Unpeeling a homoclinic banana in the FitzHugh–Nagumo system

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Abstract

The FitzHugh–Nagumo equations are known to admit fast traveling pulse solutions with monotone tails. It is also known that this system admits traveling pulses with exponentially decaying oscillatory tails. Upon numerical continuation in parameter space, it has been observed that the oscillations in the tails of the pulses grow into a secondary excursion resembling a second copy of the primary pulse. In this paper, we outline in detail the geometric mechanism responsible for this single-to-double-pulse transition, and we construct the transition analytically using geometric singular perturbation theory and blow-up techniques.

1 Introduction

The FitzHugh–Nagumo system is a singularly perturbed reaction-diffusion partial differential equation (PDE) which arose as a simplification of the Hodgkin–Huxley model [18] for the propagation of nerve impulses in axons. We consider this system in the form

\[ \begin{align*}
    u_t & = u_{xx} + f(u) - w, \\
    w_t & = \delta(u - \gamma w),
\end{align*} \] (1.1)

for \( x \in \mathbb{R} \), where \( f(u) = u(u-a)(1-u) \), with \( 0 < a < \frac{1}{2}, 0 < \delta \ll 1 \), and \( \gamma > 0 \).

We are interested in solutions resembling nerve impulses. Such solutions, which we refer to as pulses, correspond to traveling waves that propagate with constant speed and are localized, i.e. the profiles converge to zero as \( |x| \) goes to infinity.

To find traveling waves, we search for solutions of the form \((u, w)(x, t) = (u, w)(x + ct)\) for wavespeed \( c > 0 \).

Finding traveling pulse solutions to (1.1) is equivalent to finding localized solutions of the traveling wave ODE

\[ \begin{align*}
    \dot{u} & = v, \\
    \dot{v} & = cv - f(u) + w, \\
    \dot{w} & = \epsilon(u - \gamma w),
\end{align*} \] (1.2)

where we abuse notation and denote by \( \dot{\cdot} = \frac{d}{dt} \) differentiation with respect to the traveling wave variable \( \xi = x + ct \), and where \( 0 < \epsilon = \delta/c \ll 1 \). In addition, we take \( \gamma > 0 \) sufficiently small so that \((u, v, w) = (0, 0, 0)\) is the only equilibrium of the system (1.2). In all of our numerical computations, we fix \( \gamma = 0.5 \).

It is well known that for each \( 0 < a < 1/2 \) and each sufficiently small \( \epsilon > 0 \), (1.1) admits “slow” and “fast” traveling pulse solutions. Equivalently, in (1.2) this corresponds to the existence of orbits homoclinic to the only equilibrium \((u, v, w) = (0, 0, 0)\) with constant wave speeds \( c \): Slow pulses have wave speeds close to zero and arise as regular perturbations from the limit \( \epsilon \to 0 \). Fast pulses have speeds that are bounded away from zero as \( \epsilon \to 0 \) and cannot be constructed as regular perturbations from \( \epsilon = 0 \) as in the case of slow pulses. The existence
Figure 1: Shown is the bifurcation diagram indicating the known regions of existence for pulses in (1.2). Pulses on the upper branch are referred to as “fast” pulses, while those along the lower branch are called “slow” pulses. These two branches coalesce near the point \((c, a, \epsilon) = (0, 1/2, 0)\). Also shown are profiles of a fast pulse with monotone tail and fast pulse with oscillatory tail obtained numerically for the parameter values \((c, a, \epsilon) = (0.593, 0.069, 0.0036)\) and \((c, a, \epsilon) = (0.689, 0.002, 0.0036)\), respectively.

result for fast pulses has been obtained using a number of different techniques: classical singular perturbation theory [17], Conley index [5], and geometric singular perturbation theory [21]. This last viewpoint is the one we shall adopt. The main idea of geometric singular perturbation theory [11] is to use the small parameter \( \epsilon \) to separate the analysis of the system (1.2) into slow and fast components. These components are studied separately and pieced together to construct solutions to the full system.

A schematic bifurcation diagram depicting the existence results for pulses is shown in Figure 1. The existence region is composed of two branches: the upper branch represents the fast pulses, and the lower branch represents the slow pulses. It has been shown [22] that near the point \((c, a, \epsilon) = (0, 1/2, 0)\), these two branches coalesce and form a surface as shown.

In general, both slow and fast pulses as described above have monotone tails as \(x \to \pm \infty\). However, it has been shown [7] that (1.1) also admits fast traveling pulses with small amplitude, exponentially decaying oscillatory tails in the region near the point \((c, a, \epsilon) = (1/\sqrt{2}, 0, 0)\), which corresponds to the upper left corner of the bifurcation diagram in Figure 1. Figure 1 also shows profiles of a fast monotone and fast oscillatory pulse obtained numerically. Such pulse solutions with oscillatory tails are interesting due to the possibility of constructing multi-pulses, which consist of several well-separated copies of the original pulse [19, §5.1.2]. We also note that pulses with oscillatory tails have been observed in FitzHugh–Nagumo-type systems with cross diffusion terms [31].

Combining these two different existence results for fast pulses, the well known classical existence result [21] in the region where \(0 < \epsilon \ll a < \frac{1}{2}\), and the extension [7] to the regime \(0 < a, \epsilon \ll 1\), encompassing the onset of oscillations in the tails of the pulses, we have the following theorem.

**Theorem 1.1** ([7, 21]). There exists \(K^* > 0\) such that for each \(0 < \kappa < 1/2\) and \(K > K^*\) the following holds. There exists \(\epsilon_0 > 0\) such that for each \((a, \epsilon) \in \left[0, \frac{1}{2} - \kappa\right] \times (0, \epsilon_0)\) satisfying \(\epsilon < Ka^2\) system (1.2) admits a traveling-pulse solution with wave speed \(c = \tilde{c}(a, \epsilon)\) approximated uniformly in \(a\) by

\[
\tilde{c} = \sqrt{2} \left(\frac{1}{2} - a\right) + O(\epsilon).
\]

Furthermore, if we have in addition \(\epsilon > K^*a^2\), then the tail of the pulse decays in an oscillatory fashion.

We remark that while the slow pulses are known to be unstable in the PDE (1.1), it was proved independently by Jones [20] and Yanagida [30] that the fast pulses (with monotone tails) are stable for each fixed \(0 < a < \frac{1}{2}\) provided \(\epsilon > 0\) is sufficiently small. This stability result has also been recently extended [6] to all pulses in Theorem 1.1, including those with oscillatory tails.
This paper is concerned with understanding the following phenomenon associated with the above traveling pulses: when continuing these pulses numerically in the parameters (c, a) for fixed \( \epsilon \), the continuation traces out a C-shaped (or rather, backwards C-shaped) curve; see Figure 2a. This is to be expected when considering an \( \epsilon = \text{const} \) slice of the bifurcation diagram in Figure 1. When approaching the upper left corner of this bifurcation diagram, the pulses develop oscillations in the tails as described above, but the curve does not terminate; instead, as shown in Figure 2, the curve turns back sharply, and the oscillations in the tails of the pulses grow into a secondary pulse resembling the primary pulse via a mechanism resembling a canard explosion. The curve then retraces itself, and the secondary pulse transitions back into a single pulse near the lower left corner of the bifurcation diagram. Plotting the parameter \( a \) versus the \( L^2 \)-norm of the solution as in Figure 2b shows that this C-curve is indeed composed of two curves forming a so-called homoclinic banana [8]. This homoclinic C-curve and banana are shown in Figure 2, as well as a single and double pulse on either side of this sharp transition.

Numerical explorations of the FitzHugh–Nagumo system have resulted in possible explanations for the termination of the branch of pulses in the upper left corner of the C-curve and the structure of the homoclinic banana [8, 14, 15]. However, the exact nature of the sharp turn in the C-curve, and in particular the relation to the transition between the single and double pulse, is not understood. In [7], we proposed a geometric mechanism which explains this transition, best visualized in the three-dimensional phase space: Figure 3 shows a zoom of the upper left part of the banana for a lower value of \( \epsilon \) as well as six different pulses along the curve plotted.
together. See the accompanying movie for a visualization of the transition.

The goal of this paper is to construct this transition analytically using geometric singular perturbation theory and blow up techniques as in the construction of the oscillatory pulses in [7, Theorem 1.1]. Specifically we aim to extend the existence result Theorem 1.1 for fast pulses to include this transition by proving the following theorem, which for now we state informally.

**Theorem.** For each sufficiently small \( \epsilon > 0 \), there exists a one-parameter family

\[
s \to (c, a, \Gamma)(s, \sqrt{\epsilon}), \quad s \in (0, s_0)
\]

of traveling pulse solutions \( \Gamma(s, \sqrt{\epsilon}) \) to (1.2), which is \( C^1 \) in \( (s, \sqrt{\epsilon}) \). Furthermore, for \( s \) sufficiently small, the solutions \( \Gamma(s, \sqrt{\epsilon}) \) coincide with the single pulses with oscillatory tails from Theorem 1.1, while the solutions \( \Gamma(s, \sqrt{\epsilon}) \) are double pulses for \( s \) sufficiently close to \( s_0 \).

The construction is based on the slow-fast decomposition of the traveling wave ODE (1.2):

\[
\begin{align*}
\dot{u} &= v \\
\dot{v} &= cv - f(u) + w \\
\dot{w} &= \epsilon(u - \gamma w),
\end{align*}
\]

The singular \( \epsilon = 0 \) limit consists of slow flow on the critical manifold \( \{(u, v, w) = (u, 0, f(u))\} \) and fast layer dynamics restricted to planes \( w = \text{const} \) in which there exist front solutions connecting different branches of the critical manifold. The transitional pulses are then obtained as perturbations from singular limit structures obtained by concatenating fast jumps along front solutions in the layer dynamics and slow evolution along portions of the critical manifold. The geometric setup can be seen in Figure 3 where the slow (resp. fast) portions of the flow are labelled by single (resp. double) arrows. The corresponding singular limit structure is also shown in Figure 7 below.

Obtaining the transitional pulse solutions as perturbations from the singular structure poses a number of challenges, and the techniques required to construct such solutions extend beyond standard methods of geometric singular perturbation theory. A primary challenge arises from the fact that normal hyperbolicity is lost at two fold points on the critical manifold, and as the pulses in question pass near these points, blow-up techniques are needed to understand the flow in these regions. The resulting geometric matching procedure used to construct the pulses, while tailored to this particular problem, is in fact quite general: The slow/fast pieces of the pulse and passages through fold points are all matched together to construct the full solution, taking into account both hyperbolic and nonhyperbolic aspects of the flow near the folds.

We now comment on a number of phenomena that are apparent in Figure 3:

- The first is the observation that much of the transition occurs along a nearly vertical path in parameter space (see Figure 3a). We will show that this is due to the fact that the transition from a single to a double pulse is organized by a canard mechanism similar to that which generates a canard explosion of periodic orbits [24]. In this sense the transition can be viewed as a “canard explosion of homoclinic orbits,” and we show that for each fixed \( \epsilon \), much of the transition is confined to an exponentially thin region in \((a, c)\) parameter space.

- The second observation is that the transitional pulses all have oscillatory tails. As in [7], we will show that the oscillations in the tails are due to the presence of a so-called Belyakov transition occurring at the equilibrium at the origin. In the linearization of (1.2) about \((u, v, w) = (0, 0, 0)\) this transition is characterized by two real eigenvalues which collide and then split as a complex conjugate pair; when a
Figure 3: Transition from single to double pulse in the top left of the homoclinic banana for $\epsilon = 0.0036$ (a visualization of this transition is shown in the accompanying movie). The solutions labelled 1, 2 are left pulses, and those labelled 4, 5, 6 are right pulses. The solution labelled 3 lies in the transition between left/right pulses. Pulse 1 follows the sequence (a), (b), (c), (d), followed by an oscillatory tail. Pulses 2, 3, 4, 5, 6 first follow the sequence (a), (b), (c), (d), followed by a secondary pulse along the corresponding colored trajectory, followed by an oscillatory tail. Note in the zoomed in portion of (c) that pulses labelled 2, 3, 4, 5 appear to have nearly identical/overlapping tail trajectories.
Figure 4: Termination of the homoclinic banana and C-curve at the Belyakov transition for $\epsilon = 0.0036$.

A homoclinic orbit is present in this situation, it is referred to as a Belyakov transition [19, §5.1.4]. In [7], it was shown that for sufficiently small $a, \epsilon > 0$ this transition occurs when

$$\epsilon = \frac{\sqrt{2a^2}}{4} + O(a^3),$$

and proving the existence of pulses with oscillatory tails as in Theorem 1.1 amounted to showing that pulses exist on either side of this transition (see Figure 1 — the red curve denotes the location of the Belyakov transition). The presence of the oscillatory tails in the single-to-double-pulse transition indicates it occurs to the left of the Belyakov transition. For $\epsilon$ large enough, the entire banana sits to the left of the Belyakov transition. For $\epsilon$ large enough, the entire banana sits to the left of the Belyakov transition, though for sufficiently small $\epsilon$, we see numerically (Figure 4) that the banana splits, and only single pulses exist to the right of the Belyakov transition.

The remainder of this paper is organized as follows. In §2, we outline the setup and give a precise statement of the main result, Theorem 2.2. The pulse solutions are constructed in §3 save for a few technical results which are proved in §4 and §5.
2 Setup

In this section, we outline the setup and statement of the problem in the context of geometric singular perturbation theory. In §2.1, we describe properties of the reduced/layer problems, followed by §2.2, where we give a description of the geometric mechanism responsible for the single-to-double pulse transition and the definition of the singular transitional pulses from which we will construct the solutions. Then in §2.3, we give a statement of the main result of this paper, Theorem 2.2.

The remainder of this section is then devoted to collecting technical results which will be useful in the proof of Theorem 2.2: In §2.4, we collect associated results regarding persistence of invariant manifolds which follow from standard geometric singular perturbation theory. Finally, in §2.5 and §2.6 we collect results regarding the persistence of a certain extended invariant manifold and the existence of associated maximal canard solutions on this manifold.

2.1 Slow and fast subsystems

We separately consider the traveling wave ODE

\begin{align}
\dot{u} &= v, \\
\dot{v} &= cv - f(u) + w, \\
\dot{w} &= \epsilon(u - \gamma w),
\end{align}

(2.1)

which we call the fast system, and the system below obtained by rescaling time as \( \tau = \epsilon t \), which we call the slow system:

\begin{align}
\epsilon u' &= v, \\
\epsilon v' &= cv - f(u) + w, \\
w' &= (u - \gamma w),
\end{align}

(2.2)

where \( ' \) denotes \( \frac{d}{d\tau} \). The two systems (2.1) and (2.2) are equivalent for any \( \epsilon > 0 \). The idea of geometric singular perturbation theory is to determine properties of the \( \epsilon > 0 \) system by piecing together information from the simpler equations obtained by separately considering the fast and slow systems in the singular limit \( \epsilon = 0 \).

2.1.1 The critical manifold \( \mathcal{M}_0(c, a) \)

We first set \( \epsilon = 0 \) in (2.2) and obtain the reduced system

\begin{align}
0 &= v, \\
0 &= cv - f(u) + w, \\
w' &= (u - \gamma w),
\end{align}

(2.3)

where the flow is now restricted to the set \( \mathcal{M}_0(c, a) = \{(u, v, w) : v = 0, w = f(u)\} \), called the critical manifold with flow determined by the equation for \( w \).

2.1.2 The fronts \( \phi_f, \phi_b \)

We now set \( \epsilon = 0 \) in (2.1), and we obtain the layer problem

\begin{align}
\dot{u} &= v, \\
\dot{v} &= cv - f(u) + w, \\
\dot{w} &= 0,
\end{align}

(2.4)
so that $w$ becomes a parameter for the flow and $M_0(c,a)$ is the set of equilibria. Considering this system in the plane $w = 0$, we obtain the Nagumo system

$$\begin{align*}
\dot{u} &= v \\
\dot{v} &= cv - f(u).
\end{align*}$$

(2.5)

Fix $a_0$ sufficiently small. It can then be shown that for each $-a_0 \leq a \leq 1/2$, this system possesses a heteroclinic connection $\phi_f$ (the Nagumo front) between the critical points $(u,v) = (0,0)$ and $(u,v) = (1,0)$ with wavespeed $c = c^*(a) = \sqrt{2}(1/2 - a)$.

For $a > 0$, in (2.4), this manifests as a connection between normally hyperbolic segments of the left and right branches, $M_0^r(c,a)$ and $M_0^l(c,a)$, of the critical manifold $M_0(c,a)$ in the plane $w = 0$. By symmetry, there exists $w_b(a)$ such that there is a connection $\phi_b$ (the Nagumo back) in the plane $w = w_b(a)$ between the right and left branches of $M_0(c,a)$ traveling with the same speed $c = c^*(a)$ (see Figure 5). In the case $a = 1/2$, $\phi_f$ and $\phi_b$ form a heteroclinic loop, though we do not consider this case here; see [22].

For $-a_0 < a \leq 0$, this system possesses front type solutions for any $c > 1/\sqrt{2}(1 + a)$ connecting $u = 0$ to $u = 1$. For the critical value $c = c^*(a) = \sqrt{2}(1/2 - a)$ the front leaves the origin along the strong unstable manifold of the origin, and for all other values of $c$, the front leaves the origin along a weak unstable direction. In the case of $a = 0$, (2.5) reduces to a Fisher–KPP type equation

$$\begin{align*}
\dot{u} &= v \\
\dot{v} &= cv - u^2(1 - u),
\end{align*}$$

(2.6)

and the front/back solutions $\phi_f, \phi_b$ leave $M_0^r(c,a)$ and $M_0^l(c,a)$ from the nonhyperbolic fold points (see Figure 5). For the critical value $c = 1/\sqrt{2}$ the front $\phi_f$ leaves the origin along the strong unstable manifold of the origin, and for $c > 1/\sqrt{2}$, the front leaves the origin along a center manifold. We are concerned with the case of $(c,a) = (1/\sqrt{2},0)$ in which the singular fast front solution $\phi_f$ leaves the origin along the strong unstable manifold. Note that by symmetry, for $(c,a) = (1/\sqrt{2},0)$, the fast singular back solution $\phi_b$ also leaves the upper right fold point along the strong unstable direction.

**Remark 2.1.** For each $0 \leq a < 1/2$, there is a singular pulse solution obtained by following the sequence $\phi_f, M_0^r(c,a), \phi_b, M_0^l(c,a)$ (see Figure 5). The existence Theorem 1.1 is obtained by searching for perturbations from

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**Figure 5:** Shown are the singular fronts $\phi_f$ and $\phi_b$ for the layer problem (2.4) for $\epsilon = 0$ and $0 \leq a \leq 1/2$. 
these singular pulses for sufficiently small \( \epsilon > 0 \). The classical monotone pulses are obtained as perturbations from the case of \( 0 < a < 1/2 \), whereas the onset of oscillations in the tails is found by perturbing from \( \epsilon = a = 0 \).

2.1.3 The fronts \( \phi_\ell(w), \phi_r(w) \)

To understand the transitional pulses, we need more information from the layer problem (2.4), specifically regarding properties of the middle branch \( M^m_0(c,a) \) of the critical manifold \( M_0(c,a) \). For \( a = 0 \) there are also fronts connecting \( M^m_0(c,a) \) to \( M^l_0(c,a) \) and \( M^r_0(c,a) \) for values of \( w \in [0, w^\dagger] \), where \( (u^\dagger, 0, w^\dagger) = (2/3, 0, 8/27) \) denotes the location of the upper right fold point for \( (c, a) = (1/\sqrt{2}, 0) \) (see Figure 6). To see this, we study the two-dimensional system

\[
\begin{align*}
\dot{u} &= v \\
\dot{v} &= cv - u^2(1 - u) + w,
\end{align*}
\]

From §2.1.2 above, when \( w = 0 \), (2.7) has two equilibria at \( (u, v) = (0, 0) \) and \( (u, v) = (1, 0) \), and when \( c = 1/\sqrt{2} \), there is a front \( \phi_f \) connecting these two equilibria; when \( w = w^\dagger \) there is a ‘back’ \( \phi_b \) connecting the two equilibria at \( (u, v) = (2/3, 0) \) and \( (u, v) = (-1/3, 0) \).

For values of \( w \in (0, w^\dagger) \), (2.7) has three equilibria \( p_i = (u_i(w), 0) \), where \( u_i(w), i = 1, 2, 3 \) are the three solutions of \( f(u) = w \) in increasing order. The outer equilibria are saddles, while the middle equilibrium \( p_2 \) is completely unstable. To compute the type, we note that the eigenvalues at \( p_2 \) are given by

\[
\lambda = \frac{c \pm \sqrt{c^2 - 4f'(u_2(w))}}{2}.
\]

We define \( w_A \) to be the lesser of the two solutions of \( c^2 = 4f'(u_2(w)) \), and we refer to the point \( (u_2(w_A), 0, w_A) \) as the Airy point. Hence for \( (c, a) = (1/\sqrt{2}, 0) \), the equilibrium \( p_2 \) is an unstable node for \( w \in (0, w_A) \cup (w^\dagger - w_A, w^\dagger) \), a degenerate node at \( w = w_A, w^\dagger - w_A \), and an unstable spiral for \( w \in (w_A, w^\dagger - w_A) \). For each \( w \in (0, w^\dagger) \), there exists a front \( \phi_\ell(w) \) connecting the equilibria \( p_2 \) and \( p_1 \), and a front \( \phi_r(w) \), connecting the equilibria \( p_2 \) and \( p_3 \) (see Figure 6). The existence of these fronts as well as their properties are outlined in Proposition A.1.
2.2 Singular transitional pulses

In this section we construct singular transitional pulses for \((c, a, \epsilon) = (1/\sqrt{2}, 0, 0)\) using the layer analysis above, and we refer to Figure 7 for the underlying geometry. We define

\[
\mathcal{M}(u_1, u_2) := \{(u, 0, f(u)) : u \in [u_1, u_2]\}
\]

(2.9)

All singular pulses will consist of a single pulse

\[
\Gamma^1_0 = \phi_f \cup \mathcal{M}(u^\dagger, 1) \cup \phi_b \cup \mathcal{M}(u^\dagger - 1, 0),
\]

(2.10)

followed by a secondary pulse. The secondary pulse follows a canard-like explosion which we parametrize by \(s \in (0, w^\dagger)\). We define the singular secondary pulses

\[
\Gamma^2_0(s) := \begin{cases} 
\mathcal{M}(0, u_2(s)) \cup \phi_\ell(s) \cup \mathcal{M}(u_1(s), 0), & s \in (0, w^\dagger) \\
\mathcal{M}(0, u^\dagger) \cup \phi_b \cup \mathcal{M}(u^\dagger - 1, 0), & s = w^\dagger \\
\mathcal{M}(0, u_2(2w^\dagger - s)) \cup \phi_r(2w^\dagger - s) \cup \mathcal{M}(u^\dagger, u_3(2w^\dagger - s)) \cup \phi_b \cup \mathcal{M}(u^\dagger - 1, 0), & s \in (w^\dagger, 2w^\dagger).
\end{cases}
\]

(2.11)

We refer to singular transitional pulses \(\Gamma_0(s) = \Gamma^1_0 \cup \Gamma^2_0(s)\) as “left” transitional pulses for \(s \in (0, w^\dagger)\) and “right” transitional pulses for \(s \in (w^\dagger, 2w^\dagger)\). The left/right descriptor refers to whether the double pulse involves a jump from \(\mathcal{M}^m_0\) to the left branch \(\mathcal{M}^\ell_0\) or a jump to the right branch \(\mathcal{M}^r_0\). These two types of singular pulses are shown in Figure 7.

We also define the two-dimensional singular tail manifolds \(T_0(\bar{w})\) for \(\bar{w} \in (0, w_A)\) by

\[
T_0(\bar{w}) = \bigcup_{w \in (0, \bar{w})} \phi_\ell(w),
\]

where the fronts \(\phi_\ell(w)\) are defined as in Proposition A.1.

2.3 Statement of the main result

The goal of this paper is to prove the following existence theorem for a one-parameter family of homoclinic solutions to (1.2) which encompasses the transition from single pulses with oscillatory tails from [7, Theorem 1.1] to double pulse solutions comprised of a single pulse followed by a secondary excursion which is close to the original pulse.

Figure 7: Singular \(\epsilon = 0\) double pulses for \((c, a) = (1/\sqrt{2}, 0)\).
Remark 2.3. The exponential closeness of the parameter values \((c, a)\) along the transition in Theorem 2.2(iii) is related to the existence of maximal canards in a center manifold of the equilibrium, similarly to the case of the classical canard explosion [24].

Remark 2.4. The termination of the branch of transitional pulses in Theorem 2.2 at \(s = 2w^\dagger - s_{\text{end}}(\sqrt{\ell})\) is related to the presence of the Belyakov transition \((1.3)\) discussed in the introduction and the split banana appearing in the numerical continuation for sufficiently small \(\ell\) (see Figure 4). As a perturbation from the \(\ell = 0\) limit, we are able to construct pulses encompassing the transition from the single pulse to the double pulse up to a neighborhood of the Belyakov transition, but not the entire banana. We will see that our construction indeed breaks down in this region (see Remark 3.14 for further details).

Remark 2.5. We comment on the appearance of the height \(w_A\) of the Airy point in Theorem 2.2(i). Recall from §2.1 that the Airy point corresponds to the first point along the completely unstable middle branch in which the flow transitions from node to focus behavior (see Figure 6). This point turns out to be critical in understanding the phenomenon that many of the transitional pulses appear to have nearly the same tail trajectories. While Theorem 2.2(i) provides a general description on the behavior of the tails in relation to the singular pulses and singular tail manifolds, we expand on this in §4 and provide a more detailed description of the tail phenomenon in Proposition 4.4.
The remainder of this paper is devoted to the proof of Theorem 2.2. In the following, we will construct the entire transitional sequence of the upper left portion of the banana analytically using the same geometric framework used to construct the pulses with oscillatory tails in [7]. We will construct each pulse along the transition using geometric singular perturbation theory. There are several technical challenges in this construction. First, the transitional pulses pass near the two fold points where the exchange lemma fails: as in the proof of [7, Theorem 1.1], we will use blow-up techniques to analyze the flow near these fold points. Secondly, the transitional pulses jump off near the Airy point after their second excursion: understanding this phenomenon is crucial to the analysis, and we will use blow-up techniques to analyze the passage near the Airy point. Thirdly, the entire transition arises in an exponentially small parameter region, and it is therefore crucial that we identify the right variables to parametrize the branch of pulses as we cannot use the $a, c$ variables: it turns out that using the jump-off values for $w$ provides a parametrization that allows us to use $(a, c)$ as unfolding parameters for use in implicit function theorems. We remark that the first challenge mentioned above arose already in [7]; however, the remaining two challenges are unique to the construction of transitional pulses and indeed constitute the main technical innovation compared to the proofs in [7].

### 2.4 Results from geometric singular perturbation theory

We now collect a few results which follow from standard geometric singular perturbation theory and the analysis in [7]. Define the closed intervals $I_a = [-a_0, a_0]$ for sufficiently small $a_0 > 0$ and $I_c = \{c^*(a) : a \in I_a\}$; here $c^*(a) = 1/\sqrt{2}(1-2a)$ is the wavespeed for which the Nagumo front exists for this choice of $a$. Then for sufficiently small $\epsilon_0, a_0$, we have the following:

1. The origin has a strong unstable manifold $W^u_0(0; c, a)$ for $c \in I_c, a \in I_a$, and $\epsilon = 0$ which persists for $a, c$ in the same range and $\epsilon \in [0, \epsilon_0]$.

2. We consider the critical manifold $M_0(c, a) = \{(u, v, w) : v = 0, w = f(u)\}$. For each $a \in I_a$, we consider the right branch of the critical manifold $M^r_0(c, a)$ up to a neighborhood of the upper right fold point for $\epsilon = 0$. This manifold persists as a slow manifold $M^r_0(c, a)$ for $\epsilon \in [0, \epsilon_0]$. In addition, $M^r_0(c, a)$ possesses stable and unstable manifolds $W^s(M^r_0(c, a))$ and $W^u(M^r_0(c, a))$ which also persist for $\epsilon \in [0, \epsilon_0]$. In §2.5 below, we show that there is a way to extend $W^s(M^r_0(c, a))$ and $W^u(M^r_0(c, a))$.

3. In addition, we consider the left branch of the critical manifold $M^l_0(c, a)$ up to a neighborhood of the origin for $\epsilon = 0$. This manifold persists as a slow manifold $M^l_0(c, a)$ for $\epsilon \in [0, \epsilon_0]$. In addition, $M^l_0(c, a)$ possesses a stable manifold $W^s(M^l_0(c, a))$ which also persists for $\epsilon \in [0, \epsilon_0]$ as an invariant manifold which we denote by $W^s_{c, a}(c, a)$. In §2.5 below, we show that there is a way to extend $W^s_{c, a}(c, a)$ in such a manner that it also encompasses a center manifold near the origin; this will be useful in the existence proof.

4. Finally, we consider the middle branch of the critical manifold $M^m_0(c, a)$ away from neighborhoods of the origin and the upper right fold point for $\epsilon = 0$. This manifold persists as a slow manifold $M^m(c, a)$ for $\epsilon \in [0, \epsilon_0]$. In addition, $M^m(c, a)$ possesses a three-dimensional unstable manifold $W^u(M^m_0(c, a))$ which also persists for $\epsilon \in [0, \epsilon_0]$ as an invariant manifold which we denote by $W^u_{c, a}(c, a)$. The stable manifolds $W^s(M^l_0(c, a))$ and $W^s(M^r_0(c, a))$ form part of $W^u(M^m_0(c, a))$ for $\epsilon = 0$ and hence for sufficiently small $\epsilon > 0$, we have that the foliations $W^s_{c, a}(c, a)$ and $W^l_{c, a}(c, a)$ are contained in $W^u_{c, a}(c, a)$.

We also have the following proposition, which follows from the analysis in [7, §5]. The result is shown in Figure 8.

**Proposition 2.6.** There exists $\epsilon_0 > 0$ and $\mu > 0$ such that for each $a \in I_a$ and $\epsilon \in (0, \epsilon_0)$, the manifold $\bigcup_{c \in I_c} W^u_\epsilon(0; c, a)$ intersects $\bigcup_{c \in I_c} W^l_{c, a}(c, a)$ near the upper right fold point transversely in $w$-space with the intersection occurring at $c = \hat{c}(a, \epsilon)$ for a smooth function $\hat{c} : I_a \times (0, \epsilon_0) \to I_c$ where $\hat{c}(a, \epsilon) = c^*(a) - \mu \epsilon + O(\epsilon(|a| + \epsilon))$. 

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Figure 8: Shown is the geometry for the intersection of $W^u_c(0; c, a)$ and the extended center-stable manifold $W^{s, \ell}_c(c, a)$ at $c = \hat{c}(a, \epsilon)$ as in Proposition 2.6.

**Remark 2.7.** The assertions of Proposition 2.6 from [7] were shown by tracking $W^u_c(0; c, a)$ along $\phi_f$, then along $M^r_c(c, a)$ into a neighborhood of the upper right fold point using the exchange lemma. From here, since $W^u_c(0; c, a)$ transversely intersects $W^{s, \ell}_c(c, a)$ at $c = \hat{c}(a, \epsilon)$, we can then track $W^u_c(0; c, a)$ along $\phi_b$, then along $M^r_c(c, a)$ into a neighborhood of the equilibrium. Therefore, we deduce that for $a \in I_a$ and $\epsilon \in (0, \epsilon_0)$ and $c \approx \hat{c}(a, \epsilon)$, $W^u_c(0; c, a)$ follows a primary excursion which is close to $\Gamma_0^1$; see Figure 8.

### 2.5 Extending the center-stable manifold $W^{s, \ell}_c(c, a)$

There are a number of invariant manifolds near the origin which will be involved in the construction of the transitional pulses outlined above. In particular there is the local center manifold near the origin considered in [7, §6] and the stable foliation $W^{s, \ell}_c(c, a)$ of the left slow manifold $M^r_c(c, a)$. These two manifolds are only unique up to exponentially small errors and were chosen in [7] in such a manner that they overlap to form one larger extended center-type manifold.

In the following, as certain parts of our analysis are sensitive to exponentially small errors, it will be convenient to consider an even larger center manifold in this region which contains all of the essential dynamics to reduce dependence on matching conditions containing exponentially small errors.

The goal is to show that for any small $\Delta_w$, for any sufficiently small $\epsilon_0, a_0$, there exists a center type manifold at the origin (as in [7, §6]) which we can extend up to $w = w_A - \Delta_w$, where $w_A$ is the height of the Airy point, due to the consistent exponential separation away from the Airy point. We will be able to choose this manifold in such a way that it contains any part of $M^r_c(c, a)$ lying below $w = w_A - \Delta_w$ and the entirety of $W^{s, \ell}_c(c, a)$.

Hence we abuse notation and refer to this new (larger) manifold as $W^{s, \ell}_c(c, a)$; see Figure 8 for an illustration.

To see this we look at the linearization of (1.2) at $(c, a, \epsilon) = (1/\sqrt{2}, 0, 0)$ given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -f'(u) & c & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (2.15)

There are three eigenvalues $\lambda_0 = 0, \lambda_{\pm} = \frac{c \pm \sqrt{c^2 - 4f'(u)}}{2}$. A quick computation shows that $\Re(\lambda_+) > \Re(\lambda_-)$ provided $c^2 > 4f'(u)$. In particular for $(c, a) = (1/\sqrt{2}, 0)$, this holds for any $u < u_A = \frac{1}{3} \left(1 - \sqrt{\frac{5}{8}}\right)$. 

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We fix $\Delta_w$ sufficiently small and consider the union of the fronts $\phi_t$ for $w \in (0, w_A - \Delta_w)$ for $(c, a, \epsilon) = (1/\sqrt{2}, 0, 0)$ and refer to this invariant manifold as $W_{\epsilon, t}^{s, s}(c, a)$. This manifold is normally hyperbolic with the rate of expansion in the normal direction stronger than the expansion rates on $W_0^{s, s}(c, a)$. Therefore this manifold persists [4, 3] as a normally repelling locally invariant manifold $W_{\epsilon, t}^{s, s}(c, a)$ containing a neighborhood of the slow manifolds $M_t^u(c, a), M_t^\sigma(c, a)$ and the local center manifold near the origin. Taking the intersection of this manifold in a plane of fixed $w < w_A - \Delta_w$ which intersects the manifold $M_t^u(c, a)$ and evolving backwards in time determines a choice of $W_{\epsilon, t}^{s, s}(c, a)$ which also contains the strong stable fibers of the manifold $M_t^u(c, a)$ for $w > w_A - \Delta_w$. This extended center-stable manifold $W_{\epsilon, t}^{s, s}(c, a)$ is shown in Figure 8.

2.6 Existence of maximal canards

In the two-dimensional manifold $W_{\epsilon, t}^{s, s}(c, a)$, for certain parameter values, there exist canard solutions near the attracting slow manifold $M_t^u(c, a)$ which pass near the origin and then follow the repelling slow manifold $M_t^\sigma(c, a)$ for some time. The results in [23] imply that there is a maximal canard solution which, by definition, occurs when $M_t^u(c, a)$ and $M_t^\sigma(c, a)$ coincide. We have the following theorem, which will be proved in §3.1.

**Theorem 2.8.** There exists $\epsilon_0 > 0$ and a smooth function $a^C(\sqrt{\epsilon}, c) : (0, \epsilon_0) \times I_\epsilon \to I_a$ such that there is a maximal canard solution connecting the manifolds $M_t^u(c, a)$ and $M_t^\sigma(c, a)$ when $a = a^C(\sqrt{\epsilon}, c)$. We have that

$$a^C(\sqrt{\epsilon}, c) = -m(c)\epsilon + O(\epsilon^{3/2})$$

(2.16)

where $m(c)$ is positively bounded away from zero uniformly in $c \in I_\epsilon$.

3 Constructing transitional pulses

In this section, we construct transitional pulses in pieces and obtain matching conditions near the equilibrium which are solved using an implicit function theorem.

The general construction for pulses of all types involves three pieces: the primary pulse, a secondary excursion, and a tail manifold which is formed by an appropriately defined subset of the manifold $W_{\epsilon, t}^{s, s}(c, a)$. Hence the procedure involves obtaining two conditions: one which matches the primary pulse to the secondary excursion, and one matching the secondary excursion with the tail manifold. Once these matching conditions are obtained, it remains to show that solutions on the tail manifold in fact converge to the equilibrium (and are not blocked by periodic orbits, etc.). This is treated in §4, where we show that the tail manifold forms part of the stable manifold of the origin, which completes the construction.

Due to various interactions of the pulses with the fold points and the Airy point, the construction of the transitional pulses breaks down into six types. All pulse types have the same primary excursion, and hence the pulse types are determined by properties of the secondary excursion. The different pulse types are as labelled in Figure 3.

