Spectral Stability of Spiral Waves in Models of Cardiac Tissue

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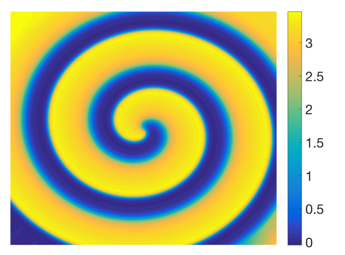
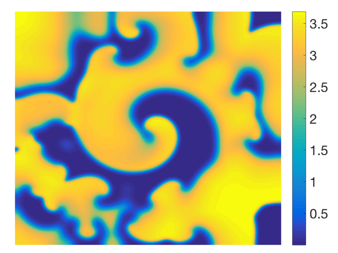
Spiral Waves in Cardiac Arrhythmias



Ventricular Tachycardia: Reentrant spiral waves create self-sustained oscillations.



Ventricular Fibrillation: Spiral wave breakup leads to unorganized self-sustained electrical activity.

- Cardiac dynamics and spiral waves described by reaction-diffusion systems

$$U_t = D\Delta U + F(U), \quad U \in \mathbb{R}^N, \quad D \in \mathbb{R}^{N\times N}, \quad F = (f_1(U), \dots, f_N(U)) \in \mathbb{R}^N$$

- Realistic ionic channel models have components without diffusion

Goals

The aim of this research is to investigate what spectral properties can tell us about the stability of spiral waves, in particular those in cardiac arrhythmias.

- Highlight differences in wave train and spiral spectra
- Investigate impact of diffusionless components on spiral spectra

Spiral Wave Properties

The Barkley Model is a simplified reaction-diffusion system for excitable media

$$\begin{cases} u_t &= \Delta u + \frac{1}{\alpha} u(1-u)\left(u - \frac{v+b}{a}\right) \\ v_t &= \delta \Delta v + u - v \end{cases}$$

Parameters a, b, δ, $\alpha \in \mathbb{R}$ control excitable threshold and fast/slow timescale.

Rigidly rotating spiral waves, $U_*(r,\psi)$, are stationary solutions in a rotating polar frame, $(r,\phi) \to (r,\psi) = (r, \phi - \omega t)$

$$0 = D\Delta_{r,\psi} U_* + \omega \partial_\psi U_* + F(U_*)$$

Spirals tend to 1D periodic **asymptotic wave trains**, U_∞, as $r \to \infty$

$$U_*(r,\psi) \to U_\infty(\kappa r + \psi) = U_\infty(\xi), \quad U_\infty(\xi) = U_\infty(\xi + 2\pi).$$

Wave trains are stationary solutions of

$$U_t = \kappa^2 D U_{\xi\xi} + \omega U_\xi + F(U).$$

Types of Spectra

Temporal Eigenvalues, λ, describe temporal growth of perturbations

$$\mathcal{L}V = D\Delta_{r,\psi} V + \omega V_\psi + F'(U_*)V = \lambda V.$$

Spatial Eigenvalues, ν, describe the spatial growth of eigenfunctions.

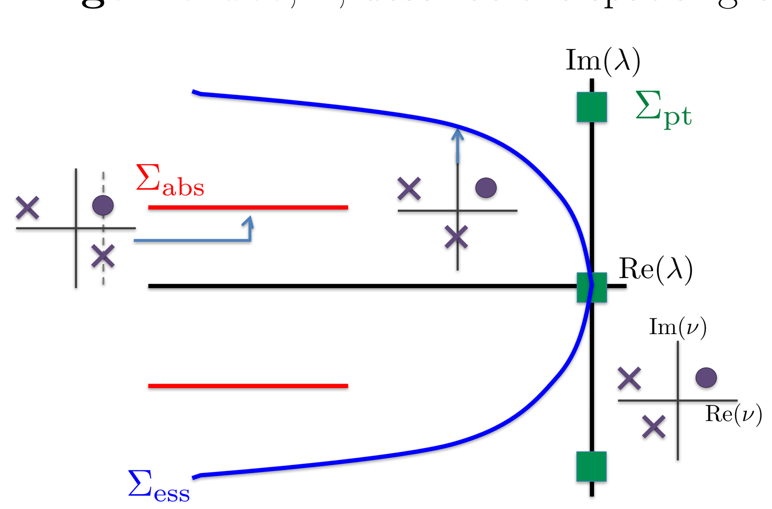


Figure: Cartoon of essential, absolute, and point spectra. Inserts show distribution of spatial eigenvalues.

