfaces

$$C_e^{-1}\eta + h.o.t \leq \|a^{1/2}(\nabla u - \nabla u_h)\|_{0,\Omega} \leq C_r\eta + h.o.t$$

- Accurateness: Observed in numerical tests that the effectivity constant is close to 1.
- Neumann BCs: Easy to incorporate Neumann Boundary Conditions into Error Estimators.
- No over refinements along the interface!

Numerical Results by Robust Flux Recovery Error Estimators

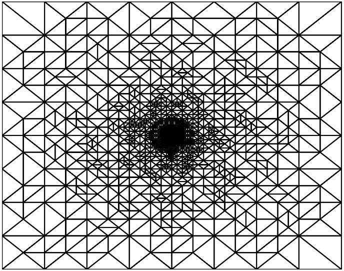

Figure: mesh generated by η

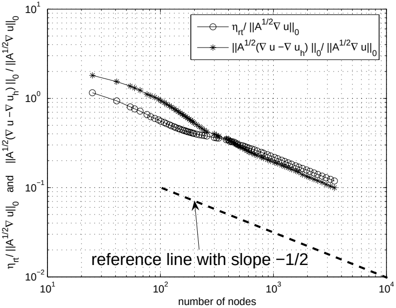

Figure: error and η

ZZ Error Estimators for Poisson Equations Revisited

- Comparison of Continuous and Discrete Solutions:

Gradient	$\nabla u \in H(\text{curl};\Omega)$	$\nabla u_h \in \nabla S_{1,0} \subset H(\text{curl};\Omega)$
Flux	$-\nabla u \in H(\text{div};\Omega)$	$-\nabla u_h \in \nabla S_{1,0} \not\subset H(\text{div};\Omega)$

- Quantity to recover: the Flux (same as the Gradient).
- Speciality for Poisson equation:
  $\nabla u \in H(\text{div};\Omega) \bigcap H(\text{curl};\Omega)$,
  $H^1(\Omega)^2 = H(\text{div};\Omega) \bigcap H(\text{curl};\Omega)$ for convex domains (or nice boundary)
  $H^1(\Omega)^2$ is a proper subset of $H(\text{div};\Omega) \bigcap H(\text{curl};\Omega)$ for non-convex domains
  $(S^1)^2$ good choice for Poisson equations on convex domains! Still has danger for non-convex domains.

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Other Finite element methods

- Mixed FEM: The numerical flux is in $H(div)$, but the numerical gradient is not in $H(curl)$, so recover the gradient in the $H(curl)$ conforming FE space.
- Nonconforming FEM: The numerical flux is in not $H(div)$, and the numerical gradient is not in $H(curl)$, so recover both the flux in the $H(div)$ conforming FE space, and the gradient in the $H(curl)$ conforming FE space.
- DG: The numerical flux is in not $H(div)$, so recover both the flux in the $H(div)$ conforming FE space. If a DG norm is used, the jump of the solutions cross the the edges can be used to measure the discontinuity of the solution.
- Elasticity, Stokes, Maxwell, more and more

Last Silde: the Source of the Error

Residual type error estimator

$$\eta^2 = \sum_{K \in \mathcal{T}} h_K^2 \| f - \nabla \cdot (A \nabla u_h) \|_{0,K}^2 + \sum_{e \in \mathcal{E}} h_K \| [\![A \nabla u_h \cdot \mathbf{n}]\!] \|_{0,e}^2$$

- The 1st term: Element residual
- The 2nd term: Edge residual, Edge jump,....
- Two sources of the error:
  - Residual in the strong form (PDE form)
  - Violations of the Intrinsic Continuities of the Physical Quantities/True Solutions
- Similar observations on mixed method, nonconforming method, DG,....

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