- **Type 1:** $\{\Gamma(s) : s \in (0, w_A + \Delta_w)\}$
  Type 1 pulses are left pulses with a secondary excursion of height $w \in (0, w_A + \Delta_w)$, where $w_A$ represents the height of the Airy point, and $\Delta_w$ is sufficiently small. For these pulses, we show that this secondary excursion already lies in the stable manifold $W_{\epsilon, t}^{s, s}(0; c, a)$, and no further matching is required.

- **Type 2:** $\{\Gamma(s) : s \in (w_A + \Delta_w, w^l - \Delta_w)\}$
  Type 2 pulses are left pulses with a secondary excursion of height $w \in (w_A + \Delta_w, w^l - \Delta_w)$, where $w_A$ represents the height of the ‘Airy’ point, and $w^l$ is the height of the upper right fold point. For these pulses, we show that a second matching condition is necessary to ensure that the secondary excursion can be matched with the tail manifold.
• Type 3: \( \{ \Gamma(s) : s \in (w^\dagger - \Delta_w, w^\dagger + \Delta_w) \} \)

Type 3 pulses pass near the upper right fold point and encompass the transition between left and right pulses. These are constructed in much the same way as type 2 pulses, but there are additional difficulties encountered in parameterizing these pulses \( (w \) is not a natural parameter in this regime), and in verifying that the interaction with the upper right fold does not cause the argument to break down.

• Type 4: \( \{ \Gamma(s) : s \in (w^\dagger + \Delta_w, 2w^\dagger - w_A - \Delta_w) \} \)

Type 4 pulses are right pulses with secondary excursion of height \( w \in (w_A + \Delta_w, 2w^\dagger - w_A - \Delta_w) \). An additional technical difficulty arises in this regime: solving the first matching condition requires showing that there is a net contraction in backwards time along the slow manifolds \( \mathcal{M}_\ell^\epsilon(c,a), \mathcal{M}_r^\epsilon(c,a) \).

• Type 5: \( \{ \Gamma(s) : s \in (2w^\dagger - w_A - \Delta_w, 2w^\dagger - \Delta_w) \} \)

Type 5 pulses are right pulses with a secondary excursion of height \( w \in (0, w_A + \Delta_w) \). For these pulses, the secondary excursions have a more delicate interaction with the Airy point and therefore introduce complications when trying to determine the final matching condition with the tail manifold.

• Type 6: \( \{ \Gamma(s) : s \in (2w^\dagger - \Delta_w, 2w^\dagger) \} \)

Type 6 pulses consist of essentially two copies of the primary pulse. In this parameter regime, we are close to the Belyakov transition at which the double pulses are expected to terminate for \( \epsilon \) sufficiently small \([8]\).

While we are able to construct some type 6 pulses, our results do not cover all type 6 pulses up to the Belyakov point. See §3.7 and Remark 3.14 for more details on where the construction breaks down.

At a technical level, the analysis in understanding the precise nature of the termination at the Belyakov point likely requires additional blow-ups near the origin to account for the changing eigenvalue structure. We believe that this regime is technically more challenging and therefore did not pursue a complete analysis of the termination of the branch of double pulses.

We begin with setting up the blown-up coordinate system near the canard point at the origin in which the matching will occur, followed by constructing pulses of type 1,2. We then outline the difficulties/differences in constructing pulses of type 3,4,5,6 and how to overcome these. The construction is then complete up to two technical results: first, the convergence of the tails, proved in §4, and second, a transversality condition which arises due to interaction with the Airy point, proved in §5.

### 3.1 Flow near the canard point

We collect some results from \([23, 7]\) which will be useful in the forthcoming analysis for obtaining matching conditions for the transitional pulses near the equilibrium. In \([7]\), it was shown that in a neighborhood of the origin, after a change of coordinates, we obtain the system

\[
\begin{align*}
\dot{x} &= -y + x^2 + O(\epsilon, xy, y^2, x^3) \\
\dot{y} &= \epsilon [x (1 + O(x, y, \alpha, \epsilon)) + \alpha (1 + O(x, y, \alpha, \epsilon)) + O(y)] \\
\dot{z} &= z \left( e^{3/2} + O(x, y, \alpha, \epsilon) \right) \\
\dot{\alpha} &= 0 \\
\dot{\epsilon} &= 0,
\end{align*}
\]

(3.1)

where \( \alpha = \frac{a}{2e^{3/2}} \). The manifold \( W_{x,y}^\alpha(c,a) \) is given by \( z = 0 \), where the strong unstable fibers have been straightened. We note that the \( (x, y) \) coordinates are in the canonical form for a canard point (compare \([23]\),
Figure 9: The local coordinates near the canard point and the section $\Sigma^m$. The manifold $W^{s,\ell}_\epsilon(c,a)$ coincides with the subspace $z = 0$.

that is,

$$\begin{align*}
\dot{x} &= -y h_1(x, y, \alpha, \epsilon, c) + x^2 h_2(x, y, \alpha, \epsilon, c) + \epsilon h_3(x, y, \alpha, \epsilon, c) \\
\dot{y} &= \epsilon (x h_4(x, y, \alpha, \epsilon, c) + \alpha h_5(x, y, \alpha, \epsilon, c) + y h_6(x, y, \alpha, \epsilon, c)) \\
\dot{z} &= z \left( c^{3/2} + O(x, y, \alpha, \epsilon) \right) \\
\dot{\alpha} &= 0 \\
\dot{\epsilon} &= 0 ,
\end{align*}$$

(3.2)

where we have

$$
\begin{align*}
h_3(x, y, \alpha, \epsilon, c) &= O(x, y, \alpha, \epsilon) \\
h_j(x, y, \alpha, \epsilon, c) &= 1 + O(x, y, \alpha, \epsilon), \quad j = 1, 2, 4, 5 .
\end{align*}
$$

(3.3)

We have now separated the hyperbolic dynamics (given by the $z$-coordinate) from the nonhyperbolic dynamics which are isolated on a four-dimensional center manifold parameterized by the variables $(x, y, \epsilon, \alpha)$ on which the origin is a canard point in the sense of [23]. Such points are characterized by “canard” trajectories which follow a strongly attracting manifold (in this case $M^\ell_\epsilon(c,a)$), pass near the equilibrium and continue along a strongly repelling manifold (in this case $M^m_\epsilon(c,a)$) for some time; see Figure 9 for an illustration. To understand the flow near this point, we use blowup methods as in [23]. Restricting to the center manifold $z = 0$, the blow up transformation is given by

$$
\begin{align*}
x &= \bar{r} \bar{x}, \quad y = \bar{r}^2 \bar{y}, \quad \alpha = \bar{r} \bar{\alpha}, \quad \epsilon = \bar{r}^2 \bar{\epsilon} ,
\end{align*}
$$

(3.4)

defined on the manifold $B_c = S^2 \times [0, \bar{r}_0] \times [-\bar{\alpha}_0, \bar{\alpha}_0]$ for sufficiently small $\bar{r}_0, \bar{\alpha}_0$ with $(\bar{x}, \bar{y}, \bar{\epsilon}) \in S^2$. There is one relevant coordinate chart which will be needed for the matching analysis. Keeping the same notation as in [23] and [24], the chart $\mathcal{K}_2$ uses the coordinates

$$
\begin{align*}
x &= r_2 x_2, \quad y = r_2^2 y_2, \quad \alpha = r_2 \alpha_2, \quad \epsilon = r_2^2 \cdot ,
\end{align*}
$$

(3.5)

Using these blow-up charts, the authors of [23] studied the behavior of the manifolds $M^\ell_\epsilon(c,a)$ and $M^m_\epsilon(c,a)$ near the equilibrium, and in particular determined conditions under which these manifolds coincide along a canard trajectory. We place a section $\Sigma^m = \{x = 0, |y| < \Delta_y, |z| \leq \Delta_z\}$ for small fixed $\Delta_z$ and $\Delta_y = 2 \Delta_w$ in which most of our computations will take place (see Figure 9).

In the chart $\mathcal{K}_2$, the section $\Sigma^m$ is given by $\Sigma^m_2 = \{(x_2 = 0, |r_2^2 y_2| < \Delta_y, |z| \leq \Delta_z\}$. It was shown in [23] that for all sufficiently small $r_2, \alpha_2$, the manifolds $M^\ell_\epsilon(c,a)$ and $M^m_\epsilon(c,a)$ reach $\Sigma^m_2$ at $y = y^M_\epsilon,\ell(c,\alpha_2, r_2)$ and
y = y^M_{l,m}(c, \alpha_2, r_2), respectively. Furthermore, we have the following result which will be useful in the coming analysis.

**Proposition 3.1.** [23, Proposition 3.5] The distance between the slow manifolds \( M^l(c, a) \) and \( M^m(c, a) \) in \( \Sigma^m \) is given by

\[
y^M_{l} - y^M_{m} = D_0(\alpha_2, r_2; c) = d_{\alpha_2} \alpha_2 + d_{r_2} r_2 + O(r_2^2 + \alpha_2^2),
\]

where the coefficients \( d_{\alpha_2}, d_{r_2} \) are positive constants. Hence we can solve for when this distance vanishes which occurs when

\[
\alpha_2 = \alpha_C = -\frac{d_{r_2}}{d_{\alpha_2}} r_2 + O(r_2^2). \tag{3.7}
\]

Theorem 2.8 then follows from the above proposition.

**Remark 3.2.** In the following, many computations will be performed in the \( K_2 \) coordinates before transforming back into the original coordinates/parameters as the results of Theorem 2.2 are stated in terms of the original parameters \((c, a, \epsilon)\), rather than \((c, \alpha_2, r_2)\). To obtain \((a, \epsilon)\) from \((\alpha_2, r_2)\), we have

\[
a = 2c^{1/2} \alpha_2 r_2
\]

\[
\epsilon = r_2^2, \tag{3.8}
\]

which are smooth functions of \((c, \alpha_2, r_2)\) for \((c, \alpha_2, r_2)\) near \((1/\sqrt{2}, 0, 0)\).

We remark that results involving transversality with respect to parameter variations due to the exchange lemma [27] which are obtained for the original system (1.2) can likewise be shown to hold in the \( K_2 \) coordinates by instead considering the system

\[
\begin{align*}
\dot{u} &= v \\
\dot{v} &= cv - f(u) + w \\
\dot{w} &= r_2^2 (u - \gamma w),
\end{align*} \tag{3.9}
\]

where \( f(u) = u(u - 2c^{1/2} \alpha_2 r_2)(1 - u) \) for \((c, \alpha_2, r_2)\) near \((1/\sqrt{2}, 0, 0)\).

## 3.2 Type 1 pulses

Type 1 pulses are the simplest of the transitional pulses and are really just single pulses with ‘large’ oscillatory tails. In this section, we deduce the existence of transitional left pulses with secondary excursion of height \( w \leq w_A + \Delta_w \) and show that these pulses are in fact a continuation of the family of pulses with oscillatory tails constructed in [7]. To construct a type 1 pulse, we need a single matching condition which matches the primary pulse with a secondary pulse of height \( w \leq w_A + \Delta_w \); see Figure 10. The fact that this secondary excursion lies in the stable manifold \( W^s_0(0; c, a) \) of the equilibrium follows from Proposition 3.3 below.

We break this into two parts. We first construct pulses of height \( w \in (\Delta_w, w_A + \Delta_w) \) and then move onto pulses with ‘small’ oscillatory tails, that is, pulses with tails of height \( w \leq \Delta_w \).

### 3.2.1 Matching condition for pulses \( \Gamma(s, \sqrt{\epsilon}), s \in (\Delta_w, w_A + \Delta_w) \)

We define the section \( \Sigma^{h,l} := \{ u = 0, \Delta_w < w < w^\dagger - \Delta_w \} \). The desired transitional pulse solution consists of a primary excursion, followed by a secondary excursion \( \gamma^{sp}(s; c, a) \) intersecting the section \( \Sigma^{h,l} \) at height \( w = s \) (see Figure 10). We can ensure that the unstable manifold \( W^u_0(0; c, a) \) completes the primary excursion and returns to a neighborhood of the equilibrium by using Proposition 2.6; see Remark 2.7 and Figure 8. The goal
is then to match \( \mathcal{W}_\epsilon^u(0; c, a) \) with \( \gamma^{sp}(s; c, a) \). The setup is shown in Figure 10. For \( w \in (\Delta_w, w_A + \Delta_w) \), the fact that this secondary excursion \( \gamma^{sp}(s; c, a) \) lies in the stable manifold \( \mathcal{W}_\epsilon^u(0; c, a) \) of the equilibrium is given by the following proposition.

**Proposition 3.3.** Fix \( \Delta_w \) sufficiently small. For each sufficiently small \( \epsilon > 0 \), consider any transitional pulse with tail landing in the manifold \( \mathcal{W}_\epsilon^{s, \ell}(c, a) \in \Sigma^{h, \ell} \) at a height \( w \leq w_A + \Delta_w \). Then the tail of this pulse in fact lies in the stable manifold \( \mathcal{W}_\epsilon^u(0; c, a) \) of the equilibrium \( (u, v, w) = (0, 0, 0) \).

The proof of this proposition will be presented in §4.

We will match the various components of the solution in the section \( \Sigma^m \) in the \( K_2 \) coordinates. Figure 11 shows the setup for the matching conditions projected onto the \((x_2, y_2)\)-plane.

**Notation.** The matching conditions in the section \( \Sigma^m \) will be determined by intersections of various invariant manifolds (e.g. \( \mathcal{W}_\epsilon^{s, \ell}(c, a), \gamma^{sp}(s; c, a) \)) evolved forwards/backwards under the flow between the sections \( \Sigma^{h, \ell}, \Sigma^m \). We will abuse notation and use the same variables to refer to these manifolds (composed of trajectories) as well as their lower dimensional intersections with specific sections (e.g. \( \Sigma^{h, \ell}, \Sigma^m \)) transverse to the flow. It will also be useful to distinguish between backwards and forwards evolution: When referring to an invariant manifold evolved in backwards time, we use the notation ‘\( \hat{\cdot} \)’, e.g. when referring to the manifold \( \hat{\mathcal{W}}_\epsilon^{s, \ell}(c, a) \), we mean the manifold \( \mathcal{W}_\epsilon^{s, \ell}(c, a) \) under the backwards evolution of (1.2).

First, in order to match \( \mathcal{W}_\epsilon^u(0; c, a) \) with \( \gamma^{sp}(s; c, a) \), we have the following lemma describing the location of \( \mathcal{W}_\epsilon^u(0; c, a) \) in \( \Sigma^m \).

**Lemma 3.4.** For each sufficiently small \( \Delta_z > 0 \), there exists \( C, q, \epsilon_0 > 0 \) and \( q_1 > q_2 > 0 \) such that the following holds. For each \( 0 < \epsilon < \epsilon_0 \) and each \( |z| < \Delta_z \), there exists \( c = c(z) \) with \( |c(z) - \hat{c}(a, \epsilon)| = \mathcal{O}(e^{-q/\epsilon}) \) such that \( \mathcal{W}_\epsilon^u(0; c(z), a) \) intersects \( \Sigma^m \) at the point \( (y_2^u(z; c(z), a), z) \) where

\[
\begin{align*}
    e^{-q_1/\epsilon}/C &\leq y_2^u(0; c(z), a) - y_2^{M, \ell} \leq Ce^{-q_2/\epsilon} \\
    |y_2^u(z; c(z), a) - y_2^u(0; c(0), a)| &\mathcal{O}(ze^{-q/\epsilon}).
\end{align*}
\]
transversely intersects $W$.

**Remark 3.5.** Geometrically, the parameter values $(c, a) = (c_E, a_E)(\sqrt{\epsilon})$ capture an intersection of the manifolds $W^u(0; c, a)$ and $W^s_{\epsilon}(c, a)$ (as in Proposition 2.6) coinciding with the existence of a maximal canard in $W^s_{\epsilon}(c, a)$ (as in Theorem 2.8).

The solution of (3.12) is then obtained by solving

$$
a = 2\epsilon^{1/2} a_2^{1/2} \alpha_2^C + O(e^{-q/\epsilon})
$$

$$
c = \tilde{c}(a, \epsilon) + O(e^{-q/\epsilon}),
$$

for which there is a solution when

$$
a = 2\epsilon^{1/2} a_2^{1/2} \alpha_2^C
$$

$$
c = \tilde{c}(a, \epsilon),
$$

which we can solve by the implicit function theorem at $(c, a) = (c_0, a_0)(\sqrt{\epsilon})$ for sufficiently small $\epsilon > 0$.

**Proof.** We have that $W^u(0; \tilde{c}(a, \epsilon), a)$ is $O(e^{-q/\epsilon})$-close to $\mathcal{M}^\ell_{\epsilon}(c, a)$ in $\Sigma^m$. Since by Proposition 2.6, $W^u(0; c, a)$ transversely intersects $W^s_{\epsilon}(c, a)$ upon varying $c \approx \tilde{c}(a, \epsilon)$, the result follows from the exchange lemma [27].

We now construct the candidate ‘secondary pulse’ solution: Consider the solution $\gamma^{sp}(s; c, a)$ on the stable foliation $W^s_{\epsilon}(c, a)$ which intersects $\Sigma^{h, \ell}$ at height $w = s \in (\Delta_w, w_A + \Delta_w)$; this intersection occurs at a point $(u, v, w) = (0, v^{sp}(s; c, a), s)$.

Considering the backwards evolution $\hat{\gamma}^{sp}(s; c, a)$ of $\gamma^{sp}(s; c, a)$ from $\Sigma^{h, \ell}$ to $\Sigma^m$, we have that $\hat{\gamma}^{sp}(s; c, a)$ is exponentially close to $\mathcal{M}^m(c, a)$ in $\Sigma^m$. Thus we have that $\hat{\gamma}^{sp}(s; c, a)$ intersects $\Sigma^m$ at a point $(y_2, z) = (y_2^b, z^b)(s; c, a)$ which satisfies

$$
|y_2^b(s; c, a) - y_2^{M, m}| = O(e^{-q/\epsilon})
$$

$$
|z^b(s; c, a)| = O(e^{-q/\epsilon}),
$$

uniformly in $(c, a)$. Thus by Proposition 3.1 we can match $W^u(0; c, a)$ with $\hat{\gamma}^{sp}(s; c, a)$ in $\Sigma^m$ by solving for when

$$
D_0(\alpha_2, r_2; c) = 0
$$

$$
c = \tilde{c}(a, \epsilon),
$$

for which there is a solution when

$$
a = 2\epsilon^{1/2} a_2^{1/2} \alpha_2^C
$$

$$
c = \tilde{c}(a, \epsilon),
$$

which we can solve by the implicit function theorem at $(c, a) = (c_E, a_E)(\sqrt{\epsilon})$ for sufficiently small $\epsilon > 0$.

See Figure 11 for the setup of the matching conditions (3.12) projected onto the $(x_2, y_2)$-plane.

Before tackling (3.12), we first consider solving the simpler equations

$$
D_0(\alpha_2, r_2; c) = 0
$$

$$
c = \tilde{c}(a, \epsilon),
$$

for which there is a solution when

$$
a = 2\epsilon^{1/2} a_2^{1/2} \alpha_2^C + O(e^{-q/\epsilon})
$$

$$
c = \tilde{c}(a, \epsilon) + O(e^{-q/\epsilon}),
$$

(3.13)
by the implicit function theorem to find \((c, a) = (c, a)(s, \sqrt{\epsilon})\) where
\[
(c, a)(s, \sqrt{\epsilon}) = (c_E, a_E)(\sqrt{\epsilon}) + \mathcal{O}(\epsilon^{-q/\epsilon}).
\] (3.16)

### 3.2.2 Connection to pulses with small oscillatory tails

We now consider the case of pulses with tails of height \(w \leq \Delta_w\). In [7], it was shown that there exists \(K^*\), such that for each \(K\) and each sufficiently small \((a, \epsilon)\) satisfying \(\epsilon < Ka^2\), there exists a pulse solution with wave speed \(c = \tilde{c}(a, \epsilon)\). For \(\epsilon > K^*a^2\), the tail of the pulse decays exponentially to zero in an oscillatory fashion.

We deduce that such pulses exist for \((\alpha_2, r_2)\) for any \(r_2 > 0\) sufficiently small and \(\alpha_2 > \frac{1}{2c\sqrt{K}} > \frac{1}{2\sqrt{K}}\), since
\[
c = 1/\sqrt{2} + \mathcal{O}(\alpha_2 r_2, r_2^2) < 1
\] (3.17)
for \(\alpha_2\) bounded and \(r_2 > 0\) sufficiently small. In this section we show that these pulses overlap with the type 1 pulses constructed above, forming a continuous one-parameter family. It turns out that this family is naturally parameterized by \(\alpha_2\).

To see this, we proceed as follows. Setting \(s = \Delta_w\), for sufficiently small \(\epsilon > 0\), we can follow the procedure above in constructing a type 1 pulse with a tail of height \(s = \Delta_w\). We note that in this case, the backwards evolution \(\tilde{\gamma}^{sp}(\Delta_w; c, a)\) of the trajectory \(\gamma^{sp}(\Delta_w; c, a)\) remains in \(\mathcal{W}^{s, \mathcal{F}}(c, a)\) until reaching the section \(\Sigma^m\) and therefore intersects this section at a point \((y_2, z) = (y^{b, z}_2)\) which satisfies
\[
e^{-q_1/\epsilon} / C \leq y^{b}_2(\Delta_w; c, a) - y^{M, m}_2 \leq Ce^{-q_2/\epsilon}
\] (3.18)
for some \(q_1 > q_2 > 0\). Thus we can match \(\mathcal{W}^{u}_0(0; c, a)\) with \(\tilde{\gamma}^{sp}(\Delta_w; c, a)\) by solving
\[
\mathcal{D}_0(\alpha_2, r_2; c) = \left( y^{M, \ell}_2 - y^{u}_2(0; c, a) \right) + \left( y^{b}_2(\Delta_w; c, a) - y^{M, m}_2 \right)
\] (3.19)
We obtain a solution by solving
\[
\alpha_2 = \alpha_2^C + \mathcal{O}(\epsilon^{-q/\epsilon})
\] (3.20)
\[
c = \tilde{c} \left( 2\epsilon^{1/2} r_2 \alpha_2, r_2^2 \right)
\]
by the implicit function theorem to find a solution at \((c, \alpha_2) = (c^u, \alpha_2^u)(r_2)\). We now consider the function \(\mathcal{D}(\alpha_2, r_2, c)\) defined to be the difference
\[
\mathcal{D}(\alpha_2, r_2, c) = y^{u}_2(0; c, a) - y^{b}_2(\Delta_w; c, a)
\] (3.21)
in \(\Sigma^m\). From the construction above for the pulse with tail of height \(\bar{w} = \Delta_w\), we have that
\[
\mathcal{D}(\alpha_2, r_2, c^u) = 0
\] (3.22)
and
\[
\mathcal{D}(\alpha_2, r_2, c) = y^{u}_2(0; c, a) - y^{b}_2(\Delta_w; c, a)
\]
\[
= y^{M, \ell}_2 - y^{M, m}_2 + \mathcal{O} \left( \epsilon^{-q/\epsilon} \right)
\]
\[
= \mathcal{D}_0(\alpha_2, r_2; c) + \mathcal{O} \left( \epsilon^{-q/\epsilon} \right)
\]
\[
= d_{\alpha_2} \alpha_2 + d_r r_2 + \mathcal{O}(\alpha_2^2, \alpha_2 r_2, r_2^2).
\]
Hence we have that
\[
\frac{\partial}{\partial \alpha_2} \mathcal{D}(\alpha_2, r_2, c) = d_{\alpha_2} + \mathcal{O}(\alpha_2, r_2) > 0
\] (3.24)
for any sufficiently small \( r_2 > 0 \) and \( |\alpha_2| \leq \kappa \), uniformly in \( c \approx 1/\sqrt{2} \).

Hence for sufficiently small \( r_2 > 0 \), for \( \alpha_2^2 < \alpha_2 < \kappa \), we can ensure that \( \mathcal{W}_c^0(0; c, a) \) lands in \( \mathcal{W}_c^{s, \ell}(c, a) \) by solving

\[
c = \bar{c}(2e^{1/2}r_2\alpha_2, r_2^2),
\]

(3.25)

for \( c = c(\alpha_2, r_2) \) by the implicit function theorem. Furthermore, we have that the distance \( \bar{D}(\alpha_2, r_2, c) \) is positive, and hence we obtain a pulse whose tail reaches a height lower than \( \Delta_w \), but remains in \( \mathcal{W}_c^{s, \ell}(c, a) \) and converges to the equilibrium. For fixed \( r_2 \), such pulses are therefore parameterized by \( \alpha_2^2 < \alpha_2 < \kappa \).

By taking \( K > \frac{1}{4\kappa^2} \) in [7, Theorem 1.1], we deduce that these pulses form a continuous family with the pulses constructed in [7] for \( \alpha_2 > \frac{1}{2\sqrt{K}} \).

### 3.3 The tail manifold

In §3.2, we proved the existence of type 1 pulses using a single matching condition. The construction of pulses of type 2 – 6 will involve two matching conditions, where the second condition guarantees that the pulses end up in the tail manifold. To prepare for the matching, we now provide a characterization of the tail manifold in which the desired pulse solutions will be trapped. In §4, we will show that this manifold indeed forms part of the stable manifold \( \mathcal{W}_c^0(0; c, a) \) of the equilibrium.

We consider the backwards evolution \( \hat{\mathcal{W}}_{c}^{s, \ell}(c, a) \) of \( \mathcal{W}_c^{s, \ell}(c, a) \) from \( \Sigma^{h, \ell} \) to \( \Sigma^m \). The manifold \( \mathcal{W}_c^{s, \ell}(c, a) \) intersects \( \Sigma^{h, \ell} \) in a curve \((u, v, w) = (0, \hat{v}^\ell(w; c, a), w)\). We have the following proposition, which states that the backwards evolution \( \hat{\mathcal{W}}_{c}^{s, \ell}(c, a) \) intersects \( \Sigma^m \) in a curve transverse to the strong unstable fibers \( y_2 = \text{const} \). Except for an exponentially thin region around \( \mathcal{M}_c^m(a, c) \), this curve coincides with \( z = 0 \). The intersection of \( \hat{\mathcal{W}}_{c}^{s, \ell}(c, a) \) with the section \( \Sigma^m \) is shown in Figure 12.

**Proposition 3.6.** For each sufficiently small \( \Delta_w > 0 \), there exists \( C, \kappa, \epsilon_0, q > 0 \) and sufficiently small choice of the intervals \( I_c, I_a \) such that for each \((c, a, \epsilon) \in I_c \times I_a \times (0, \epsilon_0) \), there exists \( w_\Delta^\alpha(c, a) \in [w_A - 3\Delta_w, w_A - \Delta_w] \) and \( w_\alpha^b(c, a) > w_A + \kappa^2/3 \) such that the following holds. Let \( \hat{\mathcal{W}}_{c}^{s, \ell}(c, a) \) denote the backwards evolution of the curve \( \{u, v, w\} = (0, \hat{v}^\ell(w; c, a), w); w^\Delta < w < w^b_\alpha \}. \) Then \( \hat{\mathcal{W}}_{c}^{s, \ell}(c, a) \) intersects \( \Sigma^m \) in a curve \( z = z^{\epsilon, \ell}(y_2; c, a) \) for \( y_2 \geq y^k_2(c, a) \) and \( y_2 \leq \hat{y}^k_2(c, a) \) where

(i) The interval \([y_2^{k, a}(c, a), y_2^{k, b}(c, a)]\) satisfies \( 0 < y_2^{k, a}(c, a) - y_2^{k, b}(c, a) < C\epsilon^{-q/\epsilon} \) uniformly in \((c, a) \in I_c \times I_a \).

(ii) There exists \( y_2^{k, a}(c, a) \in [y_2^{k, a}(c, a), y_2^{k, b}(c, a)] \) such that \( z^{\epsilon, \ell}(y_2; c, a) \equiv 0 \) for \( y_2 \in [y_2^{k, a}(c, a), y_2^{k, b}(c, a)] \).

(iii) The function \( z^{\epsilon, \ell} \) and its derivatives are \( O(\epsilon^{-q/\epsilon}) \) uniformly in \( y_2 \in [y_2^{k, a}(c, a), y_2^{k, b}(c, a)] \) and \((c, a) \in I_c \times I_a \).

Hence we have that \( \hat{\mathcal{W}}_{c}^{s, \ell}(c, a) \) intersects \( \Sigma^m \) in a curve which can be represented as a graph \( z = z^{\epsilon, \ell}(y_2; c, a) \) for \( y_2 \geq y_2^{k, a}(c, a) \) where the function \( z^{\epsilon, \ell} \) and its derivatives are \( O(\epsilon^{-q/\epsilon}) \). The proof of this proposition will be given in §3.6.

We can now define the tail manifold in the section \( \Sigma^m \). We define \( \mathcal{T}_c(c, a) \) to be the forward evolution of trajectories which intersect the section \( \Sigma^m \) in the graph of a smooth function given by \( z = z^T_c(y_2; c, a) \) for \( y_2 \geq y_2^{k, a}(c, a) \) where

\[
z^T_c(y_2; c, a) = \begin{cases} 
0 & y_2 \geq y_2^{k, a}(c, a) \\
z^{\epsilon, \ell}(y_2; c, a) & y_2^{k, a}(c, a) \leq y_2 \leq y_2^{k, b}(c, a)
\end{cases}.
\]

(3.26)

The intersection of the tail manifold with the section \( \Sigma^m \) is shown in Figure 12. By construction, the forward evolution of trajectories on \( \mathcal{T}_c(c, a) \) correspond to trajectories on \( \mathcal{W}_c^{s, \ell}(c, a) \) in \( \Sigma^{h, \ell} \) with \( w \leq w^m_\alpha \). We have the following.
Corollary 3.7. For each sufficiently small $\epsilon > 0$, consider a transitional pulse with tail landing on the tail manifold $T_{c}(c,a)$. Then the tail of this pulse in fact lies in the stable manifold $W_{c}^{s}(0; c,a)$ of the equilibrium $(u,v,w) = (0,0,0)$.

Proof. This follows directly from Proposition 3.3: By construction, the forward evolution of trajectories on $T_{c}(c,a)$ correspond to trajectories on $W_{c}^{u}(c,a)$ in $\Sigma^{h,t}$ with $w \leq w_{c}^{u}$ where $w_{c}^{u} = w_{A} + O(\epsilon^{2/3}) \leq w_{A} + \Delta_{w}$ for all sufficiently small $\epsilon > 0$.

3.4 Type 2 pulses

To construct a left transitional pulse with secondary excursion of height $s \in (w_{A} + \Delta_{w}, w_{c}^{-} - \Delta_{w})$, we consider a two dimensional manifold $B(s; c,a)$ (to be chosen below) which intersects the manifold $W_{c}^{u}(c,a)$ transversely in the section $\Sigma^{h,t} := \{u = 0, \Delta_{w} < w < w_{c}^{-} - \Delta_{w}\}$ for $w = s$. By considering the backwards evolution $\hat{B}(s; c,a)$ of $B(s; c,a)$ back to the section $\Sigma_{m}$, we show that each trajectory passing through $\hat{B}(s; c,a)$ can be matched with $W_{c}^{u}(0; c,a)$ by choosing $(c,a)$ appropriately. By evolving $B(s; c,a)$ forwards, we show that precisely one of these choices results in $W_{c}^{u}(0; c,a)$ becoming trapped in the tail manifold $T_{c}(c,a)$ as $t \to \infty$. The setup is shown in Figure 13.

Therefore, to construct a transitional pulse we need two matching conditions near the equilibrium: the first matches $W_{c}^{u}(0; c,a)$ with $\hat{B}(s; c,a)$ to guarantee height $s$ for the second excursion, and the second matches $B(s; c,a)$ with the tail manifold $T_{c}(c,a)$, which by Corollary 3.7 forms part of the two-dimensional stable manifold of the equilibrium. The local geometry for the matching conditions is shown in Figure 14. The setup for the matching conditions in the section $\Sigma_{m}$ is shown in Figure 12.

3.4.1 Matching conditions for pulses $\Gamma(s, \sqrt{s})$, $s \in (w_{A} + \Delta_{w}, w_{c}^{-} - \Delta_{w})$

We match the various components of the solution in the section $\Sigma_{m}$ in the $K_{2}$ coordinates. From Lemma 3.4, for each $|z| < \Delta_{z}$, there exists $c$ with $|c - \bar{c}(a, \epsilon)| = O(e^{-q_{1}/\epsilon})$ such that $W_{c}^{u}(0; c,a)$ intersects $\Sigma_{m}$ at the point $(y_{2}^{u}(z; c,a), z)$ where

$$e^{-q_{1}/\epsilon}/C \leq y_{2}^{u}(0; c,a) - y_{2}^{M,t} \leq C e^{-q_{2}/\epsilon}$$

$$|y_{2}^{u}(z; c,a) - y_{2}^{u}(0; c,a)| = O(ze^{-q_{2}/\epsilon}),$$

for some $q_{1} > q_{2} > 0$.

Consider the solution $\gamma^{sp}(s; c,a)$ on the stable foliation $W_{c}^{s}(c,a)$ which intersects the section $\Sigma^{h,t}_{m}$ at height $w = s$. This intersection occurs at a point $(u,v,w) = (0,v^{sp}(s; c,a), s)$. This solution is exponentially attracted in forward time to $M_{1}^{t}(c,a)$ and hence intersects $\Sigma_{m}$ at the point $(y_{2}^{sp}(s; c,a), 0)$ where

$$e^{-q_{1}/\epsilon}/C \leq y_{2}^{sp}(s; c,a) - y_{2}^{u}(0; c,a) \leq C e^{-q_{2}/\epsilon}.$$  

(3.28)

The geometry of the setup for type 2 pulses and the solution $\gamma^{sp}(s; c,a)$ is shown in Figure 13.

Define the manifold $B(s; c,a)$ to be the backwards evolution of the fiber

$$\{(0, y_{2}^{sp}(s; c,a), z) : |z| \leq \Delta_{z}\} \subseteq \Sigma_{m}.$$  

(3.29)

We parameterize $B(s; c,a)$ by $|z_B| \leq \Delta_{z}$, where $z_B$ denotes the height along the fiber. In backwards time, this fiber is exponentially contracted to the solution $\gamma^{sp}(s; c,a)$ and hence intersects $\Sigma^{h,t}_{m}$ in a one-dimensional curve which is $O(\epsilon^{-q_{2}/\epsilon})$-close to $(u,v,w) = (0,v^{sp}(s; c,a), s)$, uniformly in $(c,a)$. A schematic of the manifold $B(s; c,a)$ and its relation to $\gamma^{sp}(s; c,a)$ is shown in Figure 13.
Figure 12: Shown is the setup for the matching conditions in the section $\Sigma^m$. 
Figure 13: Shown is the geometry for constructing a type 2 left transitional pulse.

Figure 14: Shown is the setup for the matching conditions in the section $\Sigma^m$. There are two matching conditions:
(i) match $W^u(0; c, a)$ with $\hat{B}(s; c, a)$ (ii) match $\mathcal{B}(s; c, a)$ with the tail manifold $T(c, a)$.
The final matching conditions are obtained as follows: Starting in $\Sigma^{h,\ell}$, we consider the backwards evolution $\widehat{B}(s;c,a)$ of $B(s;c,a)$ back to the section $\Sigma^m$ and show that for each $|z_B| \leq \Delta_z$, $W^u_e(0;c,a)$ can be matched with the corresponding solution on $\widehat{B}(s;c,a)$ by adjusting $(c,a)$. We then evolve $B(s;c,a)$ forwards from $\Sigma^{h,\ell}$ to $\Sigma^m$ and show that $B(s;c,a)$ transversely intersects $T_e(c,a)$ for each such $(c,a)$ as $z_B$ varies. This implies the existence of parameter values $(c,a)$ for which $W^u_e(0;c,a)$ completes one full pulse and a secondary pulse of height $s$ before landing in the tail manifold $T_e(c,a)$. Convergence of the tails follows from Corollary 3.7. The setup for the matching conditions is shown in Figures 12 and 14.

Evolving $B(s;c,a)$ backwards, we have that the backwards evolution $\widehat{B}(s;c,a)$ of $B(s;c,a)$ is exponentially close to $\mathcal{M}^m_e(c,a)$ in $\Sigma^m$. Thus we have that in $\Sigma^m$, $\widehat{B}(s;c,a)$ is given by a curve $(y_2,z) = (y_2^b,z^b)(z_B,s;c,a)$ which satisfies

$$|y_2^b(z_B,s;c,a) - y_2^M m| = O(e^{-q/\epsilon})$$
$$|z^b(z_B,s;c,a)| = O(e^{-q/\epsilon}),$$

uniformly in $|z_B| \leq \Delta_z$ and $(c,a) \in I_c \times I_a$. The derivatives of the above expressions with respect to $(c,a,z_B)$ are also $O(e^{-q/\epsilon})$, by taking $q$ a bit smaller if necessary.