On the plane:

- Point spectrum, Σ_{pt}
- Essential spectrum, Σ_{ess}

On bounded domain:

- Point spectrum, Σ_{pt}
- Absolute spectrum, Σ_{abs}

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Wave Train Spectra

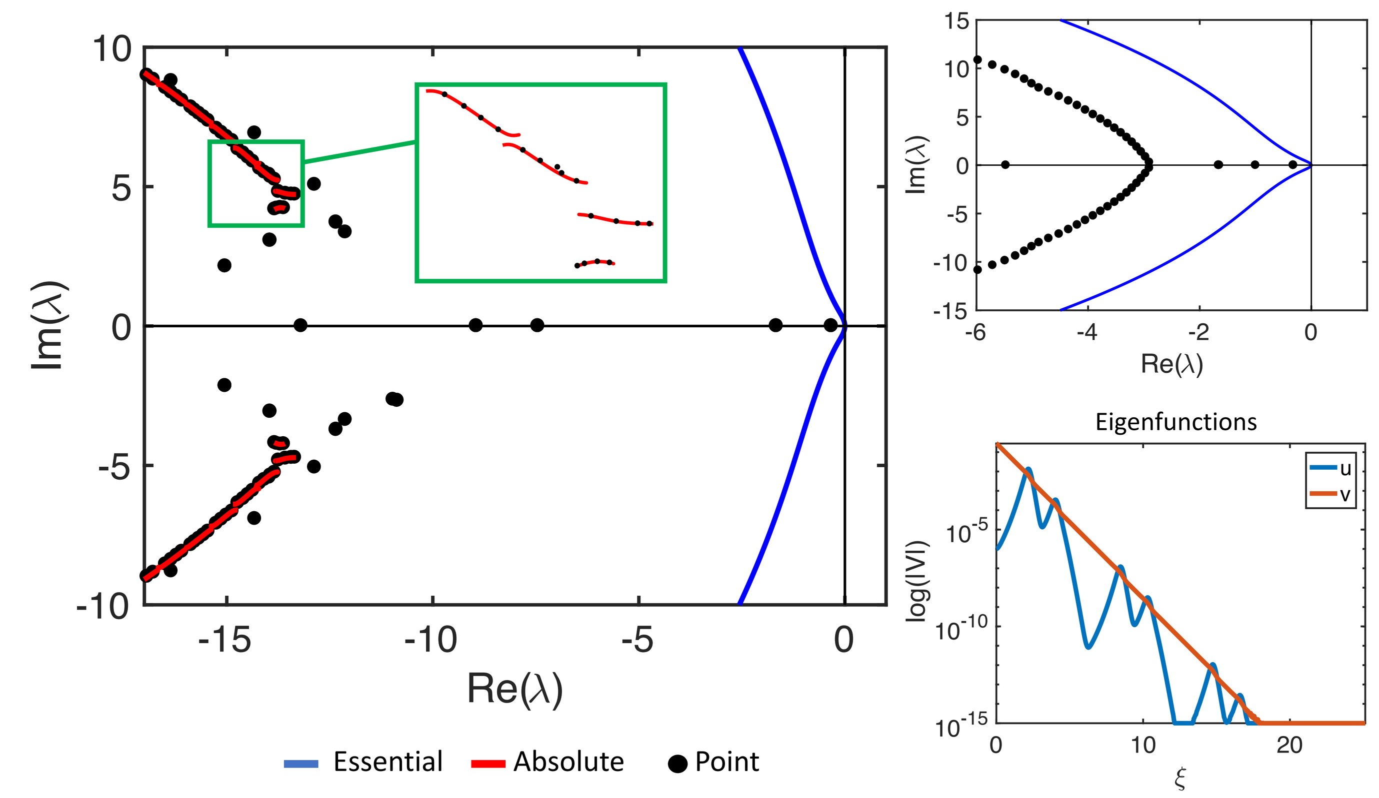


Figure: Essential, absolute, and point spectra of wave train (left). Insets show spurious eigenvalues approaching the essential spectrum (top right), and general form of eigenfunctions (bottom right). Parameters $\delta = 0.2$, $a = 0.7$, $b = 0.001$, $\alpha = 0.02$.

Observations:

- Absolute spectrum is a union of disjoint curves
- Non-normal linearized operator, $\mathcal{L}$, leads to eigenfunctions with similar spatial growth
- Direct calculation of point spectrum results in spurious eigenvalues approaching the essential spectrum