Recall from §3.3 that in the section $\Sigma^m$, the tail manifold $T_e(c,a)$ is defined by the graph of a smooth function given by $z = z_e^T(y_2;c,a)$ for $y_2 \geq y_{2,0}^e(c,a)$ where

$$z_e^T(y_2;c,a) = \begin{cases} 
0 & y_2 \geq y_{2,0}^e(c,a) \\
\ell^e,*(y_2;c,a) & y_{2,0}^e(c,a) \leq y_2 \leq y_{2,0}^e(c,a)
\end{cases}$$

and the function $z^\ell,*$ and its derivatives are $O(e^{-q/\epsilon})$. We have the following.

**Lemma 3.8.** For each sufficiently small $\Delta_w > 0$, there exists $\epsilon_0, \Delta_z, q > 0$ and sufficiently small choice of the intervals $I_c, I_a$ such that for each $s \in (w_A + \Delta_w, w^t - \Delta_w)$ and each $0 < \epsilon < \epsilon_0$, the following hold. Firstly, the backwards evolution of the manifold $B(s;c,a)$ intersects $\Sigma^m$ in a curve $(y_2,z) = (y_2^b,z^b)(z_B,s;c,a)$ which satisfies

$$|y_2^b(z_B,s;c,a) - y_2^M m| = O(e^{-q/\epsilon})$$
$$|z^b(z_B,s;c,a)| = O(e^{-q/\epsilon}),$$

uniformly in $|z_B| \leq \Delta_z$ and $(c,a) \in I_c \times I_a$; the derivatives of the above expressions with respect to $(c,a,z_B)$ are also $O(e^{-q/\epsilon})$. Secondly,

$$y_{2,0}^e(c,a) < \inf_{|z_B| \leq \Delta_z} y_{2,b}(z_B,s;c,a),$$

for all $(c,a) \in I_c \times I_a$.

The first assertion of Lemma 3.8 follows from the analysis above; the proof of the second assertion will be given in §5.6.

From Proposition 3.1, we have that the distance between the manifolds $\mathcal{M}^\ell_e(c,a)$ and $\mathcal{M}^m_e(c,a)$ in $\Sigma^m$ is given by

$$y_2^M \ell - y_2^M m = D_0(\alpha_2,r_2;c) = d_\alpha \alpha_2 + d_{r_2} r_2 + O(2).$$

Using Lemma 3.4 and Proposition 3.1, by varying $c, \alpha_2$ we can match $W^m_e(0;c,a)$ with any solution in $\widehat{B}(s;c,a)$ by solving

$$D_0(\alpha_2,r_2;c) = \left( y_2^M \ell - y_2^m(z;c,a) \right) + \left( y_2^b(z_B,s;c,a) - y_2^M m \right) = O(e^{-q/\epsilon})$$
$$z = z^b(z_B,s;c,a),$$

25
for each \(|z_B| \leq \Delta_z\). For each such \(z_B\), we obtain a solution by solving

\[
\begin{align*}
a &= 2\sqrt{\epsilon}a^1/2(\alpha_C^2 + \mathcal{O}(\epsilon^{-q/\epsilon})) \\
c &= \hat{c}(a, \epsilon) + \mathcal{O}(\epsilon^{-q/\epsilon}),
\end{align*}
\] (3.36)

by the implicit function theorem to find \((a, c) = (a^w, c^w)(z_B; s, \sqrt{\epsilon})\) where we have

\[
(c^w, a^w)(z_B; s, \sqrt{\epsilon}) = (c_E, a_E)(\sqrt{\epsilon}) + \mathcal{O}(\epsilon^{-q/\epsilon}),
\] (3.37)
uniformly in \((z_B, s)\), where \((c_E, a_E)\) is the unique solution of (3.14) (see \(\S 3.2.1\)).

We now evolve \(\mathcal{W}_c^s(0; c, a)\) forwards; for each \(z_B\) we can hit the corresponding point on \(\mathcal{B}(s; c, a)\), hence \(\mathcal{W}_c^s(0; c, a)\) intersects \(\Sigma^m\) at the point \((y_2^s(s; c, a), z_B)\) when \((c, a) = (c^w, a^w)(z_B)\) defined above. We now match with the tail manifold \(\mathcal{T}_c(c, a)\) by solving

\[
z_B = z_c^T(y_2^s(s; c, a); c^w(z_B), a^w(z_B)),
\] (3.38)
which, using Lemma 3.8, we can solve by the implicit function theorem when \(z_B = z_B^s = \mathcal{O}(\epsilon^{-q/\epsilon})\) to find the desired pulse solution when

\[
\begin{align*}
a &= a(s, \sqrt{\epsilon}) := a^w(z_B^s; s, \sqrt{\epsilon}) \\
c &= c(s, \sqrt{\epsilon}) := c^w(z_B^s; s, \sqrt{\epsilon}).
\end{align*}
\] (3.39)

### 3.5 Type 4 & 5 Pulses

Type 4 & 5 pulses correspond to \(\Gamma(s)\) for \(s \in (w^1 + \Delta_w, 2w^1 - \Delta_w)\). Type 4 pulses are right transitional pulses with a secondary pulse of heights \(w \in (w_A + \Delta_w, w^1 - \Delta_w)\), and type 5 pulses are right transitional pulses with a secondary pulse of height \(w \in (\Delta_w, w_A + \Delta_w)\). For type 4 & 5 pulses, the secondary pulses pass close to the upper right fold point. These pulses are constructed in much the same way as type 2 pulses, except with a different definition of the solution \(\gamma^{sp}(s; c, a)\) and the associated manifold \(\mathcal{B}(s; c, a)\). In terms of the actual construction of the pulses, there is no distinction between pulses of type 4 and 5. We distinguish these pulses, however, due to the technical difficulties associated with proving Lemma 3.10 below for the case of type 5 pulses, which is crucial to solving the final matching conditions.

To construct a right transitional pulse with secondary height \(w = 2w^1 - s\), we first consider the plane \(w = 2w^1 - s\) which intersects the section \(\Sigma^{h,r} := \{u = 2/3, \Delta_w < w < w^1 - \Delta_w\}\) in a line \(\{u = 2/3, w = 2w^1 - s\}\). This line transversely intersects the manifold \(\mathcal{W}_c^{s,r}(c, a)\) for all \((c, a) \in I_c \times I_a\). Using arguments similar to those in [7, \S 5] in the proof of Proposition 2.6, it follows that the forward evolution of this line transversely intersects \(\mathcal{W}_c^{s,l}(c, a)\) for each \((c, a) \in I_c \times I_a\) and each sufficiently small \(\epsilon > 0\) along a trajectory \(\gamma^{sp}(s; c, a)\). Furthermore, the solution \(\gamma^{sp}(s; c, a)\) is exponentially close to \(\mathcal{W}_c^{s,r}(c, a)\) in \(\Sigma^{h,r}\) and passes \(\mathcal{O}(\epsilon^{2/3} + |a|)\) close to the fold before intersecting \(\mathcal{W}_c^{s,l}(c, a)\).

The geometry of the setup for type 4 pulses and the solution \(\gamma^{sp}(s; c, a)\) is shown in Figure 15.

Proceeding as with type 2 pulses, we follow \(\gamma^{sp}(s; c, a)\) along \(\mathcal{W}_c^{s,l}(c, a)\) where it is exponentially contracted to \(\mathcal{M}^c(c, a)\) and intersects the section \(\Sigma^m\) at a point \((y_2^s(s; c, a), 0)\). We again define \(\mathcal{B}(s; c, a)\) to be the backwards evolution of the fiber \(\{(0, y_2^s(s; c, a), z) : |z| \leq \Delta_z\}\). We parametrize the manifold \(\mathcal{B}(s; c, a)\) by \(\{z_B, |z_B| \leq \Delta_z\}\) corresponding to the initial height along the fiber in \(\Sigma^m\). Assuming this manifold is well defined and exponentially close to \(\gamma^{sp}(s; c, a)\) in \(\Sigma^{h,r}\) (and the derivatives of the transition maps with respect to \((c, a, z_B)\) are also exponentially small), the remainder of the construction follows similarly to the case of type 2 pulses. A schematic of the manifold \(\mathcal{B}(s; c, a)\) and its relation to \(\gamma^{sp}(s; c, a)\) is shown in Figure 15.

The fact that \(\mathcal{B}(s; c, a)\) is exponentially close to \(\gamma^{sp}(s; c, a)\) in \(\Sigma^{h,r}\) is due to the following lemma, proved in Appendix B.
Figure 15: Shown is the geometry for constructing a type 4 right transitional pulse.

**Lemma 3.9.** For each sufficiently small $\Delta_w$, there exists $\epsilon_0, q > 0$ and sufficiently small choice of the intervals $I_c, I_a$, such that for each $0 < \epsilon < \epsilon_0$, each $(c, a) \in I_c \times I_a$, and each $s \in (w^\dagger + \Delta_w, 2w^\dagger - \Delta_w)$, the following holds. In $\Sigma^{h, r}$, $\mathcal{B}(s; c, a)$ is given by a set of points $(u, v, w)$ satisfying

$$(u, v, w) = \left(\frac{2}{3}, v^{sp}(s; c, a), 2w^\dagger - s\right) + (0, v_B(z_B, s; c, a), w_B(z_B, s; c, a)),$$

where

$$|v_B(z_B, s; c, a)|, |w_B(z_B, s; c, a)| = \mathcal{O}(e^{-q/\epsilon}),$$

along with their derivatives with respect to $(z_B, c, a)$ uniformly in $|z_B| \leq \Delta_z$ and $(c, a) \in I_c \times I_a$.

### 3.5.1 Matching conditions for pulses $\Gamma(s, \sqrt{\epsilon})$, $s \in (w^\dagger + \Delta_w, 2w^\dagger - \Delta_w)$

Evolving $\mathcal{B}(s; c, a)$ backwards and using Lemma 3.9, we have that the backwards evolution $\hat{\mathcal{B}}(s; c, a)$ of $\mathcal{B}(s; c, a)$ intersects $\Sigma^m$ exponentially close to $\mathcal{M}^m(c, a)$. Thus we have that in $\Sigma^m$, $\hat{\mathcal{B}}(s; c, a)$ is given by a curve $(y_2, z) = (y_2^b, z^b)(z_B, s; c, a)$ which satisfies

$$|y_2^b(z_B, s; c, a) - y_2^{M, m}| = \mathcal{O}(e^{-q/\epsilon}),$$

$$|z^b(z_B, s; c, a)| = \mathcal{O}(e^{-q/\epsilon}),$$

uniformly in $|z_B| \leq \Delta_z$.

Recall from §3.3 that in the section $\Sigma^m$, the tail manifold $\mathcal{T}_c(c, a)$ is defined by the graph of a smooth function given by $z = z^T_\epsilon(y_2; c, a)$ for $y_2 \geq y^{\ell, \ast}_{2, 0}(c, a)$ where

$$z^T_\epsilon(y_2; c, a) = \begin{cases} 0 & y_2 \geq y^{\ell, \ast}_{2, 0}(c, a) \\ z^{\ell, \ast}(y_2; c, a) & y^{\ell, \ast}_{2, 0}(c, a) \leq y_2 \leq y^{\ell}_{2, 0}(c, a) \end{cases}$$

and the function $z^{\ell, \ast}$ and its derivatives are $\mathcal{O}(e^{-q/\epsilon})$. We have the following analogue of Lemma 3.8 which will be proved in §5.
Lemma 3.10. For each sufficiently small $\Delta_w > 0$, there exists $\epsilon_0, \Delta_z, q > 0$ and sufficiently small choice of the intervals $I_c, I_a$ such that for each $s \in (w^1 + \Delta_w, 2w^1 - \Delta_w)$ and each $0 < \epsilon < \epsilon_0$, the following hold. Firstly, the backwards evolution $\mathcal{B}(s; c, a)$ of the manifold $\mathcal{B}(s; c, a)$ intersects $\Sigma^m$ in a curve $(y_2, z) = (y^b_2, z^b)(z_B, s; c, a)$ which satisfies

$$
|y^b_2(z_B, s; c, a) - y^{M,m}_2| = O(e^{-q/\epsilon})
$$

$$
|z^b(z_B, s; c, a)| = O(e^{-q/\epsilon}),
$$

uniformly in $|z_B| \leq \Delta_z$ and $(c, a) \in I_c \times I_a$; the derivatives of the above expressions with respect to $(c, a, z_B)$ are also $O(e^{-q/\epsilon})$. Secondly,

$$
y_{2,0}^L(c, a) < \inf_{|z_B| \leq \Delta_z} y^b_2(z_B, s; c, a),
$$

for all $(c, a) \in I_c \times I_a$.

The first assertion of Lemma 3.10 follows from the analysis above; the proof of the second assertion will be given in §5.6.

From Proposition 3.1, we have that the distance between the manifolds $\mathcal{M}_e^l(c, a)$ and $\mathcal{M}_e^m(c, a)$ in $\Sigma^m$ is given by

$$
y^{M,l}_2 - y^{M,m}_2 = D_0(\alpha_2, r_2; c) = d_{\alpha_2} \alpha_2 + d_{r_2} r_2 + O(2).
$$

Thus we can match $\mathcal{W}_e^m(0; c, a)$ with any solution in $\mathcal{B}(s; c, a)$ by solving

$$
D_0(\alpha_2, r_2; c) = y^u_a(z; c, a) - y^{M,l}_2 - \left(y^b_2(z_B, s; c, a) - y^{M,m}_2\right)
$$

$$
= z^b(z_B, s; c, a),
$$

for each $|z_B| \leq \Delta_z$. For each such $z_B$, we obtain a solution by solving

$$
a = 2c^{1/2} \epsilon^{1/2} (a^c_2 + O(e^{-q/\epsilon}))
$$

$$
c = \epsilon(\alpha, \epsilon) + O(e^{-q/\epsilon}),
$$

by the implicit function theorem to find $(a, c) = (a^u, c^u)(z_B; s, \sqrt{\epsilon})$ where we have

$$
(c^u, a^u)(z_B; s, \sqrt{\epsilon}) = (c_E, a_E)(\sqrt{\epsilon}) + O(e^{-q/\epsilon}),
$$

uniformly in $(z_B, s)$, where $(c_E, a_E)$ is the unique solution of (3.14).

We now evolve $\mathcal{W}_e^m(0; c, a)$ forwards; for each $z_B$ we can hit the corresponding point on $\mathcal{B}(s; c, a)$, hence $\mathcal{W}_e^m(0; c, a)$ intersects $\Sigma^m$ at the point $(y^{2p}_2(s; c, a), z_B)$ when $(c, a) = (c^u, a^u)(z_B)$ defined above. We now match with the tail manifold $\mathcal{T}_e(c, a)$ by solving

$$
z_B = z^T_e(y^{2p}_2(s; c, a); c^u(z_B), a^u(z_B)),
$$

which, using Lemma 3.10, we can solve by the implicit function theorem when $z_B = z^*_B = O(e^{-q/\epsilon})$ to find the desired pulse solution when

$$
a = a(s, \sqrt{\epsilon}) := a^u(z^*_B; s, \sqrt{\epsilon})
$$

$$
c = c(s, \sqrt{\epsilon}) := c^u(z^*_B; s, \sqrt{\epsilon}).
$$

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3.6 Type 3 pulses

We now consider type 3 pulses. Type 3 pulses are those with secondary heights which are close to the upper right fold point; these pulses encompass the transition from left pulses to right pulses and hence form a bridge between type 2 and type 4 pulses. We construct these in a manner similar to type 2 pulses, but they are not parametrized naturally by the height of the secondary pulse. To set up a parametrization of these pulses, we change to local coordinates in a neighborhood of the upper right fold point [7, §4].

The fold point is given by the fixed point \((u^*, 0, w^*)\) of the layer problem (2.4) where

\[
u^* = \frac{1}{2} \left( a + 1 + \sqrt{a^2 - a + 1} \right),
\]

and \(w^* = f(u^*)\). The linearization of (2.4) about this fixed point has one positive real eigenvalue \(c > 0\) and a double zero eigenvalue, since \(f'(u^*) = 0\). As in [7], there exists a neighborhood \(U_F\) of the upper right fold point, in which we can perform the following \(C^r\)-change of coordinates \(\Phi_c : U_F \to \mathbb{R}^3\) to (1.2), which is \(C^r\)-smooth in \(c, a\) and \(\epsilon\) for \((c, a, \epsilon)\)-values restricted to the set \(I_c \times I_a \times [0, \epsilon_0]\), where \(\epsilon_0 > 0\) is chosen sufficiently small. We apply \(\Phi_c\) in the neighborhood \(U_F\) of the fold point and rescale time by a positive constant \(\theta_0\) given by

\[
\theta_0 = \frac{1}{c^{2/3}} \left( a^2 - a + 1 \right)^{1/6} (u^* - \gamma w^*)^{1/3} > 0,
\]

uniformly in \((c, a) \in I_c \times I_a\), so that (1.2) becomes

\[
x' = (y + x^2 + h(x, y, \epsilon; c, a)),
\]

\[
y' = \epsilon g(x, y, \epsilon; c, a),
\]

\[
z' = z \left( \frac{c}{\theta_0} + O(x, y, \epsilon) \right),
\]

where \(h, g\) are \(C^r\)-functions satisfying

\[
h(x, y, \epsilon; c, a) = O(\epsilon, xy, y^2, x^3),
\]

\[
g(x, y, \epsilon; c, a) = 1 + O(x, y, \epsilon),
\]

uniformly in \((c, a) \in I_c \times I_a\). In the transformed system (3.53), the \(x, y\)-dynamics is decoupled from the dynamics in the \(z\)-direction along the straightened out strong unstable fibers. Thus, the flow is fully described by the dynamics on the two-dimensional invariant manifold \(z = 0\) and by the one-dimensional dynamics along the fibers in the \(z\)-direction.

We consider the flow of (3.53) on the invariant manifold \(z = 0\). We append an equation for \(\epsilon\), arriving at the system

\[
x' = y + x^2 + h(x, y, \epsilon; c, a),
\]

\[
y' = \epsilon g(x, y, \epsilon; c, a),
\]

\[
\epsilon' = 0.
\]

For \(\epsilon = 0\), this system possesses a critical manifold given by \(\{(x, y) : y + x^2 + h(x, y, 0, c, a) = 0\}\), which in a sufficiently small neighborhood of the origin is shaped as a parabola opening downwards. The branch of this parabola for \(x < 0\) is attracting and corresponds to the manifold \(M_0^-\). We define \(M_0^{c,+}\) to be the singular trajectory obtained by appending the fast trajectory given by the line \(\{(x, 0) : x > 0\}\) to the attracting branch \(M_0^-\) of the critical manifold. In [7] it was shown that, for sufficiently small \(\epsilon > 0\), \(M_0^{c,+}\) perturbs to a trajectory \(M_0^{c,+}\) on \(z = 0\), represented as a graph \(y = s_\epsilon(x; c, a)\), which is \(a\)-uniformly \(O(\epsilon^{2/3})\)-close to \(M_0^{c,+}\) (see Figure 16 – note that in this figure, \(x\) increases to the left). The branch of the critical manifold corresponding to \(x > 0\), which we denote by \(S_\epsilon^-(c, a)\), is repelling and corresponds to the manifold \(M_0^{c,+}\) and is normally hyperbolic.
away from the fold point. Thus by Fenichel theory, this critical manifold persists as a slow manifold $S^-_c (c, a)$ for sufficiently small $\epsilon > 0$ and consists of a single solution. This slow manifold is unique up to exponentially small errors. We will be concerned with trajectories which are exponentially contracted to $S^-_c (c, a)$ in backwards time (see Figure 16).

**Remark 3.11.** We use the notation $S^-_c (c, a)$ rather than $M^{\text{in}}_c (c, a)$ as in general these manifolds do not coincide. This is due to the fact that the choice of $M^{\text{in}}_c (c, a)$ was made so that $M^{\text{in}}_c (c, a)$ would lie in the manifold $W_{s,\epsilon} (c, a)$ in a neighborhood of the canard point at the origin.

We determine the location of $W^{s,\epsilon}_\ell (c, a)$ in the neighborhood $U_F$. From [7, §5], we know that $W^s (M^{\epsilon}_0 (c^* (0), 0))$ intersects $W^u (M^{\epsilon}_0 (c^* (0), 0))$ transversely for $\epsilon = 0$ along the Nagumo back $\phi_b$, and this intersection persists for $(c, a, \epsilon) \in I_c \times I_a \times (0, \epsilon_0)$. This means that $W^{s,\epsilon}_\ell (c, a)$ will transversely intersect the manifold $W^{u,\epsilon}_\ell (c, a)$ which is composed of the union of the unstable fibers of the continuation of the slow manifold $M^{\epsilon}_c (c, a)$ found in [7, §4]. We define the exit section $\Sigma^\text{out}$ by

$$\Sigma^\text{out} = \{ z = \Delta' \}.$$  (3.55)

For $(c, a, \epsilon) \in I_c \times I_a \times (0, \epsilon_0)$, the intersection of $W^{s,\epsilon}_\ell (c, a)$ and $W^{u,\epsilon}_\ell (c, a)$ occurs at a point

$$(x, y, z) = (x_\ell (c, a, \epsilon), s_\epsilon (x_\ell (c, a, \epsilon); c, a), \Delta') \in \Sigma^\text{out},$$  (3.56)

and thus we may expand $W^{s,\epsilon}_\ell (c, a)$ in $\Sigma^\text{out}$ as

$$(x, y) = (x_\ell (c, a, \epsilon) + O(y - s_\epsilon (x; c, a, \epsilon), y), y) \in [-\Delta_y, \Delta_y],$$  (3.57)

for some small $\Delta_y > 0$. The goal is now to parametrize the construction of type 3 pulses by the coordinate $y$, which parametrizes trajectories on the manifold $W^{s,\epsilon}_\ell (c, a)$. By taking $\Delta_w$ sufficiently small, it is clear that there is overlap with the construction of type 2 pulses. We will argue that there is also overlap with the construction of type 4 pulses by considering an appropriate range of $y$-values.

We now place a section $\Sigma^{i,-}$ defined by

$$\Sigma^{i,-} = \{ (x, y, z) : 0 \leq x \leq 2 \rho, y = -\rho^2, |z| \leq \Delta' \},$$  (3.58)
and we note that the manifold $S^+_\epsilon(c,a)$ intersects $\Sigma^{i-}$ for all sufficiently small $\epsilon > 0$. We define two other sections
\[
\Sigma^{i-} = \{(x,y,z) : -2\rho \leq x \leq 0, y = -\rho^2, |z| \leq \Delta\} \\
\Sigma^{i+} = \{(x,y,z) : x = -2\rho, |y| \leq \rho^2, |z| \leq \Delta\}. 
\]
(3.59)
(3.60)
We now note that, provided $\Delta_y$ is sufficiently small, any trajectory $W_{\epsilon}^{s,f}(c,a)$ in $\Sigma^{out}$ leaves a neighborhood of the fold point through one of the sections $\Sigma^{i-}, \Sigma^{i+}, \Sigma^{i''}$ in backwards time. We construct type 3 pulses by considering only heights $y$ which correspond to trajectories on $W_{\epsilon}^{s,f}(c,a)$ which exit via $\Sigma^{i-}$ in backwards time. The setup is shown in Figure 16.

Firstly, all solutions in $\Sigma^{i-}$ are exponentially contracted in backwards time to $M^m(c,a)$ upon arrival in the section $\Sigma^m$ near the lower left fold point. We first consider the trajectory $S^-_\epsilon(c,a)$. Tracking backwards down the middle slow manifold, we have that $S^-_\epsilon(c,a)$ is uniformly $O(e^{-q/\epsilon})$-close to $M^m(c,a)$ in $\Sigma^m$ for $(c,a) \in I_x \times I_a$.

Using similar arguments as in the construction of type 2 pulses, for each sufficiently small $\epsilon$ there exist values of $(c,a) = (c^-, a^-)(\sqrt{\epsilon})$ such that the backwards evolution of $S^-_\epsilon(c,a)$ can be matched with $W_{\epsilon}^{p}(0;c,a)$ in the section $\Sigma^m$. We now consider trajectories on $W_{\epsilon}^{s,f}(c,a)$ in $\Sigma^{out}$ which pass through $\Sigma^{i-}$ in backwards time. As stated above, trajectories on $W_{\epsilon}^{s,f}(c,a)$ in $\Sigma^{out}$ are parametrized by $y \in [-\Delta_y, \Delta_y]$. For $(c,a) = (c^-, a^-)$, for each $\bar{y}$ sufficiently small, (for instance, $-\Delta_y < \bar{y} < -\Delta_y/2$), for all $\epsilon \in (0,\epsilon_0)$, the trajectory which intersects $W_{\epsilon}^{s,f}(c,a)$ in $\Sigma^{out}$ at $\bar{y}$ contracts exponentially in backwards time to $S^-_\epsilon(c,a)$ and therefore passes through $\Sigma^{i-}$.

We can therefore define the supremum of all such values of $\bar{y}$ to be $y^-$. That is, $y^-$ is defined as the supremum of $\bar{y}$-values such that the trajectory which intersects $W_{\epsilon}^{s,f}(c^-, a^-)$ in $\Sigma^{out}$ at height $\bar{y}$ passes through $\Sigma^{i-}$ in backwards time.

We now show that for each sufficiently small $\epsilon > 0$ and each $\bar{y} \in [-\Delta_y, y^-]$, we can construct a transitional pulse with secondary excursion which passes near the upper right fold and intersects the section $\Sigma^{out}$ passing exponentially close to the point $(x,y) = (\ell(c,a,\epsilon) + \mathcal{O}(\bar{y} - s(x,c,a,\epsilon),\bar{y})$ on the manifold $W_{\epsilon}^{s,f}(c,a)$. In this sense, $\bar{y}$ will serve as a parameterization of such transitional pulses passing near the fold. We then argue below that there is no overlap between these and the construction of pulses of type 2 and 4.

To proceed, we show that for each such $\bar{y}$, a transitional pulse can be constructed following the same procedure as with type 2 pulses, though extra care is needed to make sure that the passage near the fold does not cause the argument to break down. We define the solution $\gamma^{sp}(\bar{y}; c^-, a^-)$ on the stable foliation $W_{\epsilon}^{s,f}(c^-, a^-)$ which intersects the section $\Sigma^{out}$ at height $\bar{y}$. This solution is exponentially attracted in forward time to $M^s(c^-, a^-)$ and hence intersects $\Sigma^m$ at the point $(y^{sp}_{2}(c^-, a^-), 0)$ where
\[
e^{-\eta_1/\epsilon}/C \leq y^{sp}_{2}(c^-, a^-) - y^{sp}_{2}(0;c^-, a^-) \leq C e^{-q_2/\epsilon}. 
\]
(3.61)
Define the manifold $B(\bar{y}; c^-, a^-)$ to be the backwards evolution of the fiber
\[
\{(0, y^{sp}_{2}(c^-, a^-), z) : |z| \leq \Delta_z\}. 
\]
(3.62)
In backwards time, this fiber is exponentially contracted to the solution $\gamma^{sp}(\bar{y}; c^-, a^-)$ and hence intersects $\Sigma^{out}$ in a one-dimensional curve which is $O(e^{-q/\epsilon})$-close to $\gamma^{sp}(\bar{y}; c^-, a^-)$. Because $\bar{y} < y^-$, in backwards time $\gamma^{sp}(\bar{y}; c^-, a^-)$ hits the section $\Sigma^{i-}$. The passage in backwards time from $\Sigma^{out}$ to $\Sigma^{i-}$ defines a map which is at worst expands exponentially at a rate $e^{\eta_1/\epsilon}$, where $\eta$ can be made arbitrarily small by shrinking the fold neighborhood. In particular, we can ensure that $\eta < q$. Hence we can ensure that this potential expansion is always compensated by the contraction occurring along the fibers of $W_{\epsilon}^{s,f}(c,a)$ in the passage in backwards time from $\Sigma^m$ to $\Sigma^{out}$. Hence the intersection of the backwards evolution of $B(\bar{y}; c^-, a^-)$ also defines a one-dimensional manifold in $\Sigma^{i-}$ which is $O(e^{-q/\epsilon})$-close to $\gamma^{sp}(\bar{y}; c^-, a^-)$, where $q$ may have to be slightly decreased.

We will now show that the results above hold for an interval of parameters $(c,a)$ exponentially close to $(c^-, a^-)$, that is, we write $(c,a) = (c^-, a^-) + (\hat{c}, \hat{a})$ and consider values $|\hat{c}|, |\hat{a}| \leq C e^{-2\eta/\epsilon}$. For all sufficiently small
\(e > 0\), we claim that the above assertions continue to hold uniformly for all such \((\bar{c}, \bar{a})\). We define the solution \(\gamma^{sp}(\bar{y}; c, a)\) on \(\mathcal{W}^{\beta, \ell}_{c,a}\) which intersects the section \(\Sigma^{\text{out}}\) at height \(y = \bar{y}\). This solution intersects \(\Sigma^m\) at the point \((y_2^{sp}(\bar{y}; c, a), 0)\) where

\[
e^{-q_1/\epsilon}/C \leq y_2^{sp}(\bar{y}; c, a) - y_2^{u}(0; c, a) \leq Ce^{-q_2/\epsilon}.
\]

uniformly in \((c, a)\). Again we define the manifold \(\mathcal{B}(\bar{y}; c, a)\) to be the intersection of the backwards evolution of the fiber \(\{(0, y_2^{sp}(\bar{y}; c, a), z) : |z| \leq \Delta_z\}\) with the section \(\Sigma^{\text{out}}\). In backwards time, this fiber is exponentially contracted to the solution \(\gamma^{sp}(\bar{y}; c, a)\) and hence intersects \(\Sigma^{\text{out}}\) in a one-dimensional curve which is \(\mathcal{O}(e^{-q/\epsilon})\)-close to \(\gamma^{sp}(\bar{y}; c, a)\). In backwards time, for any \(|\bar{c}|, |\bar{a}| \leq Ce^{-2n/\epsilon}\), \(\gamma^{sp}(\bar{y}; c, a)\) hits the section \(\Sigma^{i,-}\), and the backwards evolution of \(\mathcal{B}(\bar{y}; c, a)\) also defines a one-dimensional manifold in \(\Sigma^{i,-}\) which is \(\mathcal{O}(e^{-q/\epsilon})\)-close to \(\gamma^{sp}(\bar{y}; c, a)\) uniformly in \(|\bar{c}|, |\bar{a}| \leq Ce^{-2n/\epsilon}\), where \(q\) may have to be slightly decreased. Similar estimates also hold for the derivatives of the transition maps from \(\Sigma^m\) to \(\Sigma^{i,-}\).

Since \(\mathcal{B}(\bar{y}; c, a)\) intersects \(\Sigma^m\) in a vertical fiber \(\{(0, y_2^{sp}(\bar{y}; c, a), z) : |z| \leq \Delta_z\}\), we parameterize the trajectories on \(\mathcal{B}(\bar{y}; c, a)\) by \(z_B\) denotes the height along the fiber. Consider the backwards evolution \(\hat{\mathcal{B}}(\bar{y}; c, a)\) of \(\mathcal{B}(\bar{y}; c, a)\) back to \(\Sigma^m\), for any \(|\bar{c}|, |\bar{a}| \leq Ce^{-2n/\epsilon}\), we have that \(\hat{\mathcal{B}}(\bar{y}; c, a)\) is \(\mathcal{O}(e^{-q/\epsilon})\)-close to \(\mathcal{M}^{\text{sp}}_{c,a}(c, a)\) in \(\Sigma^m\). Thus we have that in \(\Sigma^m\), \(\hat{\mathcal{B}}(\bar{w}; c, a)\) is given by a curve \((y_2, z) = (y_2^b, z^b)(z_B, \bar{y}; c, a)\) which satisfies

\[
|y_2^b(z_B, \bar{y}; c, a) - y_2^{M,m}\| = \mathcal{O}(e^{-q/\epsilon}),
\]

\[
|z^b(z_B, \bar{y}; c, a)| = \mathcal{O}(e^{-q/\epsilon}),
\]

uniformly in \(z_B\) \(\leq \Delta_z\) and \(|\bar{c}|, |\bar{a}| \leq Ce^{-2n/\epsilon}\).

We can now repeat the argument in the construction of type 2 pulses, given the uniformity of the above estimates in \(|\bar{c}|, |\bar{a}| \leq Ce^{-2n/\epsilon}\) and the fact that we only need freedom in the variation in the bifurcation parameters \((c, a)\) of \(\mathcal{O}(e^{-q/\epsilon})\) for perhaps a slightly smaller value of \(q\) to solve the corresponding matching conditions.

### 3.6.1 Overlap with pulses of type 2, 4

The pulses now form a one-parameter family parametrized by the height \(\bar{y}\). By taking \(\Delta_w\) sufficiently small with respect to \(\Delta_u\), it is clear that there is overlap in the construction of type 3 pulses above and the construction of type 2 pulses. That is, type 3 pulses for \(\bar{y}\) near \(\Delta_y\) are constructed in much the same manner as type 2 pulses for \(s\) near \(w^1 - \Delta_w\). Furthermore, again with \(\Delta_w\) sufficiently small with respect to \(\Delta_y\), there is also overlap in the construction of type 4 pulses. Type 4 pulses constructed for \(s\) near \(w^1 + \Delta_w\) will pass through the section \(\Sigma^{i,-}\) before passing \(\mathcal{O}(e^{2/3} + |a|)\)-close to the fold and leaving the section \(\Sigma^{\text{out}}\) exponentially close to \(\mathcal{W}^{\beta, \ell}_{c,a}\).

These pulses therefore overlap with the construction of type 3 pulses for \(\bar{y}\) near \(y^-\).

As all pulses of types 2, 3, 4 were constructed using the implicit function theorem, by local uniqueness the one-parameter families of pulses of types 2, 3, 4 in fact form one continuous family. Hence we reparameterize the family of type 3 pulses by \(s \in (w^1 - \Delta_w, w^1 + \Delta_w)\) rather than the height \(\bar{y}\).

### 3.7 Type 6 pulses

Finally, we consider type 6 pulses, which are right transitional pulses corresponding to \(\Gamma(s)\), \(s \in (2w^1 - \Delta_w, 2w^1)\). To construct type 6 pulses, we proceed as in the case of type 4/5 pulses and analogously define the solution \(\gamma^{sp}(s; c, a)\); this solution is well defined and exponentially close to \(\mathcal{W}^{\beta, \ell}_{c,a}\) in \(\Sigma^{h,\ell}\) (and the derivatives of the transition maps with respect to \((c, a)\) are also exponentially small). However, the challenge comes from the fact that the backwards evolution \(\hat{\gamma}^{sp}(s; c, a)\) of \(\gamma^{sp}(s; c, a)\) is not \(\mathcal{O}(e^{-q/\epsilon})\)-close to \(\mathcal{M}^{\text{sp}}_{c,a}\) in \(\Sigma^m\), and hence more care is required to obtain the corresponding matching conditions.
We begin by describing the manifold $W^{n,r}_c(a) \in (c, a)$ in a neighborhood of the origin. We first write $c = (a, \epsilon) + 0 = 1/\sqrt{2} + O(\epsilon)$, for $|c|$ small and $(a, \epsilon) \in I_a \times (0, \epsilon_0)$, that is, we only allow $c$ to vary from the value $\cd(a, \epsilon)$ by an $O(\epsilon)$ amount. Since $W^{n,r}_c(0; a, \epsilon)$ lies in $W^{n}(M_{a}; c, a)$ for $(c, a) = (1/\sqrt{2}, 0)$, by standard geometric singular perturbation theory, we have that for $(a, \epsilon) \in I_a \times (0, \epsilon_0)$, in the section $\Sigma^{\epsilon} = \{ \epsilon = \Delta \}$, we can write $W^{n,r}_e(c, a)$ as a graph

$$x = x^r(y, \alpha, \epsilon, c) = O(y, \alpha, \epsilon), \quad y \in [-2\Delta_y, 2\Delta_y],$$

(3.65)

for sufficiently small $\Delta_y > 0$.

Since the coordinate transformation $x \to x + g(y, \alpha, \epsilon)$ for a function $g(y, \alpha, \epsilon) = O(y)$ does not change the form of the equations (3.2), we can assume without loss of generality that $W^{n,r}_e(c, a)$ is given by the line

$$x = x^r(\alpha, \epsilon, c) = O(\alpha, \epsilon), \quad y \in [-2\Delta_y, 2\Delta_y],$$

(3.66)

or, equivalently, in the $K_2$ coordinates

$$x_2 = x_2^r(\alpha_2, r_2, c_2) = O(\alpha_2, r_2).$$

(3.67)

### 3.7.1 Dynamics in $K_2$

The desingularized flow in the $K_2$ coordinates is given by

$$x_2' = -y_2 + x_2^2 + r_2 G_1(x_2, y_2) + O(r_2(\alpha_2 + r_2))$$

$$y_2' = x_2 + \alpha_2 + r_2 G_2(x_2, y_2) + O(r_2(\alpha_2 + r_2))$$

$$r_2' = 0$$

$$\alpha_2' = 0,$$

(3.68)

where $\tau = \frac{d}{dt_2}, t_2 = r_2 t$, and

$$G(x_2, y_2) = \left( \begin{array}{c} G_1(x_2, y_2) \\ G_2(x_2, y_2) \end{array} \right) = \left( \begin{array}{c} a_1 x_2 + a_2 x_2 y_2 + a_3 x_2^3 \\ a_4 x_2^2 + a_5 y_2 \end{array} \right),$$

(3.69)

where

$$a_1 = \frac{\partial h_2}{\partial x}(0, 0, 0, 0, c), \quad a_2 = \frac{\partial h_2}{\partial x}(0, 0, 0, 0, c), \quad a_3 = \frac{\partial h_1}{\partial x}(0, 0, 0, 0, c),$$

$$a_4 = \frac{\partial h_1}{\partial x}(0, 0, 0, 0, c), \quad a_5 = h_0(0, 0, 0, 0, c)$$

and the functions $h_1$ are as in (3.2). For $r_2 = \alpha_2 = 0$, the system is integrable with constant of motion

$$H(x_2, y_2) = \frac{1}{2} e^{-2y_2} \left( y_2^2 - x_2^2 + \frac{1}{2} \right).$$

(3.70)

The function $H$ has a continuous family of closed level curves

$$\Gamma^h_2 = \{(x_2, y_2) : H(x_2, y_2) = h\}, \quad h \in (0, 1/4)$$

(3.71)

contained in the interior of the parabola $y_2 = x_2^2 - 1/2$, which is the level curve for $h = 0$. We define the trajectories $\gamma_2^h(t) = (x_2^h(t), y_2^h(t))$ to be the closed orbits corresponding to $\Gamma^h_2$, satisfying $x_2^h(0) = 0$ and $y_2^h(0) > 0$, and we let $T^h$ denote the half period of the orbit $\Gamma^h_2$. For $h = 0$, we parametrize the parabola $\gamma_2^0(t)$ as

$$(x_2^0(t), y_2^0(t)) = \left( \frac{t}{2}, \frac{t^2}{4} - \frac{1}{2} \right).$$

(3.72)
For \((\alpha_2, r_2)\) sufficiently small and \(y^h_2(0) > 0\) bounded away from zero uniformly in \((\alpha_2, r_2)\), define \(\gamma^{h,r}_{\alpha_2,r_2}(t), \hat{\gamma}^{h,r}_{\alpha_2,r_2}(t)\) to be the forward/backward solutions of (3.68) satisfying \(\gamma^{h,r}_{\alpha_2,r_2}(0) = \hat{\gamma}^{h,r}_{\alpha_2,r_2}(0) = (x^r_2(\alpha_2, r_2, c_2), y^r_2(0))\) so that \(\gamma^{h,r}_{0,0}(0) = \hat{\gamma}^{h,r}_{0,0}(0) = \gamma^h_2(0)\). Denote by \(y^{\hat{h},r}_{\alpha_2,r_2}, \hat{y}^{\hat{h},r}_{\alpha_2,r_2} < 0\) the \(y_2\) values at which \(\gamma^{h,r}_{\alpha_2,r_2}(t), \hat{\gamma}^{h,r}_{\alpha_2,r_2}(t)\) reach \(x_2 = 0\).

For \(h \in (0, h_0)\), where \(h_0\) is sufficiently small, we are interested in the distance function

\[
D_h(\alpha_2, r_2, h) := H(0, y^h_2) - H(0, \hat{y}^{h}_2).
\]

A zero of this function corresponds to a connection between \(M^c(c, a)\) and a trajectory reaching \((x^r_2(\alpha_2, r_2, c_2), y^r_2(0))\) in the chart \(\mathcal{K}_2\). We have the following

**Proposition 3.12.** Fix \(h_0, \Delta_y > 0\) sufficiently small. There exists \(\epsilon_0 > 0\) such that for each \(r_2 \in (0, \sqrt{\epsilon_0})\), there exists \(h_L > 0\) and a curve \(\alpha_2 = \alpha^h_2(r_2, c_2)\) satisfying

\[
D_h(\alpha^h_2(r_2, c_2), r_2, h) = 0, \quad h \in (h_L, h_0).
\]

Furthermore, we have that \(r^2_2 y^{h_2}_2(0) = \Delta_y (1 + O(\Delta_y)).\)

**Proof.** The proof follows from similar arguments as in [23] and in the proof of [24, Theorem 4.2]; here we briefly outline the differences.

Proceeding as in [23, §3.6], we compute that

\[
H(0, y^h_2) = r_2 \int_{0}^{\infty} \nabla H(\gamma^h_2(t)) \cdot G(\gamma^h_2(t)) dt + \alpha_2 \int_{-\infty}^{0} \nabla H(\gamma^h_2(t)) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt + O((r_2 + |\alpha_2|)^2),
\]

and following the proof of [24, Proposition 4.1], we have that

\[
H(0, \hat{y}^{h}_2) = h - r_2 \int_{-T}^{0} \nabla H(\gamma^h_2(t)) \cdot G(\gamma^h_2(t)) dt - \alpha_2 \int_{-T}^{0} \nabla H(\gamma^h_2(t)) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt + O((r_2 + |\alpha_2|)^2).
\]

The main distinction between the argument in [24] and the derivation of the expression (3.76) above is that there are errors incurred due to the choice \(\gamma^{h}_{\alpha_2,r_2}(0) = (x^r_2(\alpha_2, r_2, c_2), y^r_2(0)) = (O(\alpha_2, r_2), y^r_2(0))\) instead of \(\hat{\gamma}^{h}_{\alpha_2,r_2}(0) = (0, y^h_2(0))\); however, by tracking the argument in the proof of [24, Proposition 4.1], we see that these errors can be absorbed into the quadratic terms, and we omit the details.

It follows that

\[
D_h(\alpha_2, r, h) = -h + d^h_{\alpha_2} \alpha_2 + d^h_{r_2} r_2 + O((r_2 + |\alpha_2|)^2),
\]

where

\[
d^h_{\alpha_2} = \int_{-T}^{0} \nabla H(\gamma^h_2(t)) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt + \int_{-T}^{0} \nabla H(\gamma^h_2(t)) \cdot G(\gamma^h_2(t)) dt,
\]

\[
d^h_{r_2} = \int_{-T}^{0} \nabla H(\gamma^h_2(t)) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt + \int_{-T}^{0} \nabla H(\gamma^h_2(t)) \cdot G(\gamma^h_2(t)) dt,
\]

with \(d^h_{\alpha_2} > 0\) uniformly in \(h \in (0, h_0)\). Though the difference function \(D_h\) differs from that in [24] due to the presence of the \(-h\) term, we may carry through the argument as in the proof of [24, Theorem 4.1] and see that analogous estimates hold for \(D_h\) and its derivatives in order to obtain the function \(\alpha^h_2(r_2, c_2)\) with the desired properties.

\[\square\]
3.7.2 Matching conditions for pulses \( \Gamma(s, \sqrt{\epsilon}) \), \( s \in (2w^\dagger - \Delta_w, 2w^\dagger) \)

Evolving \( \gamma^{sp}(s; c, a) \) backwards, we have that the backwards evolution \( \hat{\gamma}^{sp}(h; c, a) \) of \( \gamma^{sp}(s; c, a) \) is exponentially close to \( W^{s,t}_e(c, a) \) in \( \Sigma^z := \{ z = \Delta_x \} \). Rather than parameterizing \( \gamma^{sp}(s; c, a) \) by \( s \) it is more natural to use \( h \in (h_L, h_0) \) along \( W^{s,t}_e(c, a) \) from Lemma 3.12 as a parameter (this is in one-to-one correspondence with the height \( y \) along \( W^{s,t}_e(c, a) \) in the section \( \Sigma^m \)). Thus we have that in \( \Sigma^z \), \( \hat{\gamma}^{sp}(h; c, a) \) is given by a point \( (x, y) = (x^h(h; c, a, \epsilon), \epsilon y^h_2(0)) \) which satisfies

\[
|x^h(y; c, a, \epsilon) - x^r(a, \epsilon)| = \mathcal{O}(e^{-q/\epsilon}), \tag{3.79}
\]

uniformly in \((c, a)\). We have the following

**Lemma 3.13.** The backwards evolution of \( \hat{\gamma}^{sp}(s; c, a) \) intersects \( \Sigma^m \) at a point \((y_2, z) = (\hat{y}^{sp}_2(\alpha_2, r_2), \hat{z}^{sp}(\alpha_2, r_2))\) satisfying

\[
|\hat{y}^{sp}_2(\alpha_2, r_2) - \hat{y}^{h,r}_{\alpha_2,r_2}| = \mathcal{O}(e^{-q/r_2})
\]

\[
|\hat{z}^{sp}(\alpha_2, r_2)| = \mathcal{O}(e^{-q/r_2}), \tag{3.80}
\]

along with their derivatives uniformly in \((\alpha_2, r_2)\) sufficiently small.

**Proof.** To see this, we first recall that \( \hat{\gamma}^{sp}(h; c, a) \) intersects \( \Sigma^z \mathcal{O}(e^{-q/\epsilon}) \)-close to \( W^{s,t}_e(c, a) \) at a point \((x, y) = (x^h(h; c, a, \epsilon), \epsilon y^h_2(0)) \) satisfying (3.79). Recalling in the \( K_2 \) coordinates that \( t_2 = r_2 t \), we obtain the desired estimates by evolving backwards by time \( t_2 = T^h + \mathcal{O}(\alpha_2, r_2) \), noting from (3.2) that the \( z \)-coordinate evolves according to

\[
\dot{z} = z \left( e^{3/2} + \mathcal{O}(x, y, z, \alpha, \epsilon) \right), \tag{3.81}
\]

where \( \cdot = \frac{d}{dl} \), hence decaying exponentially in backwards time. \( \Box \)

The final matching conditions can then be obtained as follows. Starting in \( \Sigma^z \), we consider the backwards evolution \( \hat{\gamma}^{sp}(s; c, a) \) of \( \gamma^{sp}(s; c, a) \) back to the section \( \Sigma^m \) and show that for \( h \in (h_L, h_0) \), \( W^{s,t}_e(0; c, a) \) can be matched with \( \hat{\gamma}^{sp}(h; c, a) \) by adjusting \((c, a)\) using Lemma 3.10. We then evolve \( \gamma^{sp}(h; c, a) \) forwards from \( \Sigma^{h,r} \) to \( \Sigma^m \); by construction, here \( \gamma^{sp}(h; c, a) \) lies in \( W^{s,t}_e(c, a) \), and will not reach a height higher than \( \epsilon y^h_2(0) \). By Proposition 3.3, this gives a tail which lies \( W^{s,t}_e(0; c, a) \), completing the construction.

From Proposition 3.12, for \( h \in (h_L, h_0) \) we can match \( W^{s,t}_e(0; c, a) \) with \( \hat{\gamma}^{sp}(h; c, a) \) by solving

\[
D_h(\alpha_2, r_2, h) = \left( H(0, y^M_2) - H(0, y^a_2(z; c, a)) \right) + \left( H(0, \hat{y}^{sp}_2(\alpha_2, r_2)) - H(0, \hat{y}^{h,r}_{\alpha_2,r_2}) \right)
\]

\[
z = \hat{z}^{sp}(\alpha_2, r_2), \tag{3.82}
\]

Using Lemma 3.4, we obtain a solution by solving

\[
a = 2e^{3/2}e^{1/2}(\alpha_2^h(r_2, c_2) + \mathcal{O}(e^{-q/\epsilon}))
\]

\[
c = \tilde{c}(a, \epsilon) + \mathcal{O}(e^{-q/\epsilon}), \tag{3.83}
\]

Recalling

\[
c = \tilde{c}(a, \epsilon) + c_2 \epsilon, \tag{3.84}
\]

by the implicit function theorem we obtain a solution

\[
a = a(h, \sqrt{\epsilon})
\]

\[
c = c(h, \sqrt{\epsilon}), \tag{3.85}
\]
to (3.83) for each \( h \in (h_L, h_0) \).

We can naturally reparameterize these solutions via the invertible map \( h \to y^h_0(0) \) so that they are parameterized by the height \( y = \epsilon y^h_0(0) \) at which the secondary pulse leaves the neighborhood of the origin along \( \mathcal{W}^{\epsilon,r}(c,a) \). This \( y \)-coordinate ranges from \( y \in [y^{h_0}, y^{h_L}] \) where \( y^{h_0} = \epsilon y^{h_0}_0(0) = \mathcal{O}(\epsilon) \) and \( y^{h_L} = \epsilon y^{h_L}_0(0) = \mathcal{O}(\Delta_y(1 + \mathcal{O}(\Delta_y))) \). From this, we first deduce that for the solution in this family reaching \( \Sigma^* \) at \( (x,y) = (x^*(\alpha, \epsilon), y^{h_L}) = \mathcal{O}(\sqrt{\epsilon}) \), the secondary pulse leaves along \( \mathcal{W}^{\epsilon,r}(c,a) \) at an \( \mathcal{O}(\sqrt{\epsilon}) \) distance from the primary pulse. Secondly, from the endpoint at \( y^{h_L} = \mathcal{O}(1) \), we see that there is overlap with the construction of type 5 pulses.

Hence we can reparameterize these pulses by \( s \) so that they form one continuous family with the previously constructed type 5 pulses. For type 6 pulses, \( s \) ranges from \( s = 2w^t - \Delta_w \) to \( s = 2w^t - s_{end}(\sqrt{\epsilon}) \), the end of this family marked by the solution described above with a secondary pulse which is \( \mathcal{O}(\sqrt{\epsilon}) \)-close to the primary pulse.

**Remark 3.14.** We have shown above that type 6 pulses form a bridge between type 5 pulses at \( h = h_L \) and double pulses at \( h = h_0 \) for some small fixed \( h_0 \). Beyond this, our construction procedure breaks down. Using (3.83) and the fact that \( a^{h_0}_2(r_2, c_2) \) solves

\[
\mathcal{D}_h(a^{h_0}_2, r_2, h_0) = 0, \tag{3.86}
\]

where \( \mathcal{D}_h \) is defined by (3.77), we note that the corresponding \( a \)-value is given by

\[
a(h_0, \sqrt{\epsilon}) = 2^{3/4}h_0\sqrt{\epsilon} + \mathcal{O}(\epsilon). \tag{3.87}
\]

We note that this value is less than, but close to the value at which the Belyakov transition (1.3) occurs at

\[
a = 2^{3/4}\sqrt{\epsilon} + \mathcal{O}(\epsilon). \tag{3.88}
\]

This corresponds with the split banana we see in Figure 4. It may be possible to fully understand the termination of this branch of double pulses by unfolding the Belyakov point. We do not develop this further here, but we note that the above construction indeed fails near this region.

### 3.8 Proof of Theorem 2.2

We briefly conclude the proof of Theorem 2.2. For each sufficiently small \( \epsilon > 0 \), the existence of the desired one parameter family of pulse solutions \( \Gamma(s, \sqrt{\epsilon}) \) and their approximation by the singular pulses \( \Gamma_0(s) \) follows from the constructions of pulses of type 1, 2, 3, 4, 5, 6 in §3.2.3.7 and the convergence of the tails in Proposition 3.3 and Corollary 3.7. As all of these were constructed using the implicit function theorem, and the constructions can be taken to have overlapping regions of validity, we obtain a single continuous family of solutions which satisfy the singular approximations (i). The overlap of the family \( \Gamma(s, \sqrt{\epsilon}) \) with the pulses with small exponentially decaying oscillatory tails from [7] for sufficiently small \( s \) was shown in §3.2.2, proving (ii).

Furthermore, the estimates (iii) follow directly from the final matching conditions (3.16), (3.37), (3.49) for each of the pulse types 1, 2, 3, 4, 5, where \( (c_E, a_E) \) are as defined in §3.2.1.

### 4 Analysis of tails

In this section, we prove Proposition 3.3, that is, we show that any transitional pulse landing in \( \mathcal{W}^{\epsilon,0}(c,a) \) in the section \( \Sigma^{h_0,\epsilon} := \{u = 0, \Delta_w < w < w^t - \Delta_w\} \) at a height no higher than \( w \leq w_A + \Delta_w \) in fact lies in the stable manifold \( \mathcal{W}^{\epsilon}(0; c,a) \) of the equilibrium \( (u, v, w) = (0, 0, 0) \).

The argument is broken down as follows: In §4.1, we define a ‘way-in-way-out’ function which determines the relative contraction/expansion rates along \( \mathcal{M}^{\epsilon}_u(c,a), \mathcal{M}^{\epsilon}_v(c,a) \). In §4.2, we use this way-in-way-out function to
show that for sufficiently small $\epsilon > 0$, any transitional pulse with tail in $W_{\epsilon^s,t}(c, a)$ starting in $\Sigma_{h,t}^+$ at a height $w \leq w_A + \Delta_w$ returns at a height $w \leq w_A - \Delta_w$ and remains in $W_{\epsilon^s,t}(c, a)$ with the height on each return to $\Sigma_{h,t}^+$ monotonically decreasing until entering an arbitrarily small $O(1)$ neighborhood of the equilibrium. In §4.3, we show that any such trajectory which reaches this neighborhood in fact converges to the equilibrium, and we complete the proof of Proposition 3.3.

4.1 Way-in-way-out function

We define a way-in-way-out function as in [24] which essentially determines the difference in contraction/expansion rates along the critical manifold $M_0$ in $W_{\epsilon^s,t}(c, a)$. We consider the slow flow restricted to the critical manifold $\{v = 0, w = f(u)\}$ for $(a, \epsilon) = (0, 0)$; the flow satisfies

$$u' = \frac{u - \gamma f(u)}{f'(u)}, \quad (4.1)$$

where $f(u) = u^2(1 - u)$. For $w \in (0, w^1)$, recall the equation $f(u) - w = 0$ has three solutions $u_i(w), i = 1, 2, 3$ where the zeros are indexed in increasing order. For $u < 2/3$, we have $u' > 0$ under the flow of (4.1). Hence given $w^- \in (0, w^1)$ and $w^+ \in (0, w_A)$, there exists a solution $u = u(\tau; w^-, w^+)$ of (4.1) and time $T = T(w^-, w^+)$ such that $u(0; w^-, w^+) = u_1(w^-)$ and $u(T; w^-, w^+) = u_2(w^+)$. So for $(w^-, w^+) \in (0, w^+) \times (0, w_A)$, we now define the way-in-way-out function

$$R(w^-, w^+) := \int_0^{T(w^-, w^+)} \lambda_-(u(\tau; w^-, w^+)) d\tau, \quad (4.2)$$

where

$$\lambda_- = \lambda_-(u) = \frac{c - \sqrt{c^2 - 4f'(u)}}{2} \quad (4.3)$$

is as defined in §2.5. Hence for the solution $u = u(\tau; w^-, w^+)$ of (4.1), the quantity $R(w^-, w^+)$ measures cumulatively the contraction along the left branch $M_0^l$ of the critical manifold from $w = w^-$ to $w = 0$ and the expansion along the middle branch $M_0^m$ from $w = 0$ to $w = w^+$. For $R(w^-, w^+) > 0$, the interpretation is that the expansion along $M_0^m$ from $w = 0$ to $w = w^+$ outweighs the contraction along $M_0^l$ from $w = w^-$ to $w = 0$, yielding a net expansion, and vice versa for $R(w^-, w^+) < 0$.

Using (4.3) and changing variables, we obtain the equivalent definition

$$R(w^-, w^+) := \frac{1}{2} \int_{u_2(w^-)}^{u_2(w^+)} \left( c - \sqrt{c^2 - 4f'(u)} \right) \frac{f'(u)}{u - \gamma f(u)} du, \quad (4.4)$$

and for short, we denote $R(w) := R(w, w)$. We have the following lemma.

**Lemma 4.1.** The way-in-way-out function $R$ satisfies

(i) $R(w) > 0$ for all $w \in (0, w_A)$

(ii) $R(w^1, w_A) < 0$

**Proof.** Straightforward, but tedious computation. \hfill \Box

4.2 Tail sequence

In this section, we show that the tails of the transitional pulses $\Gamma(s, \sqrt{\epsilon})$ constructed in §3 remain in $W_{\epsilon^s,t}(c, a)$, following a sequence of excursions with the height on each return to $\Sigma_{h,t}^+$ monotonically decreasing until entering
with a critical manifold containing a turning point. We consider the flow on a piece of a sequence of excursions which the tail follows.

We need to determine the jump-off or exit height along $\Sigma^{h,\ell}$. To accomplish this, we use the way-in-way-out function $R$. The main idea is to show that the entry/exit heights are approximately related via the equation $R(w^-, w^+) = 0$. The sequence can then be found by solving for successive zeros of $R$.

We begin by discussing the concept of what we call ‘buffer heights’ and the relation to canard trajectories on $\mathcal{W}_e^{s,\ell}(c, a)$, that is, trajectories which first follow the attracting slow manifold $\mathcal{M}_e^s(c, a)$, and then the repelling slow manifold $\mathcal{M}_e^r(c, a)$ for some time. For starting heights $w^-$ below such buffer heights, the above property holds, ensuring the relation of the entry/exit heights and the function $R$. We then use this notion of buffer heights in Proposition 4.4 to show that given a transitional pulse $\Gamma(s, \sqrt{\tau})$, the function $R$ can be used to determine the sequence of excursions which the tail follows.

We primarily refer to the results of [9] in which entry-exit or way-in-way-out functions are described for systems with a critical manifold containing a turning point. We consider the flow on a piece of $\mathcal{W}_e^{s,\ell}(c, a)$ which contains everything below $w = w_a - \Delta w$ as well as trajectories to the left of $\Sigma^{h,\ell}$ for $w \in [w_a - \Delta w, w^+]$: this manifold then satisfies the assumptions T1-T6 in [9].

We now define the notion of buffer heights. These heights essentially determine how the function $R$ can be used to relate the entry/exit heights along $\mathcal{M}_e^s(c, a), \mathcal{M}_e^r(c, a)$: for entry heights below these buffer heights, the entry/exit heights are related by zeros of $R$, while the behavior varies for entry heights above the buffer heights; see Figure 17.

**Definition 4.2.** A pair $(w^-_b, w^+_b)$ satisfying $R(w^-, w^+) = 0$ are called buffer heights on $\mathcal{W}_e^{s,\ell}(c, a)$ if the following holds: Suppose for each sufficiently small $\epsilon > 0$ there exists a canard trajectory $\gamma^C_\epsilon$ at the parameter values $(c, a) = (c, a)(\epsilon)$ where $(\gamma^C_\epsilon, c, a)$ depend continuously on $\epsilon$, and suppose $\gamma^C_\epsilon$ has boundary entry point in $\Sigma^{h,\ell}$.
Figure 18: Shown is a schematic of the extended center-stable manifold $W^{s,ℓ}_ε(c,a)$ and the section $Σ^{h,ℓ}$ and the oscillatory tail sequence $w_i(s,ϵ)$ for a transitional pulse $Γ(s,√ε)$. The specific trajectory shown corresponds to case (iii) from the proof of Proposition 4.4.

given by $b^ε_ℓ$ satisfying $\lim_{ε\to 0} b^ε_ℓ = b^0_ℓ ∈ Σ^{h,ℓ} \cap \{ w = w^- \}$ so that the limit $b^0_ℓ$ lies on the stable foliation of $M^ℓ_0$. Then we have the following

(i) (Tunnel behavior): If $w^- < w^-_b$, then the corresponding exit point $b^m_ℓ$ of $γ^C_ε$ satisfies $\lim_{ε\to 0} b^m_ℓ = b^m_0 ∈ Σ^{h,ℓ} \cap \{ w = w^+ \}$ where $w^+ < w^+_b$ is the unique value satisfying $R(w^-,w^+) = 0$; see Figure 17a.

(ii) (Funnel behavior): If $w^- > w^-_b$, then the corresponding exit point $b^m_ℓ$ of $γ^C_ε$ satisfies $\lim_{ε\to 0} b^m_ℓ = b^m_0 ∈ Σ^{h,ℓ} \cap \{ w = w^+ \}$; see Figure 17b.

(iii) (Comb behavior): If $w^- = w^-_b$, then the corresponding exit point $b^m_ℓ$ of $γ^C_ε$ satisfies $\lim_{ε\to 0} b^m_ℓ = b^m_0 ∈ Σ^{h,ℓ} \cap \{ w = w^+ \}$ for some $w^+ ≥ w^+_b$. (Comb behavior corresponds to funnel behavior when considering the flow in reverse time.)

A pair $(w^-_b, w^+_b)$ satisfying only the condition (i) will be referred to as weak buffer heights.

The next result follows from [9, Theorems 6 & 7] and relates the location of such buffer heights to the existence of a canard trajectory on $W^{s,ℓ}_ε(c,a)$ with specified entry/exit heights. Given entry/exit heights, one can find such a canard solution by varying the parameter $a = a^C(√ε,c) + O(ε^{-q/ε})$ [23]. Hence the next proposition can be thought of as relating the location of the buffer heights to different values of the parameter $a$; however given that $a$ varies on an exponentially small interval, it is more natural to relate the buffer heights to the entry/exit heights of a given canard solution.

**Proposition 4.3.** Suppose for each sufficiently small $ε > 0$ there exists a canard trajectory $γ^C_ε$ at the parameter values $(c,a) = (c,a)(ε)$ where $(γ^C_ε,c,a)$ depend continuously on $ε$, and suppose $γ^C_ε$ meets the section $Σ^{h,ℓ}$ at the
entry/exit points \(b^<_c, b^>_c\) where

\[
\lim_{\epsilon \to 0} b^<_c = b^<_c \in \Sigma^{h,\ell} \cap \{w = w^-\}, \quad \lim_{\epsilon \to 0} b^>_c = b^>_c \in \Sigma^{h,\ell} \cap \{w = w^+\}.
\]

Then exactly one of the following holds:

(i) Suppose \(R(w^-, w^+) > 0\). Define \(w^+_b < w^+\) to be the unique value satisfying \(R(w^-, w^+_b) = 0\). Then \(w^-\) and \(w^+_b\) are buffer heights on \(W^{h,\ell}_{\epsilon,c,a}\).

(ii) Suppose \(R(w^-, w^+) < 0\). Define \(w^-_b < w^-\) to be the unique value satisfying \(R(w^-_b, w^+) = 0\). Then \(w^-_b\) and \(w^+\) are buffer heights on \(W^{h,\ell}_{\epsilon,c,a}\).

(iii) Suppose \(R(w^-, w^+) = 0\). Then \(w^-\) and \(w^+\) are weak buffer heights on \(W^{h,\ell}_{\epsilon,c,a}\).

From this we obtain the following proposition, constructing the sequence of excursions followed by the tail of a transitional pulse \(\Gamma(s, \sqrt{\epsilon})\). See Figure 18 for a schematic depicting the results of Proposition 4.4.

**Proposition 4.4.** Fix \(w_0 > 0\) small. For each sufficiently small \(\Delta_w > 0\), the following holds. For all sufficiently small \(\epsilon > 0\), let \((c,a) \in I_c \times I_a\) be such that there exists a transitional pulse \(\Gamma(s, \sqrt{\epsilon})\) with a tail in \(W^{h,\ell}_{\epsilon,c,a}\) starting in \(\Sigma^{h,\ell}\). There exists \(N(s) \in \mathbb{N}\) such that the tail trajectory remains in \(W^{h,\ell}_{\epsilon,c,a}\), returning to \(\Sigma^{h,\ell}\) at a sequence of heights

\[
w_1(s, \epsilon) > w_2(s, \epsilon) > \ldots \geq w_{N(s)-1}(s, \epsilon) > w_{N(s)}(s, \epsilon).
\]

We have that \(w_i(s, \epsilon) \to w_i(s, 0)\) as \(\epsilon \to 0\) where \(w_i(s, 0)\) are successive zeros of the way-in-way-out function \(R\), that is,

\[
R(w_1(s, 0), w_2(s, 0)) = R(w_2(s, 0), w_3(s, 0)) = \ldots = R(w_{N(s)-1}(s, 0), w_{N(s)}(s, 0)) = 0.
\]

Furthermore, the functions \(w_i(s, 0), i = 1, \ldots, N(s)\) are constant for \(s \in [w^0_R, 2w^+_A - w_A]\), where \(w^0_R > w_A\) is defined by \(R(w^0_R, w_A) = 0\).

**Proof.** We break the proof into four subcases:

(i) \(\Gamma(s, \sqrt{\epsilon}), s \in [w_0, w_A - \Delta_w]\)

(ii) \(\Gamma(s, \sqrt{\epsilon}), s \in [w_A - \Delta_w, w_R(\Delta_w)]\)

(iii) \(\Gamma(s, \sqrt{\epsilon}), s \in [w_R(\Delta_w), 2w^+_A - w_A + \Delta_w]\)

(iv) \(\Gamma(s, \sqrt{\epsilon}), s \in [2w^+_A - w_A + \Delta_w, 2w^+_A - w_0]\)

where \(w_R(\Delta_w) > w_A + \Delta_w\) is defined by \(R(w_R(\Delta_w), w_A - 3\Delta_w) = 0\).

We first consider case (i). In this case, the primary pulse reaches \(\Sigma^{h,\ell}\) near \(w = w^+\) and is matched with a tail which returns to \(\Sigma^{h,\ell}\) at \(w = w_1(s, \epsilon) = s\). By Lemma 4.1(i), \(R(w^+_A, w_A) < 0\); hence since \(s \in [w_0, w_A - \Delta_w]\), we also have \(R(w^+_A, w_1(s, \epsilon)) < 0\). Proposition 4.3(ii) therefore implies that the pair \((w^-_b, w_1(s, \epsilon))\) are buffer heights, where \(w^-_b < w^+_A < w^-_b < \epsilon\) satisfies \(R(w^-_b, w_1(s, \epsilon)) = 0\). Since \(R(w, w) > 0\) for \(0 < w \leq w_A\), on each subsequent return to \(\Sigma^{h,\ell}\) the tail reaches \(w = w_1(s, \epsilon)\), where \(w_{i+1}(s, \epsilon) < w_i(s, \epsilon)\) for each \(i\) and \(w_i(s, \epsilon) = w_i(s, 0)\) where \(R(w_i(s, 0), w_{i+1}(s, 0)) = 0\); after finitely many, say \(N(s)\), such excursions, we have \(w_{N(s)}(s, 0) < w_0\).

We obtain the result for case (iv) similarly to that of case (i) by applying the argument to the secondary pulse instead.

We now consider case (ii). We first claim that the trajectory on \(W^{h,\ell}_{\epsilon,c,a}\) with initial condition in \(\Sigma^{h,\ell}\) at height \(w = w_1(s, \epsilon) = s\), \(s \in [w_A - \Delta_w, w_R(\Delta_w)]\), returns to \(\Sigma^{h,\ell}\) at a height \(w \leq w_A - \Delta_w\). We first consider the
backwards evolution of the trajectory starting in $\Sigma^{h,\ell}$ at height $w = w_A - 2\Delta_w$, which ends up in $\Sigma^{h,\ell}$ at some height $w = w_C(\epsilon)$. Let $w_C = \lim w_C(\epsilon)$. We claim that $R(w_C, w_A - 2\Delta_w) = 0$ and hence by Proposition 4.3, the pair $(w_C^0, w_A - 2\Delta_w)$ are weak buffer heights on $W^{s,\ell}_c(c, a)$. To show this, we assume for contradiction that $R(w_C^0, w_A - 2\Delta_w) < 0$. By Proposition 4.3, we have that the pair $(w_C^0, w_A - 2\Delta_w)$ are buffer heights, where $w_C < w_C^0$ is the unique value satisfying $R(w_C, w_A - 2\Delta_w) = 0$. Thus any solution entering $\Sigma^{h,\ell}$ above $w = w_C^0$ exits $\Sigma^{h,\ell}$ at height $w_A - 2\Delta_w$. This gives the desired contradiction: by assumption, we know the basepoint on $W^{s,\ell}_c(c, a)$ the primary pulse, which enters $\Sigma^{h,\ell}$ near $w = w_1^+$, does not exit at this buffer height but rather continues up to have a secondary pulse at some $w_1(s, \epsilon) \in [w_A - \Delta_w, w_R(\Delta_w)]$. The case $R(w_C^0, w_A - 2\Delta_w) > 0$ can be treated similarly.

Hence the pair $(w_C^0, w_A - 2\Delta_w)$ are weak buffer heights on $W^{s,\ell}_c(c, a)$. Since $w_R(\Delta_w) < w_C^0$, the tail trajectory on $W^{s,\ell}_c(c, a)$ with initial condition in $\Sigma^{h,\ell}$ at height $w_1(s, \epsilon) = s \in [w_A - \Delta_w, w_R(\Delta_w)]$ returns to $\Sigma^{h,\ell}$ at a height $w_2(s, \epsilon) \leq w_A - 2\Delta_w$ where $\lim_{\epsilon \to 0} w_2(s, \epsilon) = w_2(s, \epsilon)$ and $R(w_1(s, 0), w_2(s, 0)) = 0$. Since the tail returns to a height $w \leq w_A - 2\Delta_w$, the remainder of the argument follows as in case (i) above.

Finally, we consider case (iii). A transitional pulse $\Gamma(s, \sqrt{\epsilon}), s \in [w_R(\Delta_w), 2w^+ - w_A + \Delta_w]$, completes a primary excursion, followed by a secondary excursion (determined by $s$), followed by a trajectory lying in the tail manifold, which intersects $W^{s,\ell}_c(c, a)$ in the section $\Sigma^{h,\ell}$ at some height $w_1(s, \epsilon) < w_C^0$ where $w_C^0 = w_A + O(\epsilon^{2/3}) \leq w_A + \Delta_w$ for all sufficiently small $\epsilon > 0$. Following a similar argument for case (ii) above, we deduce that $\lim_{\epsilon \to 0} w_1(s, \epsilon) = w_1(0) \geq w_A - 3\Delta_w$, and subsequent returns to $\Sigma^{h,\ell}$ are similarly determined by successive zeros of the way-in-way-out function $R$.

We now consider the final statement, regarding the fact that the functions $w_i(s, 0), i = 1, \ldots, N(s)$ are constant for $s \in [w_R^0, 2w^+ - w_A]$, where $w_R^0 > w_A$ is defined by $R(w_R^0, w_A) = 0$. We note that the choice of $\Delta_w$ in the above argument is arbitrary. Hence the argument for case (iii) for pulses $\Gamma(s, \sqrt{\epsilon}), s \in [w_R^0, 2w^+ - w_A]$, can be repeated for arbitrarily small $\Delta_w$, provided $\epsilon$ is correspondingly taken sufficiently small, without affecting the limits $w_i(s, 0)$. Hence $w_i(s, 0) = w_i$ for $s \in [w_R^0, 2w^+ - w_A]$ and the sequence $w_i(s), i = 2, \ldots, N(s)$ follows.

### 4.3 Convergence of tails

Once a tail trajectory ends up in $W^{s,\ell}_c(c, a)$ below $w = w_A - \Delta_w$, as shown in §4.2 above, it is stuck in the two-dimensional manifold $W^{s,\ell}_c(c, a)$ and its height is monotonically decreasing on each return to $\Sigma^{h,\ell}$, until entering a small neighborhood of the equilibrium. Hence such a trajectory is approaching the equilibrium; the only way it can fail to lie in the stable manifold $W^{s}_c(0; c, a)$ is if it is blocked by a periodic orbit (in this case $W^{s}_c(0; c, a)$ would topologically take the form of a bounded disc). The aim of this section is to show that any periodic orbit lying in $W^{s,\ell}_c(c, a)$ must be repelling.

In [24], the authors constructed periodic canard orbits in a class of planar systems. Although the entire canard explosion is not possible to construct in the same manner in our case (there is no such two-dimensional invariant manifold which contains the entire S-shaped critical manifold $M_0$), the construction procedure is valid in the two-dimensional manifold $W^{s,\ell}_c(c, a)$ for canard orbits up to height $w = w_A - \Delta_w$. We collect the following results from [24] regarding such periodic solutions.

**Proposition 4.5.** For each $c \in I_c$ and $\epsilon > 0$ sufficiently small, there exists a family of periodic orbits

$$(w, \epsilon) \to (a(w, \epsilon), \Gamma(w, \epsilon))$$

(4.7)

parameterized by the height $w \in (0, w_A - \Delta_w)$ such that $\Gamma(w, \epsilon) \subset W^{s,\ell}_c(c, a(w, \epsilon))$.