- Removing diffusion from v-equation ($\delta = 0$) results in minor spectral changes

Computation of Discrete Eigenvalues:

- Factor out spatial growth for better eigenfunction decomposition
- Let $V(\xi) = e^{w\xi}\tilde{V}(\xi)$, and form the weighted operator

$$\mathcal{L}_w \tilde{V} = \kappa^2 D (\partial_\xi + w)^2 \tilde{V} + \omega(\partial_\xi + w)\tilde{V} + F'(U_\infty)\tilde{V} = \lambda \tilde{V}$$

- Spatial growth factor, w, given by absolute spectrum

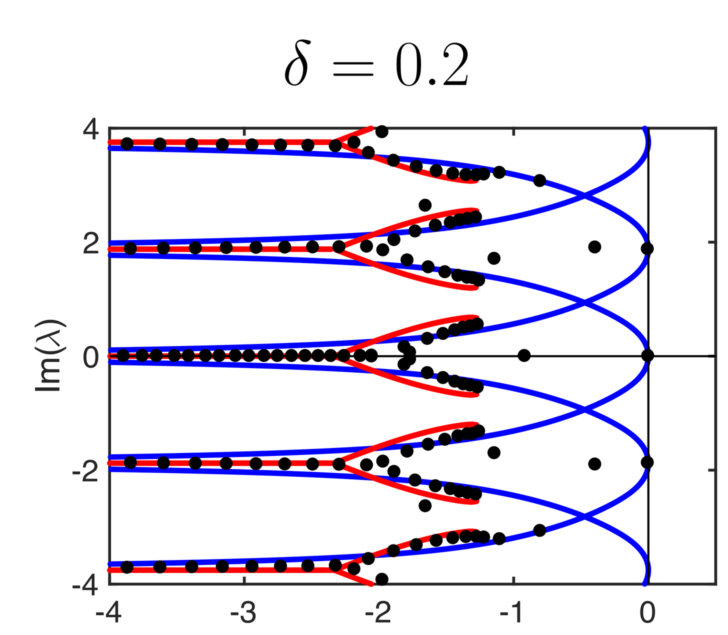
Spiral – Wave Train Relationship:

Dispersion relations of spiral, $\lambda_*(\nu_*)$, and wave train, $\lambda_\infty(\nu_\infty)$, are related via

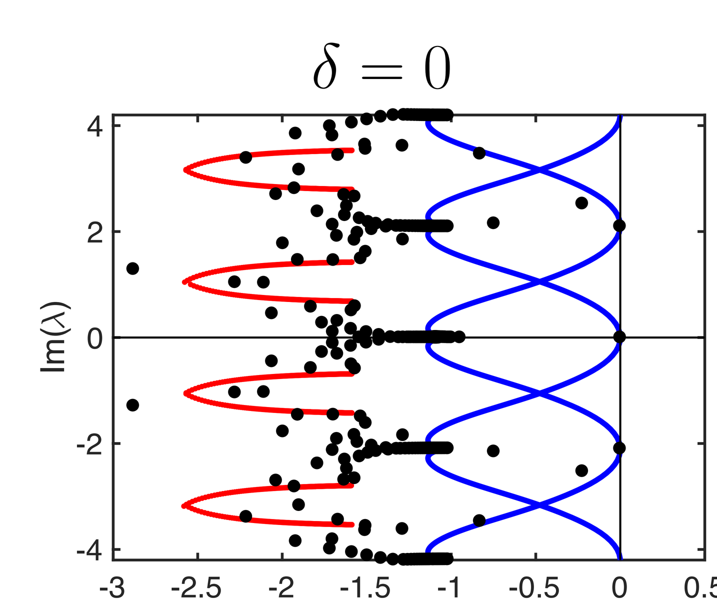
$$\lambda_*(\nu_*) = \lambda_\infty(\nu_\infty) - \omega\nu_\infty + i\omega\ell, \quad \ell \in \mathbb{Z}.$$