(i) Any periodic orbit passing near the critical manifold $M_0$ which is entirely contained in $W^{s,\ell}_c(c, a)$ for $-\Delta_w \leq w \leq w_A - \Delta_w$ is part of this family.
(ii) For $\Delta_w < w < w_A - \Delta_w$, the Floquet exponent $P(w, \epsilon)$ satisfies
\[ P(w, \epsilon) = \frac{1}{\epsilon} (R(w) + \theta(w, \epsilon)), \]
where $\theta$ and $\frac{\partial \theta}{\partial w} \to 0$ uniformly as $\epsilon \to 0$.

(iii) For $0 < w < \Delta_w$ all of the $\Gamma(w, \epsilon)$ are repelling.

In particular, the above implies that for any sufficiently small $\epsilon > 0$, there are no nonrepelling periodic orbits in $\mathcal{W}_{c,a}^s \cap (c,a)$ between $-\Delta_w \leq w \leq w_A - \Delta_w$.

We now complete the proof of Proposition 3.3.

\[ \square \]

## 5 Flow near the Airy point

The goal of this section is to prove Proposition 3.6, regarding properties of the backwards evolution of certain trajectories on $\mathcal{W}_{c,a}^s \cap (c,a)$ between the sections $\Sigma^{h,\ell}$ and $\Sigma^m$. We need to track the manifold $\mathcal{W}_{c,a}^s \cap (c,a)$ in backwards time into a neighborhood of $\mathcal{M}_c^m \cap (c,a)$ and determine its behavior near the canard point, in particular its transversality with respect to the strong unstable fibers in the section $\Sigma^m$. For trajectories on $\mathcal{W}_{c,a}^s \cap (c,a)$ for $w < w_A - \Delta_w$ this behavior is clear: these trajectories are attracted exponentially close to $\mathcal{M}_c^m \cap (c,a)$ and remain in the manifold $\mathcal{W}_{c,a}^s \cap (c,a)$ upon entering a neighborhood of the canard point. This is due to the fact that the backwards evolution of $\mathcal{W}_{c,a}^s \cap (c,a)$ from $\Sigma^{h,\ell}$ to $\Sigma^m$ and forwards evolution from $\Sigma^m$ to $\Sigma^{h,\ell}$ in fact coincide for $w < w_A - \Delta_w$. Hence there is transversality with respect to the strong unstable fibers in $\Sigma^m$.

However, for $w \geq w_A - \Delta_w$, the behavior is not so clear since near $\mathcal{M}_c^m \cap (c,a)$, the manifold $\mathcal{W}_{c,a}^s \cap (c,a)$ is not defined for $w \geq w_A - \Delta_w$ due to the lack of spectral gap in the linearization of the vector field. Such trajectories on $\mathcal{W}_{c,a}^s \cap (c,a)$ are still exponentially attracted to $\mathcal{M}_c^m \cap (c,a)$ in backwards time, but in general do not coincide with $\mathcal{W}_{c,a}^s \cap (c,a)$ upon reaching $w = w_A - \Delta_w$ due to the interaction with the focus-like properties of the manifold $\mathcal{M}_c^m \cap (c,a)$ (see Figure 19 – note that the flow direction is reversed in this figure). To understand the transition from node to focus, we must understand the flow near the Airy point. The goal is to show that even though the backwards and forward evolution of $\mathcal{W}_{c,a}^s \cap (c,a)$ between $\Sigma^{h,\ell}$ and $\Sigma^m$ do not coincide, we retain the desired transversality properties for trajectories on $\mathcal{W}_{c,a}^s \cap (c,a)$ a bit above the Airy point, specifically for $w < w_A + C\epsilon^{2/3}$ for some $C > 0$.

The Airy point $(u_A, w_A)$ is defined by the conditions $c^2 = 4f'(u_A)$ and $w_A = f(u_A)$. In a neighborhood of this point, the manifold $\mathcal{M}_c^m \cap (c,a)$ can be written as a graph $(u, v) = (u_A + h(w, \epsilon), g(w, \epsilon))$ where
\[ h(w, \epsilon) = \frac{1}{f'(u_A)} (w - w_A) + O \left( \epsilon, (w - w_A)^2 \right) \]
\[ g(w, \epsilon) = O \left( \epsilon, (w - w_A) \right). \]

We make the coordinate transformation
\[ x = \frac{2}{c} (v - g(w, \epsilon)) - (u - u_A - h(w, \epsilon)) \]
\[ y = \frac{4f''(u_A)}{c^2 f'(u_A)} (w - w_A) \]
\[ z = u - u_A - h(w, \epsilon), \]
and rescale time by \(-c/2\) to arrive at the system
\[
\begin{align*}
\dot{x} &= -x + yz + \mathcal{O}(z^2, \epsilon z, y^2 z) \\
\dot{z} &= -x - z \\
\dot{y} &= \epsilon (-k + \mathcal{O}(y, z, \epsilon)),
\end{align*}
\]
where
\[
k = \frac{4f''(u_A)}{c^2f'(u_A)} > 0.
\]
Note that the flow direction has been reversed. We make a final coordinate change \(y \to y + \mathcal{O}(y^2, \epsilon)\) to simplify the equation for \(\dot{x}\) and arrive at the system
\[
\begin{align*}
\dot{x} &= -x + yz + \mathcal{O}(z^2) \\
\dot{z} &= -x - z \\
\dot{y} &= \epsilon (-k + \mathcal{O}(x, y, z, \epsilon)),
\end{align*}
\]
We consider solutions entering via the section
\[
\Sigma_A^{in} = \{(x, y, z, \epsilon) : x = \rho^4, |y| \leq \rho^2, |z| \leq \rho^3 \mu, 0 < \epsilon \leq \rho^3 \delta \}.
\]
Such solutions exit via the section
\[
\Sigma_A^{out} = \{(x, y, z, \epsilon) : |x| \leq \rho^4, y = -\rho^2, |z| \leq \rho^3 \mu, 0 < \epsilon \leq \rho^3 \delta \}.
\]
The goal of this section is to prove Proposition 3.6, that is, we track the backwards evolution \(\hat{\mathcal{W}}_{\epsilon^*}(c, a), \hat{\mathcal{W}}_{\epsilon^*}(c, a)\) of the manifolds \(\mathcal{W}_{\epsilon^*}(c, a), \mathcal{W}_{\epsilon^*}(c, a)\) near the Airy point until they exit via \(\Sigma_A^{out}\), where we then use an exchange lemma type argument to track the rest of the way to \(\Sigma\). To start, we have the following regarding the entry of the manifolds \(\hat{\mathcal{W}}_{\epsilon^*}(c, a), \hat{\mathcal{W}}_{\epsilon^*}(c, a)\) in \(\Sigma_A^{in}\).

**Lemma 5.1.** For each sufficiently small \(\Delta_y > 0\) there exists \(\epsilon_0 > 0\) and sufficiently small choice of the intervals \(I_c \times I_a\), such that for \((c, a, \epsilon) \in I_c \times I_a \times (0, \epsilon_0)\), the manifolds \(\hat{\mathcal{W}}_{\epsilon^*}(c, a), \hat{\mathcal{W}}_{\epsilon^*}(c, a)\) intersect \(\Sigma_A^{in}\) in smooth curves \(z = z_{\epsilon^*}(y; c, a)\) for \(|y| \leq \Delta_y\). Furthermore, there exists a constant \(\kappa = \kappa(\rho) > 0\) such that
\[
z_{\epsilon^*}(y; c, a) - z_{\epsilon^*}(y; c, a) > \rho^3 \kappa(\rho)
\]
uniformly in \(|y| \leq \Delta_y\).

**Proof.** Using Proposition A.1, we find that for \((c, a, \epsilon) = (1/\sqrt{2}, 0, 0)\), the front \(\phi_{\ell}\) is asymptotic (in forward time according to 5.5) to the Airy point \((x, y, z) = (0, 0, 0)\) and satisfies
\[
\begin{align*}
x(s) &= 2\sqrt{2B_1}e^{\sqrt{2} s} + \mathcal{O}(s^2 e^{\sqrt{2} s}) \\
z(s) &= (A_\ell - B_\ell s)e^{\sqrt{2} s} + \mathcal{O}(s^2 e^{\sqrt{2} s}).
\end{align*}
\]
Therefore, \(\phi_{\ell}\) intersects \(\Sigma_A^{in}\) at the point \((x, y, z) = (\rho^4, 0, z_0^\ell)\) where
\[
\begin{align*}
z_0^\ell &= \rho^4 \left(\frac{A_\ell}{2\sqrt{2B_1}} + \log \left(\frac{\rho^4}{2\sqrt{2B_1}}\right)\right) + O(\rho^4) \\
&= 4\rho^4 \log \left(\frac{\rho}{2\sqrt{2B_1}}\right) + O(\rho^4).
\end{align*}
\]
Therefore, by a regular perturbation argument, we have for sufficiently small \(\Delta_y\) and any \((c, a) \in I_c \times I_a\), for \(|y| \leq \Delta_y\), the manifold \(\mathcal{W}_0^{x, \ell}(c, a)\) intersects \(\Sigma_A^{in}\) in a curve \(z = z_0^{x, \ell}(y; c, a)\) given by
\[
z_0^{x, \ell}(y; c, a) = z_0^\ell + \mathcal{O}(y, (c - c^*), a).
\]
Figure 19: Shown is the schematic of the flow near the Airy point. Note that the flow direction corresponds to that of (5.5), which is the reverse of (1.2).

By using standard geometric singular perturbation theory, for sufficiently small $\epsilon > 0$, this manifold perturbs to a locally invariant manifold $W^s,\ell_\epsilon(c, a)$ which intersects $\Sigma_A$ in a smooth curve $z = z^{s,\ell}_\epsilon(y; c, a)$ given by

$$z^{s,\ell}_\epsilon(y; c, a) = z^\ell_0 + O(y; (c - c^*), a, \epsilon),$$

(5.12)

for $|y| \leq \Delta_y$. We similarly obtain that $W^{s,r}_\epsilon(c, a)$ intersects $\Sigma_A^{in}$ in a smooth curve $z = z^{s,r}_\epsilon(y; c, a)$ given by

$$z^{s,r}_\epsilon(y; c, a) = z^r_0 + O(y; (c - c^*), a, \epsilon),$$

(5.13)

for $|y| \leq \Delta_y$. Using Proposition A.1, and taking $\Delta_y \ll \rho^4$ sufficiently small, we deduce that there exists $\kappa = \kappa(\rho) > 0$ such that

$$z^{s,r}_\epsilon(y; c, a) - z^{s,\ell}_\epsilon(y; c, a) > \rho^3 \kappa(\rho)$$

(5.14)

uniformly in $|y| \leq \Delta_y$.

By taking $\Delta_y := 2k\Delta_w$ sufficiently small, we reduce the study of Proposition 3.6 to just understanding the passage of trajectories on $W^{s,\ell}_\epsilon(c, a)$ which enter a neighborhood of the Airy point in backwards time in a manner governed by Lemma 5.1; these solutions interact with the flow near the Airy point in a nontrivial manner (see Figure 19). All solutions on $W^{s,\ell}_\epsilon(c, a)$ entering a neighborhood of $M^m_\epsilon(c, a)$ in backwards time at heights lower than this remain in $W^{s,\ell}_\epsilon(c, a)$ until arriving at the section $\Sigma_m^m$ due to the nature of the construction of this manifold in §2.5.

To accomplish this, we need to understand detailed properties of the flow of (5.5). Ultimately, we will show that
the flow of (5.5) is qualitatively similar to the flow of the simpler system

\[
\begin{align*}
\dot{x} &= -x + yz \\
\dot{z} &= -x - z \\
\dot{y} &= -\epsilon,
\end{align*}
\]  

(5.15)

which are essentially the Airy equations on a slow timescale coupled with exponential decay. The solutions of this system are given in terms of the Airy functions \(\text{Ai}, \text{Bi}\), and their derivatives, which are shown in Figure 20.

We begin by solving the simpler system (5.15) to demonstrate why it is reasonable to expect that the transversality properties of Proposition 3.6 should indeed hold. Then we will use blow up techniques to study (5.5) directly to show that Proposition 3.6 continues to be valid when including the higher order terms.

### 5.1 A simpler system

We are concerned with the passage of trajectories on \(\tilde{W}_\varepsilon^{s,\ell}(c, a)\) from \(\Sigma_A^{in}\) to \(\Sigma_A^{out}\). In this section, as motivation for the full proof in the following sections for the system (5.5), we first consider the simpler system (5.15). The goal is to show that \(\tilde{W}_\varepsilon^{s,\ell}(c, a)\) intersects \(\Sigma_A^{out}\) in a curve transverse to the fibers of the manifold \(W_\varepsilon^{s,\ell}(c, a)\). Then, using the exchange lemma, we deduce that the transversality holds in the section \(\Sigma^m\) as well. We note that for trajectories entering \(\Sigma_A^{in}\) for \(y < -\Delta_y\), this transversality is clear as due to the construction of the manifold \(W_\varepsilon^{s,\ell}(c, a)\) in §2.5, the forward/backward evolution of \(W_\varepsilon^{s,\ell}(c, a)\) coincide in this region. Hence we are primarily concerned with the trajectories for \(y_0 > -\Delta_y\), whose intersections with \(\Sigma_A^{out}\) we will explicitly compute below. This is precisely the regime in which the manifolds \(W_\varepsilon^{s,\ell}(c, a)\) and \(\tilde{W}_\varepsilon^{s,\ell}(c, a)\) begin to deviate.

The system (5.15) is essentially the Airy equation on a slow timescale coupled with exponential decay. To see this, we rescale \((x, z) = (e^{-t}\tilde{x}, e^{-t}\tilde{z})\) and obtain the equations

\[
\begin{align*}
\dot{\tilde{x}} &= y\tilde{z} \\
\dot{\tilde{z}} &= -\tilde{x} \\
\dot{\tilde{y}} &= -\epsilon.
\end{align*}
\]  

(5.16)
The solutions of this system can be given explicitly in terms of Airy functions $\text{Ai}, \text{Bi}$ (see Figure 20)

$$
\begin{align*}
\ddot{x}(t) &= \pi \left[ \left( \text{Ai} \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Bi}' \left( -\frac{y_0}{\epsilon^{2/3}} + \epsilon^{1/3} t \right) - \text{Bi} \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Ai}' \left( -\frac{y_0}{\epsilon^{2/3}} + \epsilon^{1/3} t \right) \right) x_0 \\
&\quad + \epsilon^{1/3} \left( \text{Ai}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Bi}' \left( -\frac{y_0}{\epsilon^{2/3}} + \epsilon^{1/3} t \right) - \text{Bi}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Ai}' \left( -\frac{y_0}{\epsilon^{2/3}} + \epsilon^{1/3} t \right) \right) z_0 \right] \\
\ddot{z}(t) &= \frac{\pi}{\epsilon^{1/3}} \left[ \left( \text{Bi} \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Ai} \left( -\frac{y_0}{\epsilon^{2/3}} + \epsilon^{1/3} t \right) - \text{Ai} \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Bi} \left( -\frac{y_0}{\epsilon^{2/3}} + \epsilon^{1/3} t \right) \right) x_0 \\
&\quad + \epsilon^{1/3} \left( \text{Bi}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Ai} \left( -\frac{y_0}{\epsilon^{2/3}} + \epsilon^{1/3} t \right) - \text{Ai}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Bi} \left( -\frac{y_0}{\epsilon^{2/3}} + \epsilon^{1/3} t \right) \right) z_0 \right] \\
y(t) &= y_0 - \epsilon t,
\end{align*}
\tag{5.17}
$$

where $y_0 = y(0), x_0 = \ddot{x}(0) = x(0), \text{ and } z_0 = \ddot{z}(0) = z(0)$. This solution reaches $y = -\rho^2$ at time $T = \frac{y_0 + \rho^2}{\epsilon}$ with

$$
\begin{align*}
x(T) &= \pi e^{-\frac{y_0 + \rho^2}{\epsilon}} \left[ \left( \text{Ai} \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Bi}' \left( \frac{\rho^2}{\epsilon^{2/3}} \right) - \text{Bi} \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Ai}' \left( \frac{\rho^2}{\epsilon^{2/3}} \right) \right) x_0 \\
&\quad + \epsilon^{1/3} \left( \text{Ai}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Bi}' \left( \frac{\rho^2}{\epsilon^{2/3}} \right) - \text{Bi}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Ai}' \left( \frac{\rho^2}{\epsilon^{2/3}} \right) \right) z_0 \right] \\
z(T) &= \frac{\pi}{\epsilon^{1/3}} e^{-\frac{y_0 + \rho^2}{\epsilon}} \left[ \left( \text{Bi} \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Ai} \left( \frac{\rho^2}{\epsilon^{2/3}} \right) - \text{Ai} \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Bi} \left( \frac{\rho^2}{\epsilon^{2/3}} \right) \right) x_0 \\
&\quad + \epsilon^{1/3} \left( \text{Bi}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Ai} \left( \frac{\rho^2}{\epsilon^{2/3}} \right) - \text{Ai}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \text{Bi} \left( \frac{\rho^2}{\epsilon^{2/3}} \right) \right) z_0 \right] \\
y(T) &= -\rho^2.
\end{align*}
\tag{5.18}
$$

Using asymptotic properties of Airy functions [1, §10.4], we have the following

**Lemma 5.2.** The Airy functions $\text{Ai}(y), \text{Bi}(y)$ have the following asymptotics for all sufficiently large $y \gg 1$

$$
\begin{align*}
\text{Ai}(y) &= e^{-\frac{2}{3} y^{3/2}} \left( 1 - \frac{15}{144 y^{3/2}} + \mathcal{O} \left( y^{-3} \right) \right), \\
\text{Ai}'(y) &= -\frac{y^{1/4} e^{-\frac{2}{3} y^{3/2}}}{2 \sqrt{\pi}} \left( 1 + \frac{21}{144 y^{3/2}} + \mathcal{O} \left( y^{-3} \right) \right), \\
\text{Bi}(y) &= \frac{e^{\frac{2}{3} y^{3/2}}}{\sqrt{\pi} y^{1/4}} \left( 1 + \frac{15}{144 y^{3/2}} + \mathcal{O} \left( y^{-3} \right) \right), \\
\text{Bi}'(y) &= \frac{y^{1/4} e^{\frac{2}{3} y^{3/2}}}{\sqrt{\pi}} \left( 1 + \frac{21}{144 y^{3/2}} + \mathcal{O} \left( y^{-3} \right) \right).
\end{align*}
\tag{5.19}
$$

Considering the linearization of (5.15) for $\epsilon = 0$ in the plane $y = -\rho^2$, we see that there are two eigenvalues

$\lambda = -1 \pm \rho$ with corresponding eigenvectors ($\mp 1, 1$). We now change coordinates $\dot{x} = x - \rho z, \dot{z} = x + \rho z$ and using Lemma 5.2 under the assumption that $0 < \epsilon^{2/3} \ll \Delta y \ll \rho^2 \ll 1$, we can expand the terms dependent on the fixed argument $\frac{\rho^2}{\epsilon^{2/3}}$ to obtain

$$
\begin{align*}
\ddot{x}(T) &= \frac{\rho^{1/2} \sqrt{\pi}}{\epsilon^{1/6}} e^{-\frac{y_0 + \rho^2}{\epsilon}} \left[ \left( x_0 \text{Ai} \left( -\frac{y_0}{\epsilon^{2/3}} \right) + \epsilon^{1/3} z_0 \text{Ai}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \right) e^{\frac{2 \rho^3}{3 \epsilon}} \left( 2 + \mathcal{O}(\epsilon) \right) \\
&\quad + \left( x_0 \text{Bi} \left( -\frac{y_0}{\epsilon^{2/3}} \right) + \epsilon^{1/3} z_0 \text{Bi}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \right) e^{-\frac{2 \rho^3}{3 \epsilon}} \left( \frac{\epsilon^2}{8 \rho^3} + \mathcal{O}(\epsilon^2) \right) \right] \\
\ddot{z}(T) &= \frac{\rho^{1/2} \sqrt{\pi}}{\epsilon^{1/6}} e^{-\frac{y_0 + \rho^2}{\epsilon}} \left[ \left( x_0 \text{Ai} \left( -\frac{y_0}{\epsilon^{2/3}} \right) + \epsilon^{1/3} z_0 \text{Ai}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \right) e^{\frac{2 \rho^3}{3 \epsilon}} \left( \frac{\epsilon^2}{24 \rho^3} + \mathcal{O}(\epsilon^2) \right) \\
&\quad + \left( x_0 \text{Bi} \left( -\frac{y_0}{\epsilon^{2/3}} \right) + \epsilon^{1/3} z_0 \text{Bi}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \right) e^{-\frac{2 \rho^3}{3 \epsilon}} \left( 1 + \mathcal{O}(\epsilon) \right) \right] \\
y(T) &= -\rho^2.
\end{align*}
\tag{5.20}
$$
We now consider solutions on $\hat{W}_c^{s,\ell}(c,a)$ which enter via $\Sigma_A^n$ with $(x,y,z)(0) = (x_0, y_0, z_0) = (\rho^4, y_0, z^{s,\ell}_c(y_0; c,a))$, where $z^{s,\ell}_c(y_0; c,a) < 0$, so that $\hat{W}_c^{s,\ell}(c,a)$ is parameterized in $\Sigma_A^n$ by $|y_0| \leq \Delta_y$. Using the above analysis, $\hat{W}_c^{s,\ell}(c,a)$ exits via $\Sigma_A^\text{out}$ in a curve $(\hat{x}, \hat{z}) = (\hat{x}^{\ell}, \hat{z}^{\ell})(y_0)$ given by

$$
\hat{x}^{\ell}(y_0) = \rho^{1/2} \sqrt{\pi} e^{-\frac{y_0 + \rho^2}{\epsilon}} \hat{X}^{\ell}(y_0)
$$

$$
\hat{z}^{\ell}(y_0) = \frac{\rho^{1/2} \sqrt{\pi}}{\epsilon^{1/6}} e^{-\frac{y_0 + \rho^2}{\epsilon}} \hat{Z}^{\ell}(y_0)
$$

where

$$
\hat{X}^{\ell}(y_0) = \left( \rho^4 \text{Ai} \left( -\frac{y_0}{\epsilon^{2/3}} \right) + e^{1/3} z^{s,\ell}_c(y_0) \text{Ai}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \right) e^{\frac{y_0^2}{\epsilon^2}} \left( 2 + O(\epsilon) \right)
$$

$$
+ \left( \rho^4 \text{Bi} \left( -\frac{y_0}{\epsilon^{2/3}} \right) + e^{1/3} z^{s,\ell}_c(y_0) \text{Bi}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \right) e^{-\frac{y_0^2}{\epsilon^2}} \left( \frac{\epsilon}{48 \rho^3} + O(\epsilon^2) \right)
$$

$$
\hat{Z}^{\ell}(y_0) = \left( \rho^4 \text{Ai} \left( -\frac{y_0}{\epsilon^{2/3}} \right) + e^{1/3} z^{s,\ell}_c(y_0) \text{Ai}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \right) e^{\frac{y_0^2}{\epsilon^2}} \left( \frac{1}{24 \rho^3} + O(\epsilon^2) \right)
$$

$$
+ \left( \rho^4 \text{Bi} \left( -\frac{y_0}{\epsilon^{2/3}} \right) + e^{1/3} z^{s,\ell}_c(y_0) \text{Bi}' \left( -\frac{y_0}{\epsilon^{2/3}} \right) \right) e^{-\frac{y_0^2}{\epsilon^2}} \left( 1 + O(\epsilon) \right).
$$

We now want to understand the transversality of this curve with respect to the fibers of the manifold $W_c^{s,\ell}(c,a)$ in the section $\Sigma_A^\text{out}$. Under the transformation to the $\sim$ coordinates corresponding to the strong/weak eigenspaces of the linearization of (5.15), the manifold $W_c^{s,\ell}(c,a)$ will manifest as a curve in $\Sigma_A^\text{out}$ aligned approximately with the subspace $\tilde{z} = 0$ and its fibers will manifest as curves aligned approximately with $\tilde{x} \approx \text{const}$. It is clear from the expressions above that the same does not hold for $\hat{W}_c^{s,\ell}(c,a)$ when $y_0$ gets too large, as the Airy functions transition to oscillatory behavior.

We compute the derivatives

$$
(\hat{x}^{\ell})'(y_0) = -\frac{\rho^{1/2} \sqrt{\pi}}{\epsilon^{1/6}} e^{-\frac{y_0 + \rho^2}{\epsilon}} \left( \hat{X}^{\ell}(y_0) + \epsilon(\hat{X}^{\ell})'(y_0) \right)
$$

$$
(\hat{z}^{\ell})'(y_0) = -\frac{\rho^{1/2} \sqrt{\pi}}{\epsilon^{1/6}} e^{-\frac{y_0 + \rho^2}{\epsilon}} \left( \hat{Z}^{\ell}(y_0) + \epsilon(\hat{Z}^{\ell})'(y_0) \right),
$$

and hence $\hat{W}_c^{s,\ell}(c,a)$ can be written as a graph $\hat{z} = \hat{z}(\hat{x})$ with

$$
\frac{d\hat{z}}{d\hat{x}} = \frac{(\hat{z}^{\ell})'(y_0)}{(\hat{x}^{\ell})'(y_0)} = \frac{\hat{Z}^{\ell}(y_0) + \epsilon(\hat{Z}^{\ell})'(y_0)}{\hat{X}^{\ell}(y_0) + \epsilon(\hat{X}^{\ell})'(y_0)}
$$

provided that the denominator does not vanish. Points at which the denominator vanishes are essentially those at which this curve becomes tangent to the fibers $\hat{x} \approx \text{const}$ of $W_c^{s,\ell}(c,a)$. Hence we reduce our study to finding zeros of this expression. This will be carried out in detail in the following sections, but we note that they occur approximately at the zeros of $\hat{X}^{\ell}(y_0)$, which are approximately the zeros of $\text{Ai} \left( -y_0/\epsilon^{2/3} \right)$ for all sufficiently small $\epsilon > 0$. Hence we are primarily concerned with studying the Airy function $\text{Ai}$. We have the following [1, §10.4]

**Lemma 5.3.** There exists $y^* < 0$ such that the Airy function $\text{Ai}$ satisfies the following

(i) $\text{Ai}(y^*) = 0$

(ii) $\text{Ai}'(y^*) > 0$

(iii) $\text{Ai}(y) > 0$ for all $y > y^*$.

We can therefore find the first zero of the denominator or equivalently, the first turning point of $\hat{W}_c^{s,\ell}(c,a)$, which occurs when $y_0 = y_0^0 \approx -y^* \epsilon^{2/3} > 0$. Therefore $\hat{W}_c^{s,\ell}(c,a)$ is transverse to the fibers of $W_c^{s,\ell}(c,a)$ in $\Sigma_A^\text{out}$ up
to the fiber passing through the point \((\tilde{x}^0, \tilde{z}(\tilde{x}^0))\) = \((\tilde{x}^\ell(y_0^0), \tilde{z}(y_0^0))\). A schematic of this result is depicted in Figure 19.

Using the exchange lemma, we continue to track \(\tilde{W}_e^{s,\ell}(c, a)\) backwards from \(\Sigma^{out}_A\) to the section \(\Sigma^m\) and deduce that this transversality holds there also.

In the coming sections, we consider the full system (5.5), and we make the above computations precise in this context.

### 5.2 Blow up transformation

To study the flow of the full equations (5.5)

\[
\begin{align*}
\dot{x} &= -x + yz + \mathcal{O}(z^2) \\
\dot{z} &= -x - z \\
\dot{y} &= \epsilon(-k + \mathcal{O}(x, y, z, \epsilon)),
\end{align*}
\]

we will use blow up techniques. The blow up is a rescaling which blows up the degenerate point \((x, y, z, \epsilon) = (0, 0, 0, 0)\) to a 3-sphere. The blow up transformation is given by

\[
x = \bar{r}^4 \tilde{x}, \quad y = \bar{r}^2 \tilde{y}, \quad z = \bar{r}^3 \tilde{z}, \quad \epsilon = \bar{r}^3 \tilde{\epsilon}.
\]  

(5.25)

Defining \(B_A = S^3 \times [0, \bar{r}_0]\) for some sufficiently small \(\bar{r}_0\), we consider the blow up as a mapping \(B_A \to \mathbb{R}^4\) with \((\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\ell}) \in S^3\) and \(\bar{r} \in [0, \bar{r}_0]\). The point \((x, y, z, \epsilon) = (0, 0, 0, 0)\) is now represented as a copy of \(S^3\) (i.e. \(\bar{r} = 0\)) in the blow up transformation. To study the flow on the manifold \(B_A\), there are three relevant coordinate charts. The first is the chart \(K_1\) which uses the coordinates

\[
x = r_1^4, \quad y = r_1^2 y_1, \quad z = r_1^3 z_1, \quad \epsilon = r_1^3 \epsilon_1,
\]  

(5.26)

the second chart \(K_2\) uses the coordinates

\[
x = r_2^4 x_2, \quad y = r_2^2 y_2, \quad z = r_2^3 z_2, \quad \epsilon = r_2^3 \epsilon_2,
\]  

(5.27)

and the third chart \(K_3\) uses the coordinates

\[
x = r_3^4 x_3, \quad y = -r_3^2, \quad z = r_3^3 z_3, \quad \epsilon = r_3^3 \epsilon_3,
\]  

(5.28)

With these three sets of coordinates, a short calculation gives the following.

**Lemma 5.4.** The transition map \(\kappa_{12} : K_1 \to K_2\) between the coordinates in \(K_1\) and \(K_2\) is given by

\[
x_2 = \frac{1}{\epsilon_1^{4/3}}, \quad y_2 = \frac{y_1}{\epsilon_1^{2/3}}, \quad z_2 = \frac{z_1}{\epsilon_1}, \quad r_2 = r_1 \epsilon_1^{1/3}, \quad \text{for } \epsilon_1 > 0,
\]  

(5.29)

the transition map \(\kappa_{13} : K_1 \to K_3\) between the coordinates in \(K_1\) and \(K_3\) is given by

\[
x_3 = \frac{1}{y_1^{1/3}}, \quad r_3 = r_1 (-y_1)^{1/2}, \quad z_3 = \frac{z_1}{(-y_1)^{3/2}}, \quad \epsilon_3 = \frac{\epsilon_1}{(-y_1)^{3/2}}, \quad \text{for } y_1 < 0,
\]  

(5.30)

and the transition map \(\kappa_{23} : K_2 \to K_3\) between the coordinates in \(K_2\) and \(K_3\) is given by

\[
x_3 = \frac{x_2}{y_2^{1/2}}, \quad r_3 = r_2 (-y_2)^{1/2}, \quad z_3 = \frac{z_2}{(-y_2)^{3/2}}, \quad \epsilon_3 = \frac{1}{(-y_2)^{3/2}}, \quad \text{for } y_2 < 0.
\]  

(5.31)

Solutions on \(\tilde{W}_e^{s,\ell}(c, a)\) will enter via the section \(\Sigma^{in}_A\) and exit via \(\Sigma^{out}_A\). During this passage, it will be necessary to track different parts of the manifold \(\tilde{W}_e^{s,\ell}(c, a)\) in the different charts \(K_1, K_2, K_3\). A diagram of the sequence through which the solutions will be tracked is shown in Figure 21. We begin in §5.3 with a study of the chart \(K_1\), where all solutions enter via the section \(\Sigma^{in}_A\).
\[ \Sigma_{in}^A = \Sigma_{in}^1 \]

\[ \begin{align*}
\Sigma_{in} & \xrightarrow{(i)} K_1 \\
& \xrightarrow{(ii)} \Sigma_{12} \\
& \xrightarrow{(iii)} \Sigma_{14} \\
& \xrightarrow{(iv)} K_2 \\
& \xrightarrow{(v)} \Sigma_{23} \\
& \xrightarrow{(vi)} K_3 \\
& \xrightarrow{(vi)} \Sigma_{3 out} = \Sigma_{A out} 
\end{align*} \]

Figure 21: Shown is the sequence of sections through which the manifold \( \widehat{W}_e^{s,f}(c,a) \) will be tracked. The table displays the charts and sections in the text in which the various transitions will be studied.

### 5.3 Dynamics in \( K_1 \)

In the \( K_1 \) coordinates, the equations are given by

\[ \begin{align*}
\dot{r}_1 &= -\frac{1}{4} r_1 + \frac{1}{4} r_1^2 y_1 z_1 + O\left(r_1^3 z_1^2 \right) \\
\dot{z}_1 &= \frac{1}{4} z_1 - r_1 - \frac{3}{4} r_1 y_1 z_1^2 + O\left(r_1^2 z_1^3 \right) \\
\dot{y}_1 &= \frac{1}{2} y_1 - k r_1 \epsilon_1 + \frac{1}{2} r_1^2 y_1 z_1 + O\left(r_1^2 y_1^2 z_1^2, r_1^2 \epsilon_1, r_1^3 \epsilon_1 y_1, r_1^4 \epsilon_1 z_1, r_1^4 \epsilon_1 \right) \\
\dot{\epsilon}_1 &= \frac{3}{4} \epsilon_1 - \frac{3}{4} r_1 y_1 z_1 \epsilon_1 + O\left(r_1^2 z_1^2 \epsilon_1 \right).
\end{align*} \] (5.32)

In these coordinates, the section \( \Sigma_{in}^i \) is given by

\[ \Sigma_{in}^i = \{(r_1, y_1, z_1, \epsilon_1) : r_1 = \rho, \ |y_1| \leq 1, \ |z_1| \leq \mu, \ 0 < \epsilon_1 \leq \delta\}. \] (5.33)

Define the set

\[ D_1 = \{(r_1, y_1, z_1, \epsilon_1) : 0 \leq r_1 \leq \rho, \ |y_1| \leq 1, \ |z_1| \leq \mu, \ 0 \leq \epsilon_1 \leq \delta\}. \] (5.34)

Under the flow of (5.32), any solution starting in \( \Sigma_{in}^i \) exits \( D_1 \) via one of the sections

\[ \begin{align*}
\Sigma_{12} &= \{(r_1, y_1, z_1, \epsilon_1) : r_1 \leq \rho, \ |y_1| \leq 1, \ |z_1| \leq \mu, \ \epsilon_1 = \delta\} \quad (5.35) \\
\Sigma_{13} &= \{(r_1, y_1, z_1, \epsilon_1) : r_1 \leq \rho, \ y_1 = -1, \ |z_1| \leq \mu, \ 0 < \epsilon_1 \leq \delta\} \quad (5.36) \\
\Sigma_{14} &= \{(r_1, y_1, z_1, \epsilon_1) : r_1 \leq \rho, \ y_1 = 1, \ |z_1| \leq \mu, \ 0 < \epsilon_1 \leq \delta\}. \quad (5.37)
\]

The setup in the chart \( K_1 \) is shown in Figure 22. It turns out that we only need to consider those solutions exiting via \( \Sigma_{12} \) and \( \Sigma_{13} \), which will be tracked in the charts \( K_2 \) and \( K_3 \), respectively (see Figure 21). Solutions exiting via \( \Sigma_{14} \) will not be analyzed.

The following result gives estimates for solutions on the manifolds \( \widehat{W}_e^{s,f}(c,a) \) and \( \widehat{W}_e^{s,r}(c,a) \) which exit via the sections \( \Sigma_{12} \) and \( \Sigma_{13} \).

**Proposition 5.5.** For each sufficiently small \( \rho, \delta > 0 \), there exists \( \epsilon_0 > 0 \) and sufficiently small choice of the intervals \( I_c, I_a \) such that the following holds. For each sufficiently small \( \Delta_{y_1} > 0 \) and each \( (c, a, \epsilon) \in I_c \times I_a \times (0, \epsilon_0) \), the manifolds \( \widehat{W}_e^{s,f}(c,a) \) and \( \widehat{W}_e^{s,r}(c,a) \) intersect \( \Sigma_{in}^i \) in smooth curves \( z_1 = z_{1,0}^f(y_1; c, a, \epsilon) \) and \( z_1 = z_{1,0}^r(y_1; c, a, \epsilon) \) for \( |y_1| \leq \Delta_{y_1} \). Furthermore, there exists \( C > 0 \) independent of \( c, a, \epsilon \) and \( 0 < \kappa(\rho) \leq C_0 \rho |\log \rho| \) where \( C_0 \) is independent of \( c, a, \epsilon, \rho, \delta \) such that for any \( (c, a, \epsilon) \in I_c \times I_a \times (0, \epsilon_0) \), the following hold...
(i) The parts of the manifolds \( W_{e}^{s, e_{1}}(c, a) \), \( W_{e}^{s, r}(c, a) \) which exit via \( \Sigma_{13} \) intersect \( \Sigma_{13} \) in curves \( z_{1} = z_{1}^{e_{1}}(r_{1}) \) which satisfy \( \frac{dz_{1}^{e_{1}}}{dr_{1}} \leq C|\log \epsilon| \) uniformly in \( y_{1} \).

(ii) The parts of the manifolds \( W_{e}^{s, e_{1}}(c, a) \), \( W_{e}^{s, r}(c, a) \) which exit via \( \Sigma_{12} \) intersect \( \Sigma_{12} \) in curves \( z_{1} = z_{1}^{e_{1}}(y_{1}) \) which satisfy

\[
|z_{1}^{e_{1}}| \leq C\epsilon^{1/3} \log \epsilon \quad (5.38)
\]

and

\[
0 < \kappa(\rho)\epsilon^{1/3} < z_{1}^{e_{1}}(y_{1}) - z_{1}^{e_{1}}(y_{1}) < C\epsilon^{1/3} \log \epsilon. \quad (5.39)
\]

uniformly in \( y_{1} \).

**Proof.** We focus on the manifold \( W_{e}^{s, e_{1}}(c, a) \); the computations for \( W_{e}^{s, r}(c, a) \) are similar.

First we consider the function \( z_{1,0}(y_{1}; c, a, \epsilon) \). By taking \( \Delta y_{1} \ll \rho^{2} \), for any sufficiently small \( \rho \) we have that

\[
\sup_{|y_{1}| \leq \Delta y_{1}} |z_{1,0}(y_{1}; c, a, \epsilon)| \leq C_{0}|\log \rho|
\]

for some \( C_{0} \) independent of \( (c, a, \epsilon, \rho, \delta) \), provided \( \epsilon \) and the intervals \( I_{c}, I_{a} \) are sufficiently small. This follows from Lemma 5.1 by taking \( \rho^{2}\Delta y_{1} = \Delta y \ll \rho^{4} \).

To prove (i), for each sufficiently small \( |y_{1,0}| \leq \Delta y_{1} \), we consider solutions starting in \( \Sigma_{A}^{in} \) with \( (r_{1}, z_{1, y_{1}, \epsilon_{1}})(0) = (\rho, z_{1,0}(y_{1,0}), y_{1,0}, \epsilon/\rho^{3}) \) which exit via \( \Sigma_{13} \) at time \( T_{1}^{c}(y_{1,0}; c, a, \epsilon) \). As the solution exits via \( \Sigma_{13} \), we must have \( y_{1}(T_{1}^{c}) = -1 \) and \( \epsilon/\rho^{3} < \epsilon_{1}(T_{1}^{c}) = \epsilon_{1}^{c} \ll \delta \).

We define \( \Phi_{1}(t, s) \) to be the linear evolution of the constant coefficient system

\[
\begin{pmatrix}
\dot{r}_{1} \\
\dot{z}_{1}
\end{pmatrix} =
\begin{pmatrix}
-1/4 & 0 \\
-1 & -1/4
\end{pmatrix}
\begin{pmatrix}
r_{1} \\
z_{1}
\end{pmatrix}.
\]

\[
(5.40)
\]
We set
\[
U_1 = \begin{pmatrix} r_1 \\ z_1 \end{pmatrix}, \quad (5.41)
\]
\[
U_{1,0} = \begin{pmatrix} r_{1,0} \\ z_{1,0}(y_{1,0}) \end{pmatrix}, \quad (5.42)
\]
and we rewrite (5.32) as the integral equation
\[
U_1(t) = \Phi_1(t,0)U_{1,0} + \int_0^t \Phi_1(t,s)g_{U_1}(r_1(s), z_1(s), y_1(s), \epsilon_1(s))ds
\]
\[
= : F_{U_1}(U_1, y_1, \epsilon_1, U_{1,0}, T_1^*; c, a)
\]
\[
y_1(t) = -e^{\frac{1}{2}(t-T_1^*)} + \int_{T_1^*}^t e^{\frac{1}{2}(t-s)}g_{y_1}(r_1(s), z_1(s), y_1(s), \epsilon_1(s))ds
\]
\[
= : F_{y_1}(U_1, y_1, \epsilon_1, U_{1,0}, T_1^*; c, a)
\]
\[
\epsilon_1(t) = \frac{\rho^3}{\epsilon_1}e^{\frac{1}{2}t} + \int_0^t e^{\frac{1}{2}(t-s)}g_{\epsilon_1}(r_1(s), z_1(s), y_1(s), \epsilon_1(s))ds
\]
\[
= : F_{\epsilon_1}(U_1, y_1, \epsilon_1, U_{1,0}, T_1^*; c, a),
\]
where
\[
g_{U_1}(r_1, z_1, y_1, \epsilon_1) = \begin{pmatrix}
\frac{1}{4} r_1^2 y_1 z_1 + O(r_1^3 z_1^2) \\
\frac{3}{4} r_1 y_1 z_1^2 + O(r_1^2 z_1^3)
\end{pmatrix}
= \mathcal{O}(|U_1|^3)
\]
\[
g_{y_1}(r_1, z_1, y_1, \epsilon_1) = -kr_1 \epsilon_1 - \frac{1}{2} r_1 y_1^2 z_1 + O(r_1^2 y_1 z_1^2, r_1^2 \epsilon_1, r_1^2 \epsilon_1 y_1, r_1^2 \epsilon_1 z_1, r_1^4 \epsilon_1)
\]
\[
= \mathcal{O}(|U_1| |\epsilon_1| + |U_1|^2 |y_1|)
\]
\[
ge_{\epsilon_1}(r_1, z_1, y_1, \epsilon_1) = -\frac{3}{4} r_1 y_1 z_1 \epsilon_1 + O(r_1^2 z_1^2 \epsilon_1)
\]
\[
= \mathcal{O}(|U_1|^2 |\epsilon_1|),
\]
and we assume $T_1^* \geq 0$ is such that $\left| \frac{\epsilon_1}{\rho^3} e^{\frac{1}{2} T_1^*} \right| \leq 2\delta$. We define the spaces
\[
V_{\frac{1}{2}}^- = \left\{ U_1 : [0, T_1^*) \to \mathbb{R}^2 : \|U_1\|_{\frac{1}{2}}^- = \sup_{t \in [0, T_1^*)} e^{\frac{1}{2} t} |U_1(t)| < \infty \right\}
\]
\[
V_{\frac{1}{2}}^+ = \left\{ y_1 : [0, T_1^*) \to \mathbb{R} : \|y_1\|_{\frac{1}{2}}^+ = \sup_{t \in [0, T_1^*)} e^{\frac{1}{2} (T_1^*-t)} |y_1(t)| < \infty \right\}
\]
\[
V_{\frac{1}{4}}^+ = \left\{ \epsilon_1 : [0, T_1^*) \to \mathbb{R} : \|\epsilon_1\|_{\frac{1}{4}}^+ = \sup_{t \in [0, T_1^*)} e^{\frac{1}{2} (T_1^*-t)} |\epsilon_1(t)| < \infty \right\},
\]
and search for solutions $(U_1, y_1, \epsilon_1) \in V_{\frac{1}{2}}^- \times V_{\frac{1}{2}}^+ \times V_{\frac{1}{4}}^+$ to (5.43). We note that
\[
\|U_1\|_{\infty} \leq \|U_1\|_{\frac{1}{2}}, \quad \|y_1\|_{\infty} \leq \|y_1\|_{\frac{1}{2}}, \quad \|\epsilon_1\|_{\infty} \leq \|\epsilon_1\|_{\frac{1}{4}},
\]
(5.46)
where $\|X\|_{\infty} = \sup_{t \in [0, T_1^*)} |X(t)|$ denotes the $C^0$-norm.

First we show that for each fixed $(U_{1,0}, T_1^*)$, the mapping
\[
(U_1, y_1, \epsilon_1) \to F_1(U_1, y_1, \epsilon_1, U_{1,0}, T_1^*; c, a),
\]
(5.47)
defined by
\[
F_i(U_1, y_1, \epsilon_1, U_{1,0}, T^*_1; c, a) = \begin{pmatrix}
F_{U_1}(U_1, y_1, \epsilon_1, U_{1,0}, T^*_1; c, a) \\
F_{y_1}(U_1, y_1, \epsilon_1, U_{1,0}, T^*_1; c, a) \\
F_{\epsilon_1}(U_1, y_1, \epsilon_1, U_{1,0}, T^*_1; c, a)
\end{pmatrix}
\]  
(5.48)

maps the space \(V_\frac{t}{2} \times V_\frac{t}{2} \times V_\frac{t}{2}^+\) into itself. We compute
\[
\|F_{U_1}(U_1, y_1, \epsilon_1, U_{1,0}, T^*_1; c, a)\|_\frac{t}{2}^+ = \sup_{t \in [0, T^*_1]} e^{\frac{\epsilon_1}{4} t} \left( e^{\frac{\epsilon_1}{4} (T^*_1 - t)} + \int_{T^*_1}^t e^{\frac{\epsilon_1}{4} (t - s)} g_{U_1}(r_1(s), z_1(s), y_1(s), \epsilon_1(s)) ds \right)
\leq C|U_{1,0}| + C \left(\|U_1\|_\frac{t}{2}\right)^3,
\]  
(5.49)

where we used (5.46) and the fact that \(|\Phi_1(t, s)| \leq |t - s| e^{-\frac{1}{4}(t-s)}\).

Similarly, we compute
\[
\|F_{y_1}(U_1, y_1, \epsilon_1, U_{1,0}, T^*_1; c, a)\|_\frac{t}{2}^+ = \sup_{t \in [0, T^*_1]} e^{\frac{\epsilon_1}{4} (T^*_1 - t)} \left( e^{\frac{\epsilon_1}{4} (T^*_1 - t)} + \int_{T^*_1}^t e^{\frac{\epsilon_1}{4} (t - s)} g_{y_1}(r_1(s), z_1(s), y_1(s), \epsilon_1(s)) ds \right)
\leq 1 + Ce^{\frac{\epsilon_1}{4} T^*_1} \int_{T^*_1}^t e^{-\frac{\epsilon_1}{4} s} \left( |U_1(s)||\epsilon_1(s)| + |U_1(s)|^2 |y_1(s)| \right) ds
\leq 1 + C \left(\|U_1\|_\frac{t}{2}\|\epsilon_1\|_\frac{t}{2} + \left(\|\epsilon_1\|_\frac{t}{2}\right)^2 \|y_1\|_\frac{t}{2}\right),
\]  
(5.50)

and
\[
\|F_{\epsilon_1}(U_1, y_1, \epsilon_1, U_{1,0}, T^*_1; c, a)\|_\frac{t}{2}^+ = \sup_{t \in [0, T^*_1]} e^{\frac{\epsilon_1}{4} (T^*_1 - t)} \left( e^{\frac{\epsilon_1}{4} (T^*_1 - t)} + \int_{T^*_1}^t e^{\frac{\epsilon_1}{4} (t - s)} g_{\epsilon_1}(r_1(s), z_1(s), y_1(s), \epsilon_1(s)) ds \right)
\leq \left(\frac{\epsilon}{\rho^2} e^{\frac{\epsilon_1}{4} T^*_1}\right) + Ce^{\frac{\epsilon_1}{4} T^*_1} \int_{T^*_1}^t e^{-\frac{\epsilon_1}{4} s} \left( |U_1(s)|^2 |\epsilon_1(s)| \right) ds
\leq \left(\frac{\epsilon}{\rho^2} e^{\frac{\epsilon_1}{4} T^*_1}\right) + C \left(\|U_1\|_\frac{t}{2}\right)^2 \|\epsilon_1\|_\frac{t}{2}.  
\]  
(5.51)

Provided \(\rho, \delta\) are sufficiently small, for each sufficiently small \(U_{1,0}\) and for \(\left|\frac{\epsilon}{\rho^2} e^{\frac{\epsilon_1}{4} T^*_1}\right| \leq 2\delta\) sufficiently small, that is \(T^*_1\) is not too large, we can solve (5.43) to find a unique solution satisfying
\[
\|U_1\|_\frac{t}{2} = \mathcal{O}(|U_{1,0}|)
\|y_1\|_\frac{t}{2} = 1 + \mathcal{O}(\delta + |U_{1,0}|)
\|\epsilon_1\|_\frac{t}{2} = \mathcal{O}(\delta)  
\]  
(5.52)

By our assumption that we consider only solutions exiting via \(\Sigma_{13}\), and so \(\epsilon_1 \leq \delta\), the time \(T^*_1\) satisfies \(0 \leq T^*_1 \leq C(\rho, \delta)|\log \epsilon|\) for all sufficiently small \(\epsilon > 0\).

To obtain estimates on the derivatives of the solutions with respect to \(U_{1,0}, c, a\), we consider the variational equation
\[
\begin{align*}
dU_1 &= \begin{pmatrix} -1/4 & 0 \\ -1 & -1/4 \end{pmatrix} dU_1 + dg_{U_1}(r_1, z_1, y_1, \epsilon_1) \\
\dot{y}_1 &= \frac{1}{2} dy_1 + dg_{y_1}(r_1, z_1, y_1, \epsilon_1) \\
\dot{\epsilon}_1 &= 3/4 d\epsilon_1 + dg_{\epsilon_1}(r_1, z_1, y_1, \epsilon_1),
\end{align*}
\]  
(5.53)
where

\[
\begin{align*}
&d_{g_U}(r_1, z_1, y_1, \epsilon_1) = O \left( |U_1|^2 dU_1, |U_1|^3 (|dy_1| + |d\epsilon_1|), |U_1|^3 \right) \\
&d_{g_Y}(r_1, z_1, y_1, \epsilon_1) = O \left( |U_1| dU_1, (|y_1| + |U_1|) dU_1, (|U_1|^2 + |U_1| |\epsilon_1|) |dy_1|, |U_1|^2 |y_1|, |U_1| |\epsilon_1| \right) \\
&d_{g_{c,a}}(r_1, z_1, y_1, \epsilon_1) = O \left( |U_1||\epsilon_1| dU_1, |U_1|^2 d\epsilon_1, |U_1|^2 |\epsilon_1| dy_1, |U_1|^2 |\epsilon_1| \right).
\end{align*}
\] (5.54)

Proceeding as above, we can rewrite this as an integral equation; using the estimates obtained for the solutions \((U_1, y_1, \epsilon_1)\) and noting that the derivatives of \(k\) with respect to \((c, a)\) are uniformly bounded, we can solve for the derivatives of the solutions on the same spaces and obtain

\[
\|D_{\nu}U_1\|_\frac{t}{\tau}, \|D_{\nu}y_1\|_\frac{t}{\tau}, \|D_{\nu}\epsilon_1\|_\frac{t}{\tau} \leq C,
\] (5.55)

\(\nu = U_{1,0}, c, a\), uniformly in \((U_{1,0}, T_1^*, c, a, \epsilon)\) for all sufficiently small \(\rho, \delta\).

We also need estimates on the derivatives with respect to \(T_1^*\). First, we show that these derivatives exist; then we show that they are in fact bounded uniformly in \(T_1^*\). To compute the derivative with respect to \(T_1^*\) at some \(T_1^* = T_0\), we rescale time by \(t = (1 + \omega)\tau\), which results in the differential equation

\[
\dot{\hat{X}} = (1 + \omega)F(\hat{X}),
\] (5.56)

where \(X = (r_1, z_1, y_1, \epsilon_1)\) and \(F(X)\) denotes the RHS of (5.32). Proceeding as above, we can now find solutions to this new system, keeping \(T_0\) fixed and allowing \(\omega\) to vary as a small parameter, with \(|\omega| \leq \omega_0\), where \(\omega_0\) is sufficiently small. We obtain a new integral equation

\[
\begin{align*}
U_1(t) &= \Phi_{1, \omega}(t, 0)U_{1,0} + \int_0^t \Phi_{1, \omega}(t, s) d_{g_U}(r_1(s), z_1(s), y_1(s), \epsilon_1(s)) ds \\
y_1(t) &= -e^{\frac{3}{4}(1+\omega)(t-T_0)} + \int_0^t e^{\frac{3}{4}(1+\omega)(t-s)} d_{g_Y}(r_1(s), z_1(s), y_1(s), \epsilon_1(s)) ds \\
\epsilon_1(t) &= \frac{\epsilon}{\rho^3}e^{\frac{3}{4}(1+\omega)t} + \int_0^t e^{\frac{3}{4}(1+\omega)(t-s)} d_{g_{c,a}}(r_1(s), z_1(s), y_1(s), \epsilon_1(s)) ds,
\end{align*}
\] (5.57)

where the functions \(d_{g_U}, d_{g_Y}, d_{g_{c,a}}\) are defined as in (5.44), and \(\Phi_{1, \omega}\) denotes the evolution of the constant coefficient system

\[
\begin{pmatrix}
\dot{r}_1 \\
\dot{z}_1
\end{pmatrix} = \begin{pmatrix}
-(1 + \omega)/4 & 0 \\
-(1 + \omega) & -(1 + \omega)/4
\end{pmatrix} \begin{pmatrix}
r_1 \\
z_1
\end{pmatrix}.
\] (5.58)

We now slightly decrease the exponential weights and solve (5.57) for \((U_1, y_1, \epsilon_1) \in V_{\frac{3}{4}(1-\omega_0)}^- \times V_{\frac{3}{4}(1-\omega_0)}^+ \times V_{\frac{3}{4}(1-\omega_0)}^+\), where the spaces \(V_{\frac{3}{4}}^\pm\) are defined analogously to (5.45). Further, as above we can use the corresponding variational equation to estimate the derivatives of the solution with respect to the parameters, including \(\omega\), noting that they are bounded uniformly in \(T_0\).

Let \(\tilde{X}(\tau; T_0, \omega, U_{1,0}, c, a) = (U_1, y_1, \epsilon_1)(\tau; T_0, \omega, U_{1,0}, c, a)\) denote a solution to (5.57), and let \(X(t; T_1^*, U_{1,0}, c, a) = (U_1, y_1, \epsilon_1)(t; T_1^*, U_{1,0}, c, a)\) denote a solution to the original equation (5.43). By uniqueness, we have that \(\tilde{X}(T_0, \omega, U_{1,0}, c, a) = X((1 + \omega)T_0, U_{1,0}, c, a)\). We now differentiate

\[
D_{\omega}\tilde{X}(\tau; T_0, \omega, U_{1,0}, c, a) = \tau \tilde{X}((1 + \omega)\tau; (1 + \omega)T_0, U_{1,0}, c, a)
+ T_0 D_{T_1^*} X((1 + \omega)\tau; (1 + \omega)T_0, U_{1,0}, c, a),
\] (5.59)

from which we deduce that the derivative \(D_{T_1^*} X\) exists and is bounded in the norms

\[
\|D_{T_1^*} U_1\|_{\frac{3}{4}(1-\omega_0)}, \|D_{T_1^*} y_1\|_{\frac{3}{4}(1-\omega_0)}, \|D_{T_1^*} \epsilon_1\|_{\frac{3}{4}(1-\omega_0)} \leq C
\] (5.60)

uniformly in \((U_{1,0}, T_1^*, c, a, \epsilon)\) for all sufficiently small \(\rho, \delta\).
We can now write the unique solution of (5.43) satisfying

\[
\begin{align*}
  r_1(0) &= \rho \\
  z_1(0) &= z_{1,0}^\ell(y_{1,0}),
\end{align*}
\]

for sufficiently small \( 0 > y_{1,0} > -\Delta_{y_1} \) so that \( z_{1,0}^\ell(y_{1,0}) = O(\rho \log \rho) \). Recalling \( U_1 = (r_1, z_1) \), we have that this solution is given by

\[
\begin{pmatrix}
  r_1(t) \\
  z_1(t)
\end{pmatrix} = \left( \begin{pmatrix}
  \rho e^{-\frac{1}{4} t} \\
  z_{1,0}^\ell(y_{1,0}) e^{-\frac{1}{4} t} + \rho \epsilon e^{-\frac{1}{4} t}
\end{pmatrix} + \int_0^t \Phi_1(t,s) g_{U_1}(r_1(s), z_1(s), y_1(s), \epsilon_1(s)) ds, \right)
\]

where

\[
y_{1,0} = -e^{-\frac{1}{2} T_1^*} + \int_{T_1^*}^0 e^{-\frac{1}{4} s} g_{y_1}(r_1(s), z_1(s), y_1(s), \epsilon_1(s)) ds.
\]

We consider only \( T_1^* \) large enough so that \( y_{1,0} \geq -\Delta_{y_1} \), and we recall that \( \Delta_{y_1} < \rho^2 \). This gives

\[
-\rho^2 \leq -e^{-\frac{1}{2} T_1^*} + \int_{T_1^*}^0 e^{-\frac{1}{4} s} g_{y_1}(r_1(s), z_1(s), y_1(s), \epsilon_1(s)) ds
\]

\[
= -e^{-\frac{1}{2} T_1^*} + O\left( e^{-\frac{1}{4} T_1^*} \| U_1 \|_1/4 \left( \| \epsilon_1 \|_3 + \| U_1 \|_1 \right) \right)
\]

\[
= -e^{-\frac{1}{2} T_1^*} \left( 1 + O(\rho^2, \rho \delta) \right),
\]

so that

\[
e^{-\frac{1}{2} T_1^*} \leq \rho^2 \left( 1 + O(\rho^2, \rho \delta) \right),
\]

and

\[
\frac{dy_{1,0}}{dT_1^*} = \frac{1}{2} e^{-\frac{1}{2} T_1^*} + \frac{e^{-\frac{1}{4} T_1^*}}{2} g_{y_1}(r_1(T_1^*), z_1(T_1^*), y_1(T_1^*), \epsilon_1(T_1^*))
\]

\[
+ \int_{T_1^*}^0 e^{-\frac{1}{4} s} \frac{dy_{1,0}}{dT_1^*} g_{y_1}(r_1(s), z_1(s), y_1(s), \epsilon_1(s)) ds
\]

\[
= e^{-\frac{1}{2} T_1^*} \left( \frac{1}{2} + O(\rho \log \rho, \delta) \right)
\]

\[
= \rho^2 \left( 1 + O(\rho \log \rho, \delta) \right)
\]

where we used (5.44), (5.60), and (5.65).

We have

\[
\begin{pmatrix}
  r_1(T_1^*) \\
  z_1(T_1^*)
\end{pmatrix} = \left( \begin{pmatrix}
  \rho e^{-\frac{1}{4} T_1^*} \\
  z_{1,0}^\ell(y_{1,0}) e^{-\frac{1}{4} T_1^*} + \rho T_1^* e^{-\frac{1}{4} T_1^*} + \int_0^{T_1^*} \Phi_1(T_1^*, s) g_{U_1}(r_1(s), z_1(s), y_1(s), \epsilon_1(s)) ds,
\end{pmatrix} \right)
\]

and we now compute

\[
\frac{d}{dT_1^*} \begin{pmatrix}
  r_1(T_1^*) \\
  z_1(T_1^*)
\end{pmatrix} = \left( \begin{pmatrix}
  -\frac{\rho}{4} e^{-\frac{1}{4} T_1^*} \\
  -\frac{\rho}{4} T_1^* e^{-\frac{1}{4} T_1^*} + \frac{\rho}{4} T_1^* + \frac{\rho}{4} T_1^* e^{-\frac{1}{4} T_1^*}
\end{pmatrix} \right)
\]

\[
+ g_{U_1}(r_1(T_1^*), z_1(T_1^*), y_1(T_1^*), \epsilon_1(T_1^*))
\]

\[
+ \int_0^{T_1^*} \frac{d}{dT_1^*} \left[ \Phi_1(T_1^*, s) g_{U_1}(r_1(s), z_1(s), y_1(s), \epsilon_1(s)) \right] ds,
\]

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where
\[ z_{1,0}(y_{1,0}) = O(\rho \log \rho) \]
\[ (z_{1,0}')(y_{1,0}) \frac{dy_{1,0}}{dT_1} = O \left( \frac{1}{\rho} e^{-\frac{1}{2}T_1} \left( 1 + O(\rho \log \rho, \delta) \right) \right) = O(\rho), \]
by (5.66), and
\[ g_{\mathcal{U}_1}(r_1(T_1^*), z_1(T_1^*), y_1(T_1^*), \epsilon_1(T_1^*)) = O \left( (\rho \log \rho)^3 e^{-\frac{3}{2}T_1^*} \right) \]
\[ \int_0^{T_1^*} \frac{d}{dT_1^*} \left[ \Phi_{\mathcal{U}_1}(T_1^*, s) g_{\mathcal{U}_1}(r_1(s), z_1(s), y_1(s), \epsilon_1(s)) \right] ds = \left( \begin{array}{c}
O \left( (\rho \log \rho)^2 e^{-\frac{3}{4}T_1^*} \right) \\
O \left( T_1^* (\rho \log \rho) e^{-\frac{3}{4}T_1^*} \right)
\end{array} \right), \]
by (5.44), (5.52), and (5.60).
Therefore we have that
\[ \frac{dr_1(T_1^*)}{dT_1^*} = \left( -\frac{\rho^4}{4} + O \left( (\rho \log \rho)^2 \right) \right) e^{-\frac{3}{4}T_1^*} \]
\[ \frac{dz_1(T_1^*)}{dT_1^*} = (O(\rho \log \rho, (1 + T_1^*)\rho)) e^{-\frac{3}{4}T_1^*} \]
(5.69)
so that in \( \Sigma_{13} \), for each fixed \( \rho, \delta \) sufficiently small, we obtain a curve \( z_1 = z_1(r_1) \) satisfying
\[ \left| \frac{dz_1}{dr_1} \right| \leq C(\rho, \delta)(1 + T_1^*), \]
(5.70)
uniformly. Using the fact that \( |T_1^*| \leq C(\rho, \delta) |\log \epsilon| \), we therefore obtain
\[ \left| \frac{dz_1}{dr_1} \right| \leq C(\rho, \delta) |\log \epsilon|, \]
(5.71)
uniformly in \((c, a, \epsilon)\), which completes the proof of (ii).
The proof of (ii) is similar and we omit the details. \( \square \)

### 5.4 Dynamics in \( \mathcal{K}_2 \)

In the \( \mathcal{K}_2 \) coordinates, the equations are given by
\[ \dot{x}_2 = -x_2 + r_2y_2z_2 + O \left( r_2^3 z_2^2 \right) \]
\[ \dot{z}_2 = -z_2 - r_2x_2 \]
\[ \dot{y}_2 = -kr_2 + O \left( r_2^3 y_2, r_2^4 \right) \]
\[ \dot{r}_2 = 0. \]
(5.72)

Solutions enter via \( \Sigma_{12} \) which is given in the \( \mathcal{K}_2 \) coordinates by
\[ \Sigma_{12} = \left\{ (x_2, y_2, z_2, r_2) : x_2 = \frac{1}{\delta^{2/3}}, |y_2| \leq \frac{1}{\delta^{2/3}}, |z_2| \leq \frac{\mu}{\delta}, 0 < r_2 \leq \rho_0^{1/3} \right\}, \]
(5.73)
and exit via
\[ \Sigma_{23} = \left\{ (x_2, y_2, z_2, r_2) : |x_2| \leq \frac{1}{\delta^{2/3}}, y_2 = -\frac{1}{\delta^{2/3}}, |z_2| \leq \frac{\mu}{\delta}, 0 < r_2 \leq \rho_0^{1/3} \right\}. \]
(5.74)
The setup in the chart \( \mathcal{K}_2 \) is shown in Figure 23.
In this chart we can determine formulae for the solutions as follows. First, we consider solutions starting in $\Sigma_{12}$ as time $t = 0$. We set $x_2 = e^{-t} \tilde{x}_2, z_2 = e^{-t} \tilde{z}_2$ and obtain the system

$$
\begin{align*}
\dot{x}_2 &= r_2 y_2 \tilde{z}_2 + O(e^{-t} r_2^2 \tilde{z}_2^2) \\
\dot{\tilde{z}}_2 &= -r_2 \tilde{x}_2 \\
\dot{y}_2 &= -kr_2 + O(r_2^3) \\
\dot{r}_2 &= 0.
\end{align*}
$$

We now rescale time by $t = t_2/r_2$ to desingularize the system

$$
\begin{align*}
\dot{x}_2' &= y_2 \tilde{z}_2 + O(e^{-t_2/r_2^2} r_2) \\
\dot{\tilde{z}}_2' &= -\tilde{x}_2 \\
\dot{y}_2' &= -k + O(r_2^3) \\
\dot{r}_2' &= 0,
\end{align*}
$$

\hspace{1cm} (5.75)

where $t$ denotes $\frac{d}{dr_2}$. Setting $r_2 = 0$ we obtain the Airy equations

$$
\begin{align*}
\dot{\tilde{x}}_2' &= y_2 \tilde{z}_2 \\
\dot{\tilde{z}}_2' &= -\tilde{x}_2 \\
\dot{y}_2' &= -k,
\end{align*}
$$

\hspace{1cm} (5.77)
which have the following explicit solutions in terms of Airy functions $A_i, B_i$

$$\tilde{x}_2 = \pi \left[ \left( A_i \left( -\frac{y_{2,0}}{k^{2/3}} \right) B_i' \left( -\frac{y_{2,0}}{k^{2/3}} + k^{1/3}t_2 \right) - B_i \left( -\frac{y_{2,0}}{k^{2/3}} \right) A_i' \left( -\frac{y_{2,0}}{k^{2/3}} + k^{1/3}t_2 \right) \right) \frac{1}{\delta^{4/3}} \right]$$

$$+ k^{1/3} \left[ A_i' \left( -\frac{y_{2,0}}{k^{2/3}} \right) B_i \left( -\frac{y_{2,0}}{k^{2/3}} + k^{1/3}t_2 \right) - B_i' \left( -\frac{y_{2,0}}{k^{2/3}} \right) A_i \left( -\frac{y_{2,0}}{k^{2/3}} + k^{1/3}t_2 \right) \right] z_{2,0}$$

$$\tilde{z}_2 = \pi \left[ \left( B_i \left( -\frac{y_{2,0}}{k^{2/3}} \right) A_i \left( -\frac{y_{2,0}}{k^{2/3}} + k^{1/3}t_2 \right) - A_i \left( -\frac{y_{2,0}}{k^{2/3}} \right) B_i' \left( -\frac{y_{2,0}}{k^{2/3}} + k^{1/3}t_2 \right) \right) \frac{1}{\delta^{4/3}} \right]$$

$$+ k^{1/3} \left[ B_i' \left( -\frac{y_{2,0}}{k^{2/3}} \right) A_i \left( -\frac{y_{2,0}}{k^{2/3}} + k^{1/3}t_2 \right) - A_i' \left( -\frac{y_{2,0}}{k^{2/3}} \right) B_i \left( -\frac{y_{2,0}}{k^{2/3}} + k^{1/3}t_2 \right) \right] z_{2,0}$$

$$y_2 = y_{2,0} - kt_2,$$

where $y_{2,0} = y_2(0)$ and $z_{2,0} = \tilde{z}_2(0) = z_2(0)$.

**Lemma 5.6.** For each fixed $\delta, \mu > 0$, there exists $r_{2,0} > 0$ such that for any $0 < r < r_{2,0}$, any solution of (5.75) with initial condition in $\Sigma_{12}$ given by $(x_2, y_2, z_2)(0) = (1/\delta^{4/3}, y_{2,0}, z_{2,0})$ reaches $\Sigma_{23}$ with

$$x_2 = \pi e^{-\frac{\pi}{k^{2/3}}y_{2,0}} \left[ \left( A_i \left( -\frac{y_{2,0}}{k^{2/3}} \right) B_i' \left( \frac{1}{k^{2/3}g^{2/3}} \right) - B_i \left( -\frac{y_{2,0}}{k^{2/3}} \right) A_i' \left( \frac{1}{k^{2/3}g^{2/3}} \right) \right) \frac{1}{\delta^{4/3}} \right]$$

$$+ k^{1/3} \left[ A_i' \left( -\frac{y_{2,0}}{k^{2/3}} \right) B_i \left( \frac{1}{k^{2/3}g^{2/3}} \right) - B_i' \left( -\frac{y_{2,0}}{k^{2/3}} \right) A_i \left( \frac{1}{k^{2/3}g^{2/3}} \right) \right] z_{2,0} + O(r_2^2) \right]$$

$$z_2 = \pi e^{-\frac{\pi}{k^{2/3}}y_{2,0}} \left[ \left( B_i \left( -\frac{y_{2,0}}{k^{2/3}} \right) A_i \left( \frac{1}{k^{2/3}g^{2/3}} \right) - A_i \left( -\frac{y_{2,0}}{k^{2/3}} \right) B_i' \left( \frac{1}{k^{2/3}g^{2/3}} \right) \right) \frac{1}{\delta^{4/3}} \right]$$

$$+ k^{1/3} \left[ B_i' \left( -\frac{y_{2,0}}{k^{2/3}} \right) A_i \left( \frac{1}{k^{2/3}g^{2/3}} \right) - A_i' \left( -\frac{y_{2,0}}{k^{2/3}} \right) B_i \left( \frac{1}{k^{2/3}g^{2/3}} \right) \right] z_{2,0} + O(r_2^2) \right]$$

$$y_2 = -\frac{1}{\delta^{2/3}}.$$

**Proof.** Considering the equations (5.77), solutions given by (5.78) with initial conditions $(x_2, y_2, z_2)(0) = (1/\delta^{4/3}, y_{2,0}, z_{2,0})$ in $\Sigma_{12}$ exit $\Sigma_{23}$ in time

$$T_2 = \frac{1}{k} \left( y_{2,0} + \frac{1}{\delta^{2/3}} \right).$$

For each fixed $\delta > 0$, $T_2$ is bounded uniformly in $|y_{2,0}| \leq 1/\delta^{2/3}$. Hence by a regular perturbation argument, and returning to the original coordinates $x_2, z_2$, we obtain the result. \hfill \Box

### 5.5 Dynamics in $\mathcal{K}_3$

In the $\mathcal{K}_3$ coordinates, the equations are given by

$$\dot{x}_3 = -x_3 - r_3 z_3 - 2kr_3 x_3 z_3 + O \left( r_3^2 z_3^2, r_3^3 x_3 \right)$$

$$\dot{z}_3 = -z_3 - r_3 x_3 - \frac{3}{2}kr_3 z_3 z_3 + O \left( r_3^3 z_3^2 \right)$$

$$\dot{r}_3 = \frac{1}{2} r_3^2 z_3 \left( k + O \left( r_3^2 \right) \right)$$

$$\dot{\epsilon}_3 = -\frac{3}{2} r_3^2 \epsilon_3 \left( k + O \left( r_3^2 \right) \right)$$

Solutions enter via $\Sigma_{13}$ or $\Sigma_{23}$ which are given in the $\mathcal{K}_3$ coordinates by

$$\Sigma_{13} = \{ (x_3, z_3, r_3, \epsilon_3) : x_3 = 1, \ |z_3| \leq \mu, \ 0 < r_3 \leq \rho, \ 0 < \epsilon_3 \leq \delta \}$$

$$\Sigma_{23} = \{ (x_3, z_3, r_3, \epsilon_3) : |x_3| \leq 1, \ |z_3| \leq \mu, \ 0 < r_3 \leq \rho, \ \epsilon_3 = \delta \},$$

(5.83)
Figure 24: Shown is the setup in the chart $\mathcal{K}_3$, including a schematic of the results of Lemma 5.10, Lemma 5.11, and Lemma 5.12.

respectively, and exit via

$$
\Sigma_3^{out} = \{(x_3, z_3, r_3, \epsilon_3) : |x_3| \leq 1, |z_3| \leq \mu, r_3 = \rho, 0 < \epsilon_3 \leq \delta\}. \quad (5.84)
$$

We need to determine the behavior of solutions which enter via $\Sigma_{13}$ or $\Sigma_{23}$ upon exit in $\Sigma_3^{out}$. The setup is shown in Figure 24. Between these sections, due to the relation $r_3^3 \epsilon_3 = \epsilon$, such solutions are restricted to the region $(\epsilon/\delta)^{1/3} \leq r_3 \leq \rho$ in which $r_3$ is strictly increasing. The linearization of (5.81) in the $(x_3, z_3)$-plane has approximate eigenvalues $(-1 \pm r_3)$. Hence, informally one expects that the flow should separate into strong and weak stable directions with an exponential separation that is initially $O(\epsilon^{1/3})$ and grows to $O(1)$ at $\Sigma_3^{out}$. We begin by deriving a change of coordinates $(x_3, z_3) \rightarrow (\tilde{x}_3, \tilde{z}_3)$ which more clearly separates these strong/weak directions.