Effect of Diffusion on Spiral Spectra

Removing diffusion in v-equation changes properties of essential, absolute, and point spectra



- Many discrete eigenvalues approach absolute spectrum
- Few point eigenvalues near imaginary axis
- Expected behavior



- Few discrete eigenvalues near absolute spectrum
- Many eigenvalues approach endpoints of essential spectrum
- Unexpected behavior

Preliminary Explanation of Spiral Spectra for $\delta = 0$

- Using spiral eigenfunctions of the form

$$V_*(r,\psi;\lambda) = e^{\nu r} V_\infty(\kappa r + \psi) + \mathcal{O}\left(\frac{1}{r}\right),$$

the spiral eigenvalue problem reduces to the far-field form

$$\lambda V_\infty = D(\kappa\partial_\xi + \nu)^2 V_\infty + \omega \partial_\xi V_\infty + F'(U_\infty) V_\infty.$$

- As $\lambda \to -1 + i\ell\omega$, $\ell \in \mathbb{Z}$, we have $\nu \to i\gamma$, with $\gamma \in \mathbb{R}$, $\gamma \to \infty$.
- Let $x = \gamma\xi$, $\epsilon = 1/\gamma$, and seek eigenfunctions of the form $V_\infty(x) = (u, w, v)(x)$,

$$\begin{cases} u' &= w \\ w' &= -\epsilon^2 \frac{1}{\kappa^2}[\partial_u f_1(U_\infty)u + \partial_v f_1(U_\infty)v - \lambda u] - \epsilon\frac{\omega}{\kappa^2} w - \frac{1}{\kappa^2}[2\kappa w + u] \\ v' &= \frac{\epsilon}{\omega}[\partial_u f_2(U_\infty)u + \partial_v f_2(U_\infty)v + \lambda v] \end{cases}$$

- Using methods from Geometric Singular Perturbation Theory, u, $w \to 0$, and we are left with the slow dynamics, given by

$$\omega v' = (\partial_v f_2(U_\infty) + \lambda)\, v.$$

Conclusions

- Behavior of wave train and spiral spectra can be very different
- Observe unexpected behavior of spectrum in case with diffusionless components
- Non-normality of operator needs to be taken into account in spectral calculations
- Similar results for spiral spectra in the more realistic Karma model