To see this, we add an equation for the ratio $\theta_3 := z_3/x_3$

$$
\dot{\theta}_3 = \frac{\tilde{x}_3^i - x_3^i}{x_3} - \theta_3 \frac{\tilde{x}_3^i}{x_3} = -r_3 + r_3 \theta_3^2 + \frac{3}{2} kr_3 \epsilon_3 \theta_3 + 2 kr_3 \epsilon_3 \theta_3 + O \left(r_3^3 \epsilon_3 \theta_3, r_3^2 \theta_3^2\right) \quad (5.85)
$$

and we consider the extended system

$$
\begin{align*}
\dot{x}_3 &= -x_3 - r_3 z_3 - 2 kr_3 x_3 \epsilon_3 + O \left(r_3^2 (|x_3| + |z_3|)\right) \\
\dot{z}_3 &= -z_3 - r_3 x_3 - \frac{3}{2} kr_3 \epsilon_3 z_3 + O \left(r_3^3 \epsilon_3\right) \\
\dot{\theta}_3 &= r_3 \left(\theta_3^2 - 1 + \frac{1}{2} k \epsilon_3 \theta_3\right) + O \left(r_3^2\right) \\
\dot{r}_3 &= \frac{1}{2} r_3^2 \epsilon_3 \left(k + O \left(r_3^2\right)\right) \\
\dot{\epsilon}_3 &= -\frac{3}{2} r_3 \epsilon_3 \left(k + O \left(r_3^2\right)\right).
\end{align*} \quad (5.86)
$$
Figure 25: Shown are the invariant manifolds $M^\pm_3(c,a)$ corresponding to the dynamics of (5.88).

Solutions are exponentially attracted to the subspace $x_3 = z_3 = 0$ on which the flow is given by

$$\begin{align*}
\dot{\theta}_3 &= r_3 \left( \theta_3^2 - 1 + \frac{1}{2} k \epsilon_3 \theta_3 \right) + O(r_3^2) \\
\dot{r}_3 &= \frac{1}{2} r_3^2 \epsilon_3 \left( k + O(r_3^2) \right) \\
\dot{\epsilon}_3 &= -\frac{3}{2} r_3^2 \epsilon_3 \left( k + O(r_3^2) \right).
\end{align*}$$

(5.87)

Rescaling time by $t_3 = r_3 t$, we obtain

$$\begin{align*}
\dot{\theta}'_3 &= \theta_3^2 - 1 + \frac{1}{2} k \epsilon_3 \theta_3 + O(r_3) \\
\dot{r}'_3 &= \frac{1}{2} r_3^2 \epsilon_3 \left( k + O(r_3^2) \right) \\
\dot{\epsilon}'_3 &= -\frac{3}{2} \epsilon_3^2 \left( k + O(r_3^2) \right).
\end{align*}$$

(5.88)

Firstly, there are two invariant subspaces for the dynamics of (5.88): the plane $r_3 = 0$ and the plane $\epsilon_3 = 0$. Their intersection is the invariant line $l_3 = \{ (\theta_3, 0, 0) : \theta_3 \in \mathbb{R} \}$, and the dynamics on $l_3$ evolve according to $\theta'_3 = -1 + \theta_3^2$. There are two equilibria $p^- = (-1, 0, 0)$ and $p^+ = (1, 0, 0)$, with eigenvalues $-2$ and $2$, respectively, for the flow along $l_3$. In the plane $\epsilon_3 = 0$, the dynamics are given by

$$\begin{align*}
\dot{\theta}'_3 &= \theta_3^2 - 1 + O(r_3) \\
\dot{r}'_3 &= 0.
\end{align*}$$

(5.89)

This system has normally hyperbolic curves $S^\pm_{0,3}(c,a)$ of equilibria emanating from $p^\pm$ (see Figure 25). Along $S^\pm_{0,3}(c,a)$ the linearization has one zero eigenvalue and one eigenvalue close to $\pm 2$ for small $r_3$.

In the invariant plane $r_3 = 0$, the dynamics are given by

$$\begin{align*}
\dot{\theta}'_3 &= \theta_3^2 - 1 + \frac{1}{2} k \epsilon_3 \theta_3 \\
\dot{\epsilon}'_3 &= -\frac{3}{2} \epsilon_3^2.
\end{align*}$$

(5.90)
Restricting attention to the set along which $\epsilon$. The corresponding eigenvector is ($\ldots$).

Here we still have the equilibria $p^\pm$ which now have an additional zero eigenvalue due to the second equation. The corresponding eigenvector is $(-k, 4)$ and hence there exist one-dimensional center manifolds $N^\pm_3(c, a)$ at $p^\pm$ along which $\epsilon_3$ decreases. Note that the branch of $N^+_3(c, a)$ in the half space $\epsilon_1 > 0$ is unique.

Restricting attention to the set

$$D_3 = \{ (\theta_3, r_3, \epsilon_3) : \theta_3 \in \mathbb{R}, 0 \leq r_3 \leq \rho, 0 \leq \epsilon_1 \leq \delta \},$$

we have the following (see Figure 25).

**Proposition 5.7.** For any $(c, a) \in I_c \times I_a$ and any sufficiently small $\rho, \delta > 0$, the following assertions hold for the dynamics of (5.88):

1. There exists a repelling two-dimensional center manifold $M^+_3(c, a)$ at $p^+$ which contains the line of equilibria $S^+_{0,3}(c, a)$ and the center manifold $N^+_3(c, a)$. In $D_3$, $M^+_3(c, a)$ is given as a graph $\theta_3 = h^+(r_3, \epsilon_3, c, a) = 1 + O(r_3, \epsilon_3)$. The branch of $N^+_3(c, a)$ in $r_3 = 0, \epsilon_3 > 0$ is unique.

2. There exists an attracting two-dimensional center manifold $M^-_3(c, a)$ at $p^-$ which contains the line of equilibria $S^-_{0,3}(c, a)$ and the center manifold $N^-_3(c, a)$. In $D_3$, $M^-_3(c, a)$ is given as a graph $\theta_3 = h^-(r_3, \epsilon_3, c, a) = -1 + O(r_3, \epsilon_3)$.

We now return to the full system (5.86), in which the flow on the subspace $x_3 = z_3 = 0$ is foliated by flow along strong stable fibers. Hence in the full five-dimensional space, there exist four-dimensional invariant manifolds $\tilde{M}^\pm_3(c, a)$ (see Figure 26) given by the strong stable foliations of the two-dimensional manifolds $M^\pm_3(c, a)$. The manifolds $\tilde{M}^\pm_3(c, a)$ can be written as graphs $\theta_3 = H^\pm(x_3, z_3, r_3, \epsilon_3, c, a) = \pm 1 + O(r_3, \epsilon_3)$.

Now using the relation $\theta_3 = z_3/x_3$, we see that the dynamics are in fact restricted to three-dimensional invariant submanifolds $\tilde{M}^\pm_3(c, a)$ of $M^\pm_3(c, a)$. The manifolds $\tilde{M}^\pm_3(c, a)$ are given by $z_3 = x_3 H^\pm(x_3, z_3, r_3, \epsilon_3, c, a)$. By the implicit function theorem, for any sufficiently small $\rho, \delta > 0$, we can now solve to find $\tilde{M}^\pm_3(c, a)$ as graphs

$$z_3 = F^-(x_3, r_3, \epsilon_3, c, a) = x_3(-1 + O(r_3, \epsilon_3))$$

$$x_3 = F^+(z_3, r_3, \epsilon_3, c, a) = z_3(1 + O(r_3, \epsilon_3)).$$

Figure 26: Shown are the invariant manifolds $\tilde{M}^\pm_3(c, a)$ corresponding to the dynamics of (5.86).
We now change coordinates by
\[ \tilde{x}_3 = x_3 - F^{-}(x_3, r_3, \epsilon_3, c, a) = x_3 + x_3(1 + O(r_3, \epsilon_3)) \]
\[ \tilde{z}_3 = x_3 - F^{+}(z_3, r_3, \epsilon_3, c, a) = x_3 - z_3(1 + O(r_3, \epsilon_3)) \]

In these coordinates, (5.81) becomes
\[ \begin{align*}
\dot{\tilde{x}}_3 &= (-1 + r_3 + r_3 h_+(\tilde{x}_3, \tilde{z}_3, r_3, \epsilon_3)) \tilde{x}_3 \\
\dot{\tilde{z}}_3 &= (-1 - r_3 + r_3 h_- (\tilde{x}_3, \tilde{z}_3, r_3, \epsilon_3)) \tilde{z}_3 \\
\dot{r}_3 &= \frac{1}{2} r_3^2 \epsilon_3 (k + g_1(\tilde{x}_3, \tilde{z}_3, r_3, \epsilon_3)) \\
\dot{\epsilon}_3 &= -\frac{3}{2} r_3 \epsilon_3^2 (k + g_2(\tilde{x}_3, \tilde{z}_3, r_3, \epsilon_3))
\end{align*} \tag{5.94} \]

where
\[ h_+(\tilde{x}_3, \tilde{z}_3, r_3, \epsilon_3) = O(r_3, \epsilon_3) \]
\[ h_- (\tilde{x}_3, \tilde{z}_3, r_3, \epsilon_3) = O(r_3, \epsilon_3) \]
\[ g_1(\tilde{x}_3, \tilde{z}_3, r_3, \epsilon_3) = O(r_3^2) \]
\[ g_2(\tilde{x}_3, \tilde{z}_3, r_3, \epsilon_3) = O(r_3^2). \tag{5.95} \]

In (5.94), it is clear that the strong attraction in the variables \((x_3, z_3)\) splits into strong/weak directions where the exponential splitting increases as \(r_3\) increases. By changing coordinates to \((\tilde{x}_3, \tilde{z}_3)\), we straighten out the invariant manifolds \(\tilde{M}_3^\pm(c, a)\) (see Figure 27).

### 5.5.1 Solutions with initial conditions in \(\Sigma_{23}\)

We first consider solutions entering \(K_3\) via \(\Sigma_{23}\). Using \(r_3^3 \epsilon_3 = \epsilon\), we have that such solutions satisfy \(\epsilon_3 = \delta, r_3 = (\epsilon/\delta)^{1/3}\) in \(\Sigma_{23}\). We have the following.

**Lemma 5.8.** For all sufficiently small \(\rho, \delta > 0\), any solution to (5.94) with initial condition \((\tilde{x}_3, \tilde{z}_3, r_3, \epsilon_3)(0) = (\tilde{x}_{3,0}, \tilde{z}_{3,0}, (\epsilon/\delta)^{1/3}, \delta) \in \Sigma_{23}\) which reaches the section \(\Sigma_{3}^{out}\) at time \(t = T^* = T^* (\rho, \delta, \tilde{x}_{3,0}, \tilde{z}_{3,0}, \epsilon)\) satisfies
\[ \begin{align*}
\tilde{x}_3(T^*) &= \tilde{x}_{3,0} \exp \left( \beta_2^0 (\rho, \delta, \epsilon) + \eta_2^0 (\rho, \delta, \tilde{x}_{3,0}, \tilde{z}_{3,0}, \epsilon) \right) \\
\tilde{z}_3(T^*) &= \tilde{z}_{3,0} \exp \left( \beta_2^0 (\rho, \delta, \epsilon) + \eta_2^0 (\rho, \delta, \tilde{x}_{3,0}, \tilde{z}_{3,0}, \epsilon) \right) \\
r_3(T^*) &= \rho \\
\epsilon_3(T^*) &= \frac{\epsilon}{\rho^3}.
\end{align*} \tag{5.96} \]
where
\[ \beta_2^2(\rho, \delta, \epsilon) = \frac{\rho^2}{\epsilon} \left( -1 - \frac{2\rho}{3} + \mathcal{O}(\rho^2, \rho\delta) \right) \]
\[ \beta_2^2(\rho, \delta, \epsilon) = \frac{\rho^2}{\epsilon} \left( -1 + \frac{2\rho}{3} + \mathcal{O}(\rho^2, \rho\delta) \right) \]
\[ \eta_2^2(\rho, \delta, \tilde{x}_3, \tilde{z}_3, \epsilon) = \mathcal{O}\left( \left( \frac{\epsilon}{\delta} \right)^{1/3} \right)(|\tilde{x}_3| + |\tilde{z}_3|) \].

Proof. It is clear from (5.94) that the \((\tilde{x}_3, \tilde{z}_3)\)-coordinates decay exponentially for all sufficiently small \(\rho, \delta > 0\).

By directly integrating (5.94), we obtain the following expressions
\[ \tilde{x}_3(T^*) = \tilde{x}_{3,0} \exp \left( -T^* + \int_0^{T^*} r_3(t) (1 + h_+(\tilde{x}_3(t), \tilde{z}_3(t), r_3(t), \epsilon_3(t))) \, dt \right) \]
\[ \tilde{z}_3(T^*) = \tilde{z}_{3,0} \exp \left( -T^* - \int_0^{T^*} r_3(t) (1 + h_-(\tilde{x}_3(t), \tilde{z}_3(t), r_3(t), \epsilon_3(t))) \, dt \right) \]  
(5.98)
\[ r_3(T^*) = \rho \]
\[ \epsilon_3(T^*) = \frac{\epsilon}{\rho^3} \].

We determine the functions \(\beta_2^2, \eta_2^2\). The computation of \(\beta_2^2, \eta_2^2\) is similar. We now write
\[ T^* = \int_0^\rho \frac{1}{r_3} \, dr_3 \]
\[ = \frac{2}{\epsilon} \int_0^\rho r_3 \left( 1 + \mathcal{O}(r_3^2) \right) \, dr_3, \]
(5.99)
using \(r_3^2 \epsilon_3 = \epsilon\). We also have
\[ \int_0^{T^*} r_3(t) (1 + h_+(\tilde{x}_3(t), \tilde{z}_3(t), r_3(t), \epsilon_3(t))) \, dt = \frac{2}{\epsilon} \int_0^\rho r_3^2 (1 + \mathcal{O}(r_3, \epsilon_3)) \, dr_3, \]
(5.100)
and hence
\[ -T^* + \int_0^{T^*} r_3(t) (1 + \mathcal{O}(r_3(t), \epsilon_3(t))) \, dt = -\frac{2}{\epsilon} \int_0^\rho r_3^2 + \mathcal{O}(r_3^3, r_3^2 \epsilon_3) \, dr_3 \]
\[ = -\frac{2}{\epsilon} \int_0^\rho r_3^2 + h_1(r_3, \epsilon_3) + h_2(r_3, \epsilon_3, \tilde{x}_3, \tilde{z}_3) \, dr_3, \]
(5.101)
where we have separated out the \(\tilde{x}_3, \tilde{z}_3\) dependence through the functions \(h_1, h_2\). That is, we have \(\partial_{\tilde{x}_3} h_1 = \partial_{\tilde{z}_3} h_1 = 0\) and
\[ h_1(r_3, \epsilon_3) = \mathcal{O}\left( r_3^3, r_3^2 \epsilon_3 \right) \]
\[ h_2(r_3, \epsilon_3, \tilde{x}_3, \tilde{z}_3) = \mathcal{O}\left( r_3^2 (|r_3| + |\epsilon_3|) (|\tilde{x}_3| + |\tilde{z}_3|) \right). \]
(5.102)

We now define
\[ \beta_2^2(\rho, \delta, \epsilon) = -\frac{2}{\epsilon} \int_0^\rho r_3^2 + h_1(r_3, \epsilon_3) \, dr_3 \]
\[ = -\frac{2}{\epsilon} \left( \frac{\rho^2}{2} - \frac{\rho^3}{3} + \mathcal{O}(\rho^4, \rho^3 \delta) \right) \]
(5.103)
\[ \eta_2^2(\rho, \delta, \tilde{x}_3, \tilde{z}_3, \epsilon) = -\frac{2}{\epsilon} \int_0^\rho h_2(r_3, \epsilon_3, \tilde{x}_3, \tilde{z}_3) \, dr_3. \]
To estimate $\eta_1^2$, we first note that for any sufficiently small $\rho, \delta$, we can bound

$$
|\ddot{\bar{x}}_3(t)| \leq \ddot{x}_{3,0} \exp(-t/2)
$$
$$
|\ddot{\bar{z}}_3(t)| \leq \ddot{z}_{3,0} \exp(-t/2),
$$

(5.104)

for any $0 \leq t \leq T^*$. Furthermore, we have

$$
t = \int_{(\epsilon, \delta)_{1/3}}^{r_{3}(t)} \frac{1}{r_{3}} dr_{3}
$$
$$
= \frac{2}{\epsilon} \int_{(\epsilon, \delta)_{1/3}}^{r_{3}(t)} r_{3} \left(1 + O \left(r_{3}^{2}\right)\right) dr_{3}
$$
$$
> \frac{1}{2\epsilon} \left(r_{3}(t)^2 - (\epsilon/\delta)^{2/3}\right),
$$

(5.105)

for each sufficiently small fixed $\rho, \delta > 0$. Hence we have

$$
\eta_1^2(\rho, \delta, \bar{x}_{3,0}, \bar{z}_{3,0}, \epsilon) = -\frac{2}{\epsilon} \int_{(\epsilon, \delta)_{1/3}}^{\rho} h_2(r_{3}, \epsilon_3, \bar{x}_3, \bar{z}_3) dr_{3}
$$
$$
= -\frac{2}{\epsilon} \int_{(\epsilon, \delta)_{1/3}}^{\rho} \mathcal{O} \left(\frac{r_{3}^2}{k_{\delta}}(r_{3}) + |\epsilon_3|(|\bar{x}_3| + |\bar{z}_3|)\right) dr_{3}
$$
$$
= -\frac{2}{\epsilon} \int_{(\epsilon, \delta)_{1/3}}^{\rho} \mathcal{O} \left(\frac{r_{3}^2}{k_{\delta}}(r_{3}) + |\epsilon_3|(|\bar{x}_3| + |\bar{z}_3|) \exp \left(-\frac{1}{4\epsilon} \left(r_{3}^2 - (\epsilon/\delta)^{2/3}\right)\right)\right) dr_{3}
$$
$$
= \mathcal{O} \left(\left(\frac{\epsilon}{\delta}\right)^{1/3}(|\bar{x}_3| + |\bar{z}_3|)\right)
$$

(5.106)

that is, the dependence on the initial $(\bar{x}_{3,0}, \bar{z}_{3,0})$ of the exponential contraction between $\Sigma_{23}$ and $\Sigma_{3}^{out}$ is very small.

We now consider solutions on $\hat{\Sigma}_{e,j}(c, a)$, $j = \ell, r$ passing through $\Sigma_{\ell}^{in} = \Sigma_{1}^{in} \rightarrow \Sigma_{12} \rightarrow \Sigma_{23}$. We obtain estimates for these solutions upon entry in the chart $K_{3}$ in $\Sigma_{23}$ and exit via $\Sigma_{3}^{out}$.

**Lemma 5.9.** Solutions on the manifolds $\hat{\Sigma}_{e,j}(c, a)$, $j = \ell, r$, which have initial conditions in $\Sigma_{23}$ define curves in $\Sigma_{23}$ parametrized by $|y_{2,0}| \leq 1/\delta^{2/3}$ given by

$$
\ddot{\bar{x}}_{3,0}(y_{2,0}) = \sqrt{\pi e^{-\frac{y_{2,0}^2}{k_{1/6}^{3/2}}}} \bar{X}_{3}^{j}(y_{2,0})
$$
$$
\ddot{\bar{z}}_{3,0}(y_{2,0}) = \sqrt{\pi e^{-\frac{y_{2,0}^2}{k_{1/6}^{3/2}}}} \bar{Z}_{3}^{j}(y_{2,0}),
$$

(5.107)

where

$$
\bar{X}_{3}^{j}(y_{2,0}) = \left(\mathrm{Ai} \left(-\frac{y_{2,0}}{k_{2/3}^{3/2}}\right) + k_{1/3}^{1/3} \delta^{4/3} \mathrm{Ai}^{'} \left(-\frac{y_{2,0}}{k_{2/3}^{3/2}}\right) \ddot{z}_{2,0}(y_{2,0})\right) e^{\frac{2}{3} \frac{\epsilon}{\delta}} (2 + \mathcal{O}(\delta))
$$
$$
+ \mathcal{O}(\delta) \left(\mathrm{Bi} \left(-\frac{y_{2,0}}{k_{2/3}^{3/2}}\right) + k_{1/3}^{1/3} \delta^{4/3} \mathrm{Bi}^{'} \left(-\frac{y_{2,0}}{k_{2/3}^{3/2}}\right) \ddot{z}_{2,0}(y_{2,0})\right) e^{-\frac{2}{3} \frac{\epsilon}{\delta}} + \mathcal{O}(\epsilon^{2/3})
$$
$$
\bar{Z}_{3}^{j}(y_{2,0}) = \left(\mathrm{Bi} \left(-\frac{y_{2,0}}{k_{2/3}^{3/2}}\right) + k_{1/3}^{1/3} \delta^{4/3} \mathrm{Bi}^{'} \left(-\frac{y_{2,0}}{k_{2/3}^{3/2}}\right) \ddot{z}_{2,0}(y_{2,0})\right) e^{-\frac{2}{3} \frac{\epsilon}{\delta}} + \mathcal{O}(\epsilon^{2/3})
$$
$$
+ \mathcal{O}(\delta) \left(\mathrm{Ai} \left(-\frac{y_{2,0}}{k_{2/3}^{3/2}}\right) + k_{1/3}^{1/3} \delta^{4/3} \mathrm{Ai}^{'} \left(-\frac{y_{2,0}}{k_{2/3}^{3/2}}\right) \ddot{z}_{2,0}(y_{2,0})\right) e^{\frac{2}{3} \frac{\epsilon}{\delta}} + \mathcal{O}(\epsilon^{2/3}),
$$

(5.108)

for $j = \ell, r$, where

$$
\left|\ddot{z}_{2,0}(y_{2,0})\right| \leq C \epsilon^{1/3} |\log \epsilon|
$$
$$
\left|\frac{d\ddot{z}_{2,0}}{dy_{2,0}}\right| \leq C \epsilon^{1/3} |\log \epsilon|
$$

(5.109)
and

\[ \kappa \epsilon^{1/3} < z_{2,0}^r(y_{2,0}) - z_{2,0}^s(y_{2,0}) < C \epsilon^{1/3} | \log \epsilon |. \]  \hspace{1cm} (5.110)

uniformly in \( y_{2,0} \) for some \( C, \kappa > 0 \) independent of \( c, a, \epsilon \).

Proof. Using the analysis in §5.4, Lemma 5.6 and the estimates in Proposition 5.5 (ii), we deduce that solutions on the manifolds \( \tilde{\mathcal{W}}_c^{+j}(c, a) \) define curves in \( \Sigma_{33} \) parametrized by \( |y_{2,0}| \leq 1/\delta^{2/3} \) as

\[ x_3 = x_{3,0}(y_{2,0}) = \pi e^{-\frac{2 + y_{2,0}}{k_{1/2}^2 \delta^{1/2}}}, \]

\[ + k^{1/3} \delta^{4/3} \left( \text{Ai}' \left( -\frac{y_{2,0}}{k^{2/3}} \right) - \text{Bi}' \left( -\frac{y_{2,0}}{k^{2/3}} \right) \right) z_{2,0}^j(y_{2,0}) + O(r_2^2) \]

\[ z_3 = z_{3,0}(y_{2,0}) = \pi e^{-\frac{2 + y_{2,0}}{k_{1/2}^2 \delta^{1/2}}} \left( \text{Bi} \left( -\frac{y_{2,0}}{k^{2/3}} \right) - \text{Ai} \left( -\frac{y_{2,0}}{k^{2/3}} \right) \right) z_{2,0}^j(y_{2,0}) + O(r_2^2) \]

\[ \epsilon_3 = \delta, \]

for \( j = \ell, r \), where

\[ \left| z_{2,0}^j(y_{2,0}) \right| \leq C \epsilon^{1/3} | \log \epsilon | \]

\[ \left| \frac{dz_{2,0}^j}{dy_{2,0}} \right| \leq C | \log \epsilon | \]  \hspace{1cm} (5.112)

and

\[ \kappa \epsilon^{1/3} < z_{2,0}^r(y_{2,0}) - z_{2,0}^s(y_{2,0}) < C \epsilon^{1/3} | \log \epsilon |. \]  \hspace{1cm} (5.113)

uniformly in \( y_{2,0} \) for some \( C, \kappa > 0 \) independent of \( c, a, \epsilon \). Using asymptotic properties of Airy functions (5.19), we have

\[ x_{3,0}^j(y_{2,0}) = \sqrt{\pi} e^{-\frac{2 + y_{2,0}}{k_{1/6}^2 \delta^{1/6}}} X_{3,0}^j(y_{2,0}) \]

\[ z_{3,0}^j(y_{2,0}) = \sqrt{\pi} e^{-\frac{2 + y_{2,0}}{k_{1/6}^2 \delta^{1/6}}} Z_{3,0}^j(y_{2,0}), \]  \hspace{1cm} (5.114)

where

\[ X_{3,0}^j(y_{2,0}) = \left( \text{Ai} \left( -\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Ai}' \left( -\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^j(y_{2,0}) \right) e^{\frac{2}{3} \frac{k^2}{\delta}} \left( 1 + O(\delta) \right) \]

\[ + \left( \text{Bi} \left( -\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Bi}' \left( -\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^j(y_{2,0}) \right) e^{-\frac{2}{3} \frac{k^2}{\delta}} \left( 1 + O(\delta) \right) + O(r_2^2) \]  \hspace{1cm} (5.115)

\[ Z_{3,0}^j(y_{2,0}) = -\left( \text{Ai} \left( -\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Ai}' \left( -\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^j(y_{2,0}) \right) e^{\frac{2}{3} \frac{k^2}{\delta}} \left( 1 + O(\delta) \right) \]

\[ + \left( \text{Bi} \left( -\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Bi}' \left( -\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^j(y_{2,0}) \right) e^{-\frac{2}{3} \frac{k^2}{\delta}} \left( 1 + O(\delta) \right) + O(r_2^2) \].
Using (5.93), in the ‘˜’ coordinates we have

\[
\begin{align*}
\tilde{z}_{3,0}^0(y_2,0) &= z_{3,0}^0(y_2,0) - F^- \left( x_{3,0}^0(y_2,0), (\epsilon/\delta)^{1/3}, \delta, c, a \right) \\
\tilde{x}_{3,0}^0(y_2,0) &= x_{3,0}^0(y_2,0) - F^+ \left( z_{3,0}^0(y_2,0), (\epsilon/\delta)^{1/3}, \delta, c, a \right),
\end{align*}
\]

(5.116)

from which the result follows, noting \( r_2 = \epsilon^{1/3} \).

We now obtain estimates for solutions on \( \tilde{W}_c^{s,\ell}(c, a) \) with initial conditions in \( \Sigma_{23} \) upon exit in \( \Sigma_3^{\text{out}} \). We have the following lemma regarding \( \tilde{W}_c^{s,\ell}(c, a) \) (an analogous result holds for \( \tilde{W}_c^{s,T}(c, a) \)).

**Lemma 5.10.** Consider solutions on the manifold \( \tilde{W}_c^{s,\ell}(c, a) \), with initial conditions as in Lemma 5.9 parameterized by \( |y_{2,0}| \leq 1/\delta^{2/3} \). Such solutions exit \( \Sigma_3^{\text{out}} \) in time \( T^* = T^*(y_{2,0}) \) in a curve \((\tilde{x}_3^0(T^*(y_{2,0})), \tilde{z}_3^0(T^*(y_{2,0})))\). For each sufficiently small \( \delta, \rho > 0 \), there exists \( C > 0 \) independent of \( (c, a, \epsilon) \) and \( y_{2,0}^0 > 0 \) such that the following holds. Let \( \tilde{x}_3 = \tilde{x}_3(T^*(-1/\delta^{2/3})) \), and let \( \tilde{x}_3^0 = \tilde{x}_3(T^*(y_{2,0})) \). Then

\[
\tilde{x}_3^0 \leq -C \frac{\epsilon^{1/3}}{e} \exp \left( \frac{\beta^2 \rho, \delta, \epsilon}{e} - \frac{C}{e^{1/3}} \right),
\]

(5.117)

and for \( y_{2,0} \in (-1/\delta^{2/3}, y_{2,0}^0) \), the curve \((\tilde{x}_3^0(T^*(y_{2,0})), \tilde{z}_3^0(T^*(y_{2,0})))\) can be expressed as a graph \( \tilde{z}_3 = \tilde{z}_3(\tilde{x}_3; c, a, \epsilon) \) for \( \tilde{x}_3 \in (\tilde{x}_3^0(c, a, \epsilon), \tilde{x}_3^0(c, a, \epsilon)) \) and \( (c, a, \epsilon) \).

**Proof.** Using Lemma 5.8, we have that solutions with initial conditions given by Lemma 5.9 for \( |y_{2,0}| \leq 1/\delta^{2/3} \) reach \( \Sigma_3^{\text{out}} \) at time \( T^* = T^*(y_{2,0}) \) in curves

\[
\begin{align*}
\tilde{x}_3^0(T^*(y_{2,0})) &= \tilde{x}_3^0(y_{2,0}) \exp \left( \beta^2 \rho, \delta, \epsilon \right) + \eta^2 \left( \rho, \delta, \tilde{x}_3^0(y_{2,0}), \tilde{z}_3^0(y_{2,0}), \epsilon \right) \\
\tilde{z}_3^0(T^*(y_{2,0})) &= \tilde{z}_3^0(y_{2,0}) \exp \left( \beta^2 \rho, \delta, \epsilon \right) + \eta^2 \left( \rho, \delta, \tilde{x}_3^0(y_{2,0}), \tilde{z}_3^0(y_{2,0}), \epsilon \right),
\end{align*}
\]

(5.119)

where \( \beta^2, \eta^2 \) are given by the integrals (5.103) (and analogously for \( \beta^2, \eta^2 \)). It remains to estimate the derivatives

\[
\frac{dx_3^0(T^*)}{dy_{2,0}}, \frac{dz_3^0(T^*)}{dy_{2,0}}.
\]

To obtain estimates on the derivatives of the solutions with respect to \( y_{2,0}, c, a \), we consider the variational equation of (5.94). Using the estimates (5.119), for \( K = 1/2 \) and each small \( \kappa > 0 \), there exists \( C \) such that for all sufficiently small \( \rho, \delta \), we can estimate

\[
\begin{align*}
\frac{dx_3^0(t)}{dy_{2,0}} &\leq C \left( |(\tilde{x}_3^0)'(y_{2,0})| + |(\tilde{z}_3^0)'(y_{2,0})| \right) e^{-Kt} \\
\frac{dz_3^0(t)}{dy_{2,0}} &\leq C \left( |(\tilde{x}_3^0)'(y_{2,0})| + |(\tilde{z}_3^0)'(y_{2,0})| \right) e^{-Kt} \\
\frac{dr_3^0(t)}{dy_{2,0}} &\leq C \left( |(\tilde{x}_3^0)'(y_{2,0})| + |(\tilde{z}_3^0)'(y_{2,0})| \right) e^{\kappa t} \\
\frac{d\epsilon(t)}{dy_{2,0}} &\leq C \left( |(\tilde{x}_3^0)'(y_{2,0})| + |(\tilde{z}_3^0)'(y_{2,0})| \right) e^{\kappa t},
\end{align*}
\]

(5.120)

for solutions on \( \tilde{W}_c^{s,\ell}(c, a) \) with initial conditions

\[
(\tilde{x}_3, \tilde{z}_3, r_3, \epsilon_3)(0) = (\tilde{x}_3^0(y_{2,0}), \tilde{z}_3^0(y_{2,0}), (\epsilon/\delta)^{1/3}, \delta) \in \Sigma_{23}.
\]

(5.121)
Differentiating (5.119), we have that
\[
\frac{d\tilde{x}_3^\ell(T^*)}{dy_{2,0}} = \left( (\ell^*_{x,3,0}'(y_{2,0}) + \ell^*_{x,3,0}) \left( \frac{2}{\epsilon} \int_{r_{3,0}}^{\rho} \mathcal{O}(r_{3,0}^3(|\ell^*_{x,3,0}| + |\ell^*_{z,3,0}|), r_{3,0}^2 \ell_3(|\ell^*_{x,3,0}| + |\ell^*_{z,3,0}|)) dr_{3} \right) \right)
\times \exp \left( \beta_{2}^+ (\rho, \delta, \epsilon) + \eta_{2}^+ (\rho, \delta, \ell^*_{x,3,0}, \ell^*_{0,0}, \epsilon) \right),
\]
where \( r_{3,0} = (\epsilon/\delta)^{1/3} \). It remains to prove the following estimate for the second term
\[
\left| \frac{d}{dy_{2,0}} \left( \frac{2}{\epsilon} \int_{r_{3,0}}^{\rho} \mathcal{O}(r_{3,0}^3(|\ell^*_{x,3,0}| + |\ell^*_{z,3,0}|), r_{3,0}^2 \ell_3(|\ell^*_{x,3,0}| + |\ell^*_{z,3,0}|)) dr_{3} \right) \right| \leq C \delta \left( |(\ell^*_{x,3,0})'(y_{2,0})| + |(\ell^*_{0,0})'(y_{2,0})| \right),
\]
where \( C > 0 \) is independent of \( \delta, \rho, \epsilon \). We begin with estimating a term of the form
\[
\frac{d}{dy_{2,0}} \left( \frac{2}{\epsilon} \int_{r_{3,0}}^{\rho} r_{3}^2 \ell_{x,3,0} \ell_{3,0}^3 dr_{3} \right),
\]
as the others will be similar. As the endpoints of the integral are fixed, we obtain
\[
\frac{d}{dy_{2,0}} \left( \frac{2}{\epsilon} \int_{r_{3,0}}^{\rho} r_{3}^2 \ell_{x,3,0} \ell_{3,0}^3 dr_{3} \right) = \frac{2}{\epsilon} \int_{r_{3,0}}^{\rho} r_{3}^2 \frac{d\ell_{x,3,0}}{dy_{2,0}} \ell_{3,0}^3 dr_{3} + \frac{2}{\epsilon} \int_{r_{3,0}}^{\rho} 3r_{3}^2 \frac{d\ell_{3,0}}{dy_{2,0}} \ell_{x,3,0}^3 dr_{3}.
\]
Using the estimates (5.120) and noting that \( t > \frac{1}{2\epsilon} (r_3(t)^2 - r_{3,0}^2) \) (as in the proof of Lemma 5.8), we see that we can bound the above integrals by an integral of the form
\[
\frac{2}{\epsilon} \left( |(\ell^*_{x,3,0})'(y_{2,0})| + |(\ell^*_{0,0})'(y_{2,0})| \right) \int_{r_{3,0}}^{\rho} r_{3}^2 \exp \left( -\frac{1}{4\epsilon} (r_{3}^2 - r_{3,0}^2) \right) dr_{3}
\leq C \left( |(\ell^*_{x,3,0})'(y_{2,0})| + |(\ell^*_{0,0})'(y_{2,0})| \right) (r_{3,0} + \mathcal{O}(r_{3,0}^3)),
\]
where \( C \) is independent of \( \rho, \delta, \epsilon \). Proceeding similarly, we estimate
\[
\frac{d}{dy_{2,0}} \left( \frac{2}{\epsilon} \int_{r_{3,0}}^{\rho} r_{3} \ell_{x,3,0} \ell_{3,0}^3 dr_{3} \right) \leq C \delta \left( |(\ell^*_{x,3,0})'(y_{2,0})| + |(\ell^*_{0,0})'(y_{2,0})| \right) (1 + \mathcal{O}(r_{3,0})),
\]
where \( C \) is independent of \( \rho, \delta, \epsilon \). Using the fact that \( r_{3,0} = \left( \frac{\epsilon}{\delta} \right)^{1/3} \), we obtain
\[
\frac{d\tilde{x}_3^\ell(T^*)}{dy_{2,0}} = \left( (\ell^*_{x,3,0})'(y_{2,0}) + \mathcal{O}(\delta\ell_{x,3,0}(y_{2,0}) \left( |(\ell^*_{x,3,0})'(y_{2,0})| + |(\ell^*_{0,0})'(y_{2,0})| \right)) \right)
\times \exp \left( \beta_{2}^+ (\rho, \delta, \epsilon) + \eta_{2}^+ (\rho, \delta, \ell^*_{x,3,0}(y_{2,0}), \ell^*_{0,0}(y_{2,0}), \epsilon) \right),
\]
for \( j = \ell, r \), where \( \beta_{2,0}^+, \eta_{2}^+ \) are given by Lemma 5.8.

We now can compute the slope of \( \tilde{W}_\ell^{\ell,\ell}(c, a) \) in \( \Sigma_3^{out} \)
\[
\frac{d\tilde{\ell}_3}{dy_{3}} = \frac{(\ell^*_{0,0})'(y_{2,0}) + \mathcal{O}(\delta\ell_{x,3,0}(y_{2,0}) \left( |(\ell^*_{x,3,0})'(y_{2,0})| + |(\ell^*_{0,0})'(y_{2,0})| \right))}{\left( \ell^*_{x,3,0}(y_{2,0}) + \mathcal{O}(\delta\ell_{x,3,0}(y_{2,0}) \left( |(\ell^*_{x,3,0})'(y_{2,0})| + |(\ell^*_{0,0})'(y_{2,0})| \right)) \right)} \left( e^{-\frac{\epsilon^2}{4\epsilon} (1 + \mathcal{O}(\rho, \delta))} \right)
\]
\[
= \frac{\tilde{Z}_3 - ke^{1/3}\tilde{Z}_3'} + \mathcal{O} \left( \delta\ell_{x,3,0}(y_{2,0}) \left( |\tilde{X}_3| + |\tilde{Z}_3| + \mathcal{O}(e^{1/3}) \right) \right) e^{-\frac{\epsilon^2}{4\epsilon} (1 + \mathcal{O}(\rho, \delta))}
\]
where we used that \( r_2 = e^{1/3} \). For each fixed small \( \delta, \rho > 0 \), the numerator in the above fraction is a bounded function for sufficiently small \( \epsilon > 0 \). Hence it is clear that the tangent space to \( \tilde{W}_\ell^{\ell}(c, a) \) has exponentially small
slope except near points where the denominator is also exponentially small. Hence we proceed by obtaining a lower bound for the denominator for an appropriate range of $y_{2,0}$.