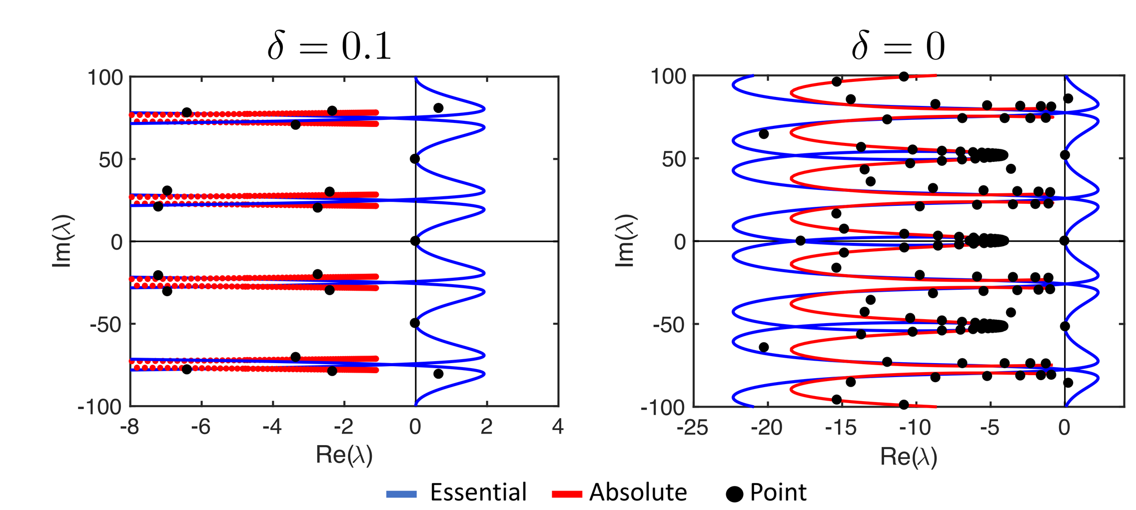


Figure: Essential, absolute, and point spectra for spiral in the Karma model.

Future Work

- Analyze more realistic cardiac models with and without full diffusion
- Investigate use of weighted operator in calculation of spiral spectra
- Apply knowledge to understand the alternans instability
- Determine if stable alternans patterns exist

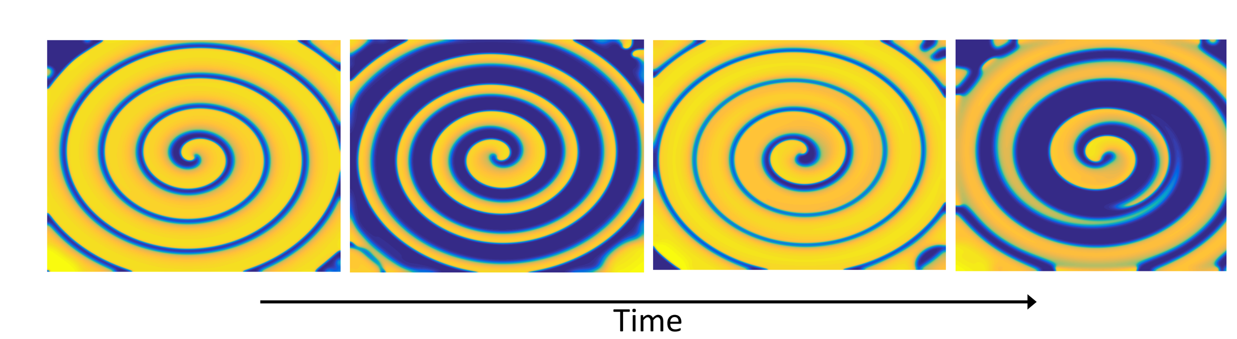


Figure: Spiral breakup by alternans in the Karma Model. 16 cm square with Neumann boundary conditions.

References

[1] D. Barkley, *Euclidean symmetry and the dynamics of rotating spiral waves*, Phys. Rev. Lett. **72** (1994), 164-167.

[2] K. J. Groot, *et. al.*, *Global stability spectar of absolute and convective instabilities*, Journal of Fluid Mechanics, Submitted (2017).

[3] A. Karma, *Electrical alternans and spiral wave breakup in cardiac tissue*, Chaos, **4**(3) (1994), 461 - 472.

[4] J. D. M. Rademacher, *et. al*, *Computing absolute and essential spectra using continuation,* Physica D **229** (2007), 166-183.

[5] D. S. Rosenbaum, *et. al. Electrical alternans and vulnerability to ventricular arrhythmias.* New England Journal of Medicine. (1994), 330(4): 235-241.

[6] M. Rubart, D. P. Zipes, *Mechanisms of sudden cardiac death,* Journal of Clinical Investigation. (2005); 115(9): 2305-2315.

[7] B. Sandstede and A. Scheel, *Absolute and convective instabilities of spiral waves,* Phys. Rev. E, **62** (2000), 7708-7714.