From properties of Airy functions in Lemma 5.3 and the bounds in Lemma 5.9, we know that the function $\tilde{X}_3^\ell(y_{2,0})$ is smooth and is positive for $y_{2,0} \leq 0$. For $y_{2,0} > 0$, $\tilde{X}_3^\ell(y_{2,0})$ transitions to oscillatory behavior: the first zero occurs at $y_{2,0} = y_{2,0}^* > 0$ and here $(\tilde{X}_3^\ell)'(y_{2,0}^*) < 0$. Hence by the implicit function theorem we can solve for the first zero of the denominator

$$\left(\tilde{X}_3^\ell - ke^{1/3}(\tilde{X}_3^\ell)' + \mathcal{O}\left(\delta \tilde{X}_3^\ell + |\tilde{Z}_3^\ell| + \mathcal{O}(\epsilon^{1/3})\right)\right) = 0. \quad (5.130)$$

We first argue that the $\mathcal{O}$-term does not contribute to finding zeros in this expression. To see this, we note that for $\delta$ fixed sufficiently small, we can bound

$$|\tilde{X}_3^\ell| + |\tilde{Z}_3^\ell| + \mathcal{O}(\epsilon^{1/3}) \leq 4|\tilde{X}_3^\ell| \quad (5.131)$$

uniformly in $y_{2,0} \in (-1/\delta^{2/3}, -1/\delta^{2/3} + \delta)$, provided $\epsilon$ is taken sufficiently small. Hence there are no zeros of $(5.130)$ for $y_{2,0} \in (-1/\delta^{2/3}, -1/\delta^{2/3} + \delta)$ and $\epsilon$ sufficiently small. For $y_{2,0} > -1/\delta^{2/3} + \delta$, we have that

$$\tilde{x}_{3,0}^\ell = \sqrt{\pi e^{-\frac{1}{3\delta} + y_{2,0}}} \frac{1}{k^{1/6} \delta^{1/6}} \tilde{X}_3^\ell(y_{2,0}), \quad \text{(5.132)}$$

where we used the fact that $r_2 = e^{1/3}$. Hence by taking $\epsilon$ sufficiently small, we can bound $\tilde{x}_{3,0}^\ell = \mathcal{O}(\epsilon^{2/3})$. Hence the first zero of $(5.130)$ occurs when

$$y_{2,0} = y_{2,0}^* + ke^{1/3} + \mathcal{O}(\epsilon^{2/3}). \quad (5.133)$$

Hence there exists $C$ such that for all

$$y_{2,0} \leq y_{2,0}^* := y_{2,0}^* + ke^{1/3} - Ce^{2/3}, \quad (5.134)$$

the slope $\frac{d\tilde{z}}{dx}(y_{2,0})$ of the manifold $\tilde{W}_x^\alpha(c, a)$ in $\Sigma_{3,0}^{\text{out}}$ is exponentially small. We now define

$$\tilde{x}_{3,0}^\eta(c, a, \epsilon) = \tilde{x}_{3,0}(y_{2,0}^*) \exp \left(\beta_{\eta}(\rho, \delta, \epsilon) + \eta_{\eta}(\rho, \delta, \tilde{x}_{3,0}(y_{2,0}^*), \tilde{x}_{3,0}(y_{2,0}^*), \epsilon)\right)$$

$$\leq -\frac{\epsilon^{1/3}}{C} \exp \left(\beta_{\eta}(\rho, \delta, \epsilon) - \frac{C}{\epsilon^{1/3}}\right) \quad (5.135)$$

for some $C > 0$ independent of $(c, a, \epsilon)$. The result follows. \hfill \Box

### 5.5.2 Solutions with initial conditions in $\Sigma_{13}$

The above takes care of the solutions on $\tilde{W}_x^\alpha(c, a)$ entering via $\Sigma_{23}$, but we still have those which enter via $\Sigma_{13}$ directly from chart $K_1$. We have the following.

**Lemma 5.11.** For each sufficiently small $\rho, \delta > 0$, there exists $C, \eta, \epsilon_0 > 0$ and sufficiently small choice of the intervals $I_{\alpha}, I_{c}$ such that for each $(c, a, \epsilon) \in I_{\alpha} \times I_{c} \times (0, \epsilon_0)$, there exists $x_{3,0}^\alpha(c, a, \epsilon) > 0$ such that the following holds. Consider solutions on the manifold $\tilde{W}_x^\alpha(c, a)$, with initial conditions in $\Sigma_{13}$. All such solutions exit $\Sigma_{3,0}^{\text{out}}$ in a curve which can be represented as a graph $\tilde{z}_3 = \tilde{z}_3(\tilde{x}_3; c, a, \epsilon)$ for $\tilde{x}_3 \in (\tilde{x}_{3,0}^\ell(c, a, \epsilon), \tilde{x}_{3}^\Delta(c, a, \epsilon))$ which satisfies

$$|\tilde{z}_3(\tilde{x}_3; c, a, \epsilon)| \leq Ce^{-\eta/\epsilon},$$

$$\left|\frac{d\tilde{z}_3}{d\tilde{x}_3}(\tilde{x}_3; c, a, \epsilon)\right| \leq Ce^{-\eta/\epsilon}, \quad (5.136)$$

uniformly in $\tilde{x}_3 \in (\tilde{x}_{3,0}^\ell(c, a, \epsilon), \tilde{x}_{3}^\Delta(c, a, \epsilon))$ and $(c, a, \epsilon) \in I_{\alpha} \times I_{c} \times (0, \epsilon_0)$.
Lemma 5.12. For each sufficiently small $\rho, \delta$, we now fix \( r_3, 3 \) sufficiently small and combine the results of Lemma 5.9 and Lemma 5.11 into the following.

\[
\hat{x}_3, 3 (r_3, 3) = 1 - F^+ (z_3, 3 (r_3, 3), r_3, 3, \epsilon_3, 3 (r_3, 3), c, a)
\]

\[
\hat{z}_3, 3 (r_3, 3) = z_3, 3 (r_3, 3) - F^- (1, r_3, 3, \epsilon_3, 3 (r_3, 3), c, a)
\]

where

\[
\epsilon_3, 3 (r_3, 3) = \frac{\epsilon}{r_3, 3}.
\]

Similarly to the proof of Lemma 5.8 we track these solutions through to \( \Sigma_{3, 3}^{\text{out}} \), where they intersect in a curve

\[
\hat{x}_3 (r_3, 3) = \hat{x}_3, 3 (r_3, 3), \exp \left( \beta_+ (r, 3, 3, \epsilon) \right)
\]

\[
\hat{z}_3 (r_3, 3) = \hat{z}_3, 3 (r_3, 3), \exp \left( \beta_- (r, 3, 3, \epsilon) \right),
\]

where

\[
\beta_+ (r, 3, 3, \epsilon) = \frac{2}{\epsilon} \int_{r, 3}^{\rho} (-r_3^3 + O(r_3^2 \epsilon_3)) dr_3
\]

\[
\beta_- (r, 3, 3, \epsilon) = \frac{2}{\epsilon} \int_{r, 3}^{\rho} (-r_3^3 + O(r_3^2 \epsilon_3)) dr_3
\]

Using similar arguments as in the proof of Lemma 5.10, we estimate

\[
\frac{d\hat{x}_3}{dr_3, 3} = \frac{d\hat{x}_3, 3}{dr_3, 3} + \frac{2}{\epsilon} \hat{x}_3, 3 (r_3, 3, O(r_3^2)) + O \left( \hat{x}_3, 3 \left( \epsilon^{-1/3} + \left| \frac{d\hat{x}_3, 3}{dr_3, 3} \right| \right) \right)
\]

\[
\times \exp \left( \beta_+ (r, 3, 3, \epsilon) \right)
\]

\[
\frac{d\hat{z}_3}{dr_3, 3} = \frac{d\hat{z}_3, 3}{dr_3, 3} + \frac{2}{\epsilon} \hat{x}_3, 3 (r_3, 3, O(r_3^2)) + O \left( \hat{x}_3, 3 \left( \epsilon^{-1/3} + \left| \frac{d\hat{z}_3, 3}{dr_3, 3} \right| \right) \right)
\]

\[
\times \exp \left( \beta_- (r, 3, 3, \epsilon) \right),
\]

and we define

\[
\hat{x}_3^c (c, a, \epsilon) := \hat{x}_3, 3 \left( \epsilon^1 \delta \right) \exp \left( \beta_+ (r, 3, 3, (\epsilon/\delta)^{1/3}, \epsilon) \right)
\]

\[
\hat{x}_3^\Delta (c, a, \epsilon) := \hat{x}_3, 3 (\Delta, 3) \exp \left( \beta_+ (r, 3, 3, \Delta, 3, \epsilon) \right),
\]

noting that the definition of \( \hat{x}_3^c (c, a, \epsilon) \) coincides with that in Lemma 5.10. We therefore can estimate

\[
\frac{d\hat{x}_3}{dr_3} (r_3, 3) = \frac{\hat{x}_3}{\hat{x}_3, 3} + O \left( \epsilon^{1/3} \log \epsilon \right) \exp \left( -\frac{4}{3\epsilon} (\rho^3 - r_3^3) + O \left( \rho^4, \rho^3 \delta, r_3^4 \right) \right),
\]

from which we obtain the required estimates, recalling the choice of \( 0 < \Delta, 3 \ll \rho \).

\[\square\]

5.5.3 Estimates for \( \hat{W}_3(c, a) \) in \( \Sigma_{3, 3}^{\text{out}} \)

We now fix \( \rho, \delta \) sufficiently small and combine the results of Lemma 5.9 and Lemma 5.11 into the following.

Lemma 5.12. For each sufficiently small \( \Delta_\eta > 0 \), there exists \( C, \eta, \epsilon_0 > 0 \) and sufficiently small choice of the intervals \( I_\alpha, I_\epsilon \) such that for each \( (c, a, \epsilon) \in I_\epsilon \times I_\alpha \times (0, \epsilon_0) \), there exists \( y_\epsilon^c (c, a) > \epsilon^{2/3}/C \) such that the following
holds. Define $\hat{W}^{s,t,*}_\epsilon(c,a)$ to be the backwards evolution of initial conditions $\{ (\rho^t, y, z, \epsilon) : z = z^{s,t}_\epsilon(y; c, a), -\Delta y \leq y \leq y^*_{\epsilon}(c, a) \}$ on $\hat{W}^{s,t}_\epsilon(c, a)$ in $\Sigma^{out}_A$. The intersection of $\hat{W}^{s,t,*}_\epsilon(c, a)$ with $\Sigma^{out}_A = \Sigma^{out}_3$ is given by a curve $\hat{z}_3 = \hat{z}_3(\hat{x}_3; c, a, \epsilon)$ in the $K_3$ coordinates for $\hat{x}_3 \in (\hat{x}^{\beta}_3(c, a, \epsilon), \hat{x}^{\Delta}_3(c, a, \epsilon))$ which satisfies
\[
|\hat{z}_3(\hat{x}_3; c, a, \epsilon)| \leq C e^{-\eta/\epsilon},
\]
\[
\frac{d\hat{z}_3}{d\hat{x}_3}(\hat{x}_3; c, a, \epsilon) \leq C e^{-\eta/\epsilon},
\] (5.143)
uniformly in $\hat{x}_3 \in (\hat{x}^{\beta}_3(c, a, \epsilon), \hat{x}^{\Delta}_3(c, a, \epsilon))$ and $(c, a, \epsilon)$. Here $\hat{x}^{\beta}_3(c, a, \epsilon), \hat{x}^{\Delta}_3(c, a, \epsilon)$ are defined as in Lemma 5.9 and Lemma 5.11 and
\[
\hat{x}^{\beta}_3(c, a, \epsilon) \leq -\frac{\epsilon^{1/3}}{C} \exp\left(\beta^2_3(\rho, \delta, \epsilon) - \frac{C}{\epsilon^{1/3}}\right).
\] (5.144)

5.6 Proofs of transversality results
To measure the transversality properties of $\hat{W}^{s,t,*}_\epsilon(c, a)$ with respect to the strong unstable fibers of $W^{s,t}_\epsilon(c, a)$ in the section $\Sigma^m$, we use the transversality results for the backwards evolution of fibers on $\hat{W}^{s,t}_\epsilon(c, a)$ with height $y > -\Delta y$ obtained above in the section $\Sigma^{out}_A$ combined with the fact that the backwards/forwards evolution of $W^{s,t}_\epsilon(c, a)$ between the sections $\Sigma^m, \Sigma^{h,t}$ coincide for $y < -\Delta_w$.

Proof of Proposition 3.6. We note that the manifold $\hat{W}^{s,t,*}_\epsilon(c, a)$ is defined in terms of the $(u, v, w)$-coordinates in $\Sigma^{h,t}$ in Proposition 3.6, but defined in terms of the $(x, y, z)$-coordinates in $\Sigma^{in}_A$ in Lemma 5.12. In the analysis below, it is more natural to determine transversality properties in the section $\Sigma^{out}_3$, and hence also more natural to represent this manifold in the $(x, y, z)$-coordinates near the Airy point. To obtain the corresponding definition in $\Sigma^{h,t}$, we evolve $\hat{W}^{s,t,*}_\epsilon(c, a)$ forwards from $\Sigma^{in}_A$ to $\Sigma^{h,t}$ to obtain the $w$-coordinate endpoints $w^\Delta_\epsilon$ and $w^\beta_\epsilon$ corresponding to the solutions on $\hat{W}^{s,t,*}_\epsilon(c, a)$ with initial conditions in $\Sigma^{in}_A$ at $y = -\Delta y$ and $y = y^*_{\epsilon}(c, a)$, respectively.

Using Lemma 5.12, we are able to track trajectories on $\hat{W}^{s,t}_\epsilon(c, a)$ with initial conditions in $\Sigma^{in}_A$ with $y \in (-\Delta y, y^*_{\epsilon})$ down to the section $\Sigma^{out}_3$. In the chart $K_3$, the intersection of this manifold with $\Sigma^{out}_3$ is given by a curve $\hat{z}_3 = \hat{z}_3(\hat{x}_3; c, a, \epsilon)$ for $\hat{x}_3 \in (\hat{x}^{\beta}_3(c, a, \epsilon), \hat{x}^{\Delta}_3(c, a, \epsilon))$ which satisfies
\[
|\hat{z}_3(\hat{x}_3; c, a, \epsilon)| \leq C e^{-\eta/\epsilon},
\]
\[
\frac{d\hat{z}_3}{d\hat{x}_3}(\hat{x}_3; c, a, \epsilon) \leq C e^{-\eta/\epsilon},
\] (5.145)
uniformly in $\hat{x}_3 \in (\hat{x}^{\beta}_3(c, a, \epsilon), \hat{x}^{\Delta}_3(c, a, \epsilon))$ and $(c, a, \epsilon)$.

We now investigate the intersection of $W^{s,t}_\epsilon(c, a)$ (integrated forwards from $\Sigma^m$ up to the section $\Sigma^{out}_3$). This manifold will intersect $\Sigma^{out}_3$ in a curve $\hat{z}_3 = \hat{z}_3(\hat{x}_3; c, a, \epsilon)$ which satisfies
\[
\hat{z}_3(\hat{x}_3; c, a, \epsilon) = \hat{z}_3(0; c, a, \epsilon) + \frac{d\hat{z}_3}{d\hat{x}_3}(0; c, a, \epsilon)\hat{x}_3 + o(\hat{x}_3)
\] (5.146)
where
\[
\hat{z}_3(0; c, a, \epsilon) = O(e^{-q/\epsilon}), \quad \frac{d\hat{z}_3}{d\hat{x}_3}(0; c, a, \epsilon) = O(e^{-q/\epsilon}),
\]
uniformly in $(c, a, \epsilon)$. This follows from the fact that $W^{s,t}_\epsilon(c, a)$ contains a (non-unique) choice of the slow unstable manifold $\mathcal{M}^s(c, a)$ which will be exponentially close to the point $(\hat{x}, \hat{z}) = (0, 0)$. Furthermore, at this point, $W^{s,t}_\epsilon(c, a)$ will be (up to exponentially small errors) tangent to the weak unstable subspace $\hat{z} = 0$, and the strong unstable fiber at this point will be (up to exponentially small errors) tangent to the strong unstable subspace $\hat{x} = 0$. Therefore, the strong unstable fiber of a basepoint $(\hat{x}_3, \hat{z}_3(\hat{x}_3))$ on $W^{s,t}_\epsilon(c, a)$ is given by a graph
\[
\dot{x}_3^b(\tilde{x}_3; x_3, c, a, \epsilon) = \dot{x}_3 + \frac{d\hat{x}_3}{d\hat{z}_3}(\tilde{x}_3^b(\hat{x}_3; x_3, c, a, \epsilon)(\hat{z}_3 - \hat{z}_3^b) + O\left((\hat{z}_3 - \hat{z}_3^b)^2\right)
\]
where
\[
\frac{d\hat{x}_3}{d\hat{z}_3}(\tilde{x}_3^b; x_3, c, a, \epsilon) = O(\hat{x}_3, \hat{z}_3^b(\hat{x}_3), e^{-q/\epsilon}).
\]

Finally, since the forward/backward evolution of \(\mathcal{W}_{\epsilon}^{\rho, \ell}(c, a)\) coincide for \(y < -\Delta_y\), we have that \(\hat{z}_3^b(\hat{x}_3^b; c, a, \epsilon) = \hat{x}_3(\hat{x}_3^b; c, a, \epsilon)\) and trajectories on \(\mathcal{W}_{\epsilon}^{\rho, \ell}(c, a)\) evolved backwards from \(\Sigma_{out}^b\) for \(y < -\Delta_y\) in fact again land in \(\mathcal{W}_{\epsilon}^{\rho, \ell}(c, a)\). Since \(\mathcal{W}_{\epsilon}^{\rho, \ell}(c, a)\) is certainly transverse to its own strong unstable fibers, we are only concerned for values \(\hat{x}_3 \in (\hat{x}_3^b(c, a, \epsilon), \hat{x}_3^b(c, a, \epsilon))\) as here the backwards/forwards evolution of \(\mathcal{W}_{\epsilon}^{\rho, \ell}(c, a)\) separates in the section \(\Sigma_{out}^b\) into curves given by the two functions described above.

From the estimates above, we can deduce that the backwards evolution \(\hat{\mathcal{W}}_{\epsilon}^{\rho, \ell, *}(c, a)\) given by the curve \(\hat{z}_3 = \hat{z}_3(\hat{x}_3; c, a, \epsilon)\) in the section \(\Sigma_{out}^b\) remains transverse to the strong unstable fibers of \(\mathcal{W}_{\epsilon}^{\rho, \ell}(c, a)\) at least up to the fiber which passes through the point \((\hat{x}_3^b, \hat{z}_3(\hat{x}_3^b))\).

We now evolve the manifold \(\hat{\mathcal{W}}_{\epsilon}^{\rho, \ell, *}(c, a)\) backwards from \(\Sigma_{out}^b\) to \(\Sigma^m\). Using the exchange lemma, we deduce that the above transversality also holds in the section \(\Sigma^m\) and all trajectories are exponentially contracted to \(\mathcal{M}_{\epsilon}^m(c, a)\).

We denote by \(\hat{y}_{2,0}^b(c, a)\) the \(y_2\) coordinate in \(\Sigma^m\) of the backwards evolution of the solution \(\hat{\mathcal{W}}_{\epsilon}^{\rho, \ell, *}(c, a)\) passing through \((\hat{x}_3^b(c, a, \epsilon), \hat{z}_3(\hat{x}_3^b(c, a, \epsilon)))\) in \(\Sigma_{out}^b\), and we denote by \(\hat{y}_{2,0}^b(c, a)\) the \(y_2\) coordinate in \(\Sigma^m\) of the backwards evolution of the basepoint on \(\mathcal{W}_{\epsilon}^{\rho, \ell}(c, a)\) corresponding to the fiber containing the solution on \(\hat{\mathcal{W}}_{\epsilon}^{\rho, \ell, *}(c, a)\) passing through \((\hat{x}_3^b, \hat{z}_3(\hat{x}_3^b))\) in \(\Sigma_{out}^b\). With these definitions, we see that the assertions of the proposition hold in the section \(\Sigma^m\).

\textbf{Proof of Lemma 3.8}. The estimates (3.32) are shown in §3.4.1. Hence it remains to show that the transversality of \(\hat{\mathcal{W}}_{\epsilon}^{\rho, \ell}(c, a)\) with respect to the fibers of \(\mathcal{W}_{\epsilon}^{\rho, \ell, *}(c, a)\) in \(\Sigma^m\) includes the fibers through all points on the backwards evolution of \(\mathcal{B}(s; c, a)\). By Proposition 3.6, this amounts to proving (3.33). As in the proof of Proposition 3.6, we work in a neighborhood of the Airy point and determine transversality conditions in the section \(\Sigma_{out}^b\) and use this information to deduce what happens in \(\Sigma^m\).

Here we consider pulses of Type 2, 3 so \(s \in (w_A + \Delta_w, w^+ + \Delta_w)\). Evolving \(\mathcal{B}(s; c, a)\) backwards from \(\Sigma^{b, \ell}\), these solutions are already exponentially contracted to \(\mathcal{M}_{\epsilon}^m(c, a)\) above the Airy point, and we see that they eventually enter the chart \(K_3\) via the section \(\Sigma_3^b\) where their \((\hat{x}, \hat{z})\)-coordinates are already \(O(e^{-q/\epsilon})\) uniformly in \((c, a)\).

Suppose we take any such solution which enters \(\Sigma_3^b\) at a point \((\hat{x}_3^b, \hat{z}_3^b, 0, 0, 0, 0, 0, 0, 0, 0) = O(e^{-q/\epsilon})\) where we drop the dependence on \((c, a)\). This solution reaches \(\Sigma_{out}^b\) at \((\hat{x}_3, \hat{z}_3) = (\hat{x}_3^b, \hat{z}_3^b, 0, 0, 0, 0, 0, 0, 0, 0)\) where
\[
\begin{align*}
\hat{x}_3^b &= \hat{x}_3^b(\rho, \delta, \epsilon), \\
\hat{z}_3^b &= \hat{z}_3^b(\rho, \delta, \epsilon) = \hat{z}_3(\rho, \delta, \epsilon) + O(e^{-q/\epsilon}).
\end{align*}
\]

We then need to show that \(\hat{\mathcal{W}}_{\epsilon}^{\rho, \ell, *}(c, a)\) is transverse to the fiber in \(\Sigma_{out}^b\) passing through the point \((\hat{x}_3^b, \hat{z}_3^b)\). One way to do this is to find the intersection of this fiber with \(\hat{\mathcal{W}}_{\epsilon}^{\rho, \ell, *}(c, a)\) and show that it occurs for some \(\hat{x}_3 > \hat{x}_3^b\), where we know this transversality holds.

The fiber through \((\hat{x}_3, \hat{z}_3) = (\hat{x}_3^b, \hat{z}_3^b)\) is given by the set of \((\hat{x}_3, \hat{z}_3)\) satisfying
\[
\hat{x}_3 = \hat{x}_3^b + O\left(\left(\left|\hat{x}_3^b\right| + \left|\hat{z}_3^b\right| + |e^{-q/\epsilon}|\right)\left((\hat{z}_3 - \hat{z}_3^b)_1, (\hat{z}_3 - \hat{z}_3^b)_2\right)^2\right).
\]

We can solve for when this intersects \(\hat{\mathcal{W}}_{\epsilon}^{\rho, \ell, *}(c, a)\) by substituting the expressions \((\hat{x}_3, \hat{z}_3) = (\hat{x}_3, \hat{z}_3(\hat{x}_3; c, a, \epsilon))\) to obtain
\[
\hat{x}_3 = \hat{x}_3^b + O\left(\left(\left|\hat{x}_3^b\right| + \left|\hat{z}_3^b\right| + |e^{-q/\epsilon}|\right)\left((\hat{z}_3(\hat{x}_3) - \hat{z}_3^b)_1, (\hat{z}_3(\hat{x}_3) - \hat{z}_3^b)_2\right)^2\right),
\]
which we can solve by the implicit function theorem to find an intersection at
\[ \tilde{x}_3^* = \mathcal{O} \left( \exp \left( -\frac{4}{c} + \beta_2^2 (\rho, \delta, \epsilon) \right) \right), \] (5.153)
which indeed satisfies \( \tilde{x}_3^* > \tilde{x}_3^0 \). As the chosen solution on \( \mathcal{B}(s; c, a) \) was arbitrary, this shows that \( \hat{W}_{e}^{s,t,*}(c,a) \) is transverse to the fibers passing through each solution on \( \mathcal{B}(s; c, a) \) in the section \( \Sigma_3^{out} \) for all \((c,a) \in I_c \times I_a\).

As in the proof of Proposition 3.6, we track these solutions in backwards time from \( \Sigma_3^{out} \) to \( \Sigma^m \) to deduce that the transversality holds there also. We recall that \( y_{2,0}^*(c,a) \) denotes the \( y_2 \) coordinate in \( \Sigma^m \) of the backwards evolution of the basepoint on \( \mathcal{W}^{s,t}_r(c,a) \) in \( \Sigma_3^{out} \) corresponding to the fiber containing the solution on \( \hat{W}_{e}^{s,t,*}(c,a) \) passing through \((\tilde{x}_3^0, \tilde{z}_3(\tilde{x}_3^0))\). Hence following the solutions on \( \mathcal{B}(s;c,a) \) from \( \Sigma_3^{out} \) to \( \Sigma^m \) in backwards time gives the result (3.33).

\[ \square \]

**Proof of Lemma 3.10.** For the case of type 4 pulses, the argument proceeds as in the proof of Lemma 3.8. To treat the case of Type 5 pulses, more care is needed. Using Proposition A.1 and the fact that the forward/backward evolution of \( \mathcal{W}^{s,t}_r(c,a) \) coincide for \( w \leq w_A - \Delta_w \), the transversality result (3.45) hold easily for type 5 pulses with \( s \in (2w^t - w_A + \Delta_w, 2w^t - \Delta_w) \), that is, with secondary right pulses of height \( w \in (\Delta_w, w_A - \Delta_w) \). For type 5 pulses with \( s \in (2w^t - w_A - \Delta_w, 2w^t - w_A + \Delta_w) \), that is, with secondary height \( w \in (w_A - \Delta_w, w_A + \Delta_w) \) the backwards evolution of \( \mathcal{B}(s;c,a) \) interacts with the Airy point before reaching the section \( \Sigma^m \), and hence the result (3.45) is not clear.

For type 5 pulses with secondary height \( w \in (w_A - \Delta_w, w_A + \Delta_w) \), the manifolds \( \mathcal{B}(s;c,a) \) in fact approach the Airy point exponentially close to \( \hat{W}_{e}^{s,t,r}(c,a) \) in backwards time. Hence these trajectories reach \( \Sigma_3^{out} \) after passing through different charts, as with \( \hat{W}_{e}^{s,t,*}(c,a) \). We need to ensure that \( \hat{W}_{e}^{s,t,*}(c,a) \) is transverse to the fibers in \( \Sigma_3^{out} \) passing through each point on the intersection of \( \hat{W}_{e}^{s,t}(c,a) \) with \( \Sigma_3^{out} \). Similar to the above analysis for tracking \( \hat{W}_{e}^{s,t,*}(c,a) \), the manifold \( \hat{W}_{e}^{s,t,r}(c,a) \) intersects \( \Sigma_3^{out} \) curve defined in terms of Airy functions which winds around the origin in an exponentially decaying manner.

We focus on the part of \( \hat{W}_{e}^{s,t,r}(c,a) \) which reaches \( \Sigma_3^{out} \) after passing through the charts \( \mathcal{K}_1 \rightarrow \mathcal{K}_2 \rightarrow \mathcal{K}_3 \) (see §5.5.1) as solutions entering \( \mathcal{K}_3 \) via different charts do not cause issues. From Lemma 5.9, we have that \( \hat{W}_{e}^{s,t}(c,a) \) intersects \( \Sigma_3^{out} \) in a curve parameterized by \( y_{2,0} \) as
\[ \tilde{x}_3^{*,1}(y_{2,0}) = \tilde{x}_3^{*,0}(y_{2,0}) \exp \left( \beta_2^2 (\rho, \delta, \epsilon) + \eta_2^2 (\rho, \delta, \tilde{x}_3^{*,0}, \tilde{z}_3^{*,0}, \epsilon) \right), \] (5.154)
where
\[ \tilde{x}_3^{*,0}(y_{2,0}) = \sqrt{\pi} e^{-\frac{y_{2,0}^2 + 2}{k + \frac{3}{2 \pi}}} \tilde{X}_3^{*,0}(y_{2,0}), \quad \tilde{z}_3^{*,0}(y_{2,0}) = \sqrt{\pi} e^{-\frac{y_{2,0}^2 + 2}{k + \frac{3}{2 \pi}}} \tilde{Z}_3^{*,0}(y_{2,0}), \] (5.155)
and
\[ \tilde{X}_3^{*,0}(y_{2,0}) = \left( \text{Ai} \left( -\frac{y_{2,0}}{k^{1/3}} \right) + k^{1/3} \frac{1}{\delta^{1/3}} \right) e^{\frac{2}{\pi} \frac{y_{2,0}}{k^{1/3}}} (1 + O(\delta)) \]
\[ + O(\delta) \left( \text{Bi} \left( -\frac{y_{2,0}}{k^{1/3}} \right) + k^{1/3} \frac{1}{\delta^{1/3}} \right) e^{-\frac{2}{\pi} \frac{y_{2,0}}{k^{1/3}}} + O(\epsilon^{2/3}) \]
\[ \tilde{Z}_3^{*,0}(y_{2,0}) = \left( \text{Bi} \left( -\frac{y_{2,0}}{k^{1/3}} \right) + k^{1/3} \frac{1}{\delta^{1/3}} \right) e^{-\frac{2}{\pi} \frac{y_{2,0}}{k^{1/3}}} (1 + O(\delta)) \]
\[ + O(\delta) \left( \text{Ai} \left( -\frac{y_{2,0}}{k^{1/3}} \right) + k^{1/3} \frac{1}{\delta^{1/3}} \right) e^{\frac{2}{\pi} \frac{y_{2,0}}{k^{1/3}}} + O(\epsilon^{2/3}). \] (5.156)

The fiber through \((\tilde{x}_3, \tilde{z}_3) = (\tilde{x}_3^{*,1}(y_{2,0}), \tilde{z}_3^{*,1}(y_{2,0}))\) is given by \((\tilde{x}_3, \tilde{z}_3)\) satisfying
\[ \tilde{x}_3 = \tilde{x}_3^{*,1}(y_{2,0}) + O \left( \left( |\tilde{x}_3^{*,1}(y_{2,0})| + |\tilde{z}_3^{*,1}(y_{2,0})| + |e^{-y_{2,0}^2/2}| \right) (\tilde{z}_3 - \tilde{z}_3^{*,1}(y_{2,0})), (\tilde{z}_3 - \tilde{z}_3^{*,1}(y_{2,0}))^2 \right). \] (5.157)
We can solve for when this intersects \( \hat{W}_{\epsilon}^{\alpha, \ell, \varepsilon}(c, a) \) by plugging in \((\hat{x}_3, \hat{z}_3) = (\hat{x}_3, \hat{z}_3(\hat{x}_3))\) to obtain
\[
\hat{x}_3 = \hat{x}_{3,1}^\alpha(y_{2,0}) + \mathcal{O}\left( \left( |\hat{x}_{3,1}^\alpha(y_{2,0})| + |\hat{z}_{3,1}^\alpha(y_{2,0})| + |e^{-q'/\ell^2}| \right) \left( \hat{z}_3(\hat{x}_3) - \hat{z}_{3,1}^\alpha(y_{2,0}) \right) \right),
\]
which we can solve by the implicit function theorem to find an intersection at
\[
\hat{x}_3^\alpha(y_{2,0}) = \hat{x}_{3,1}^\alpha(y_{2,0}) + \mathcal{O}\left( e^{2\delta^2(\rho, \delta, \varepsilon)} \right),
\]
\(\text{provided } \hat{x}_3^\alpha > \hat{x}_3^\beta \) (i.e. we need to be careful not to leave the domain on which \( \hat{z}_3(\hat{x}_3; c, a, \varepsilon) \) is both well-defined and transverse to the fibers of \( W_{\epsilon}^{\alpha, \ell}(c, a) \)). To determine this, we note that the minimum possible \( \hat{x}_3^\alpha(y_{2,0}) \)-value achieved is at a value of \( y_{2,0} \) which is exponentially close to that which gives the minimum value of \( \hat{x}_{3,1}^\alpha(y_{2,0}) \). We hence proceed as above by computing the first ‘turning point’ on this curve, that is, the minimum (or largest negative) \( \hat{x}_3 \)-value achieved by \( \hat{x}_{3,1}^\alpha(y_{2,0}) \).

Similar to the proof of Lemma 5.10, we search for the first zero of \((\hat{x}_{3,1}^\alpha)'(y_{2,0})\), which amounts to solving for the first zero of
\[
\left( \hat{X}_3^\alpha(y_{2,0}) - k e^{1/3}(\hat{X}_3^\alpha)'(y_{2,0}) + \mathcal{O}\left( \delta \hat{x}_{3,0}^\alpha(y_{2,0}) \left( |\hat{X}_3^\alpha'| + |\hat{Z}_3^\alpha| + \mathcal{O}(e^{1/3}) \right) \right) \right) = 0,
\]
which occurs when
\[
y_{2,0} = y_{2,0}^\alpha + k e^{1/3} + \mathcal{O}(e^{2/3}),
\]
where \( y_{2,0}^\alpha \) is the first zero of \( \hat{X}_3^\alpha(y_{2,0}) \). Hence the minimum of \( \hat{x}_3^\alpha(y_{2,0}) \) occurs at some
\[
y_{2,0}^\alpha = y_{2,0}^\alpha + k e^{1/3} + \mathcal{O}(e^{2/3}).
\]

We now note that for \( y_{2,0} \) near \( y_{2,0}^\alpha \), for all sufficiently small \( \varepsilon \), we have that
\[
\hat{X}_3^\alpha(y_{2,0}) = \hat{X}_3^\alpha(y_{2,0}) + \left( k^{1/3} \delta^4/3 \right) \left( -\frac{y_{2,0}}{k^{2/3}} \right) e^{\frac{x}{4} \frac{2}{\pi}} \left( 2 + \mathcal{O}(\delta) \right)
\]
\[+ \mathcal{O}\left( \delta e^{-\frac{x}{4} \frac{2}{\pi} k^{1/3} \delta^4/3} \right) \left( z_{2,0}^\alpha(y_{2,0}) - z_{2,0}^\alpha(y_{2,0}) \right) + \mathcal{O}(e^{2/3}),
\]
and hence
\[
0 = \hat{X}_3^\alpha(y_{2,0}) = (\hat{X}_3^\alpha)'(y_{2,0}) (y_{2,0} - y_{2,0}^\alpha) + \mathcal{O}\left( (y_{2,0} - y_{2,0}^\alpha)^2, e^{2/3} \right)
\]
\[+ \left( k^{1/3} \delta^4/3 \right) \left( -\frac{y_{2,0}}{k^{2/3}} \right) e^{\frac{x}{4} \frac{2}{\pi}} \left( 2 + \mathcal{O}(\delta) \right)
\]
\[+ \mathcal{O}\left( \delta e^{-\frac{x}{4} \frac{2}{\pi} k^{1/3} \delta^4/3} \right) \left( z_{2,0}^\alpha(y_{2,0}) - z_{2,0}^\alpha(y_{2,0}) \right),
\]
from which we deduce that
\[
y_{2,0} - y_{2,0}^\alpha = \mu \left( z_{2,0}^\alpha(y_{2,0}) - z_{2,0}^\alpha(y_{2,0}) \right) + \mathcal{O}\left( (z_{2,0}^\alpha(y_{2,0}) - z_{2,0}^\alpha(y_{2,0}))^2, e^{2/3} \right),
\]
for some constant \( \mu > 0 \) bounded away from zero uniformly in \( \varepsilon \). Hence we have
\[
y_{2,0}^\alpha - y_{2,0}^\alpha = \mu \left( z_{2,0}^\alpha(y_{2,0}) - z_{2,0}^\alpha(y_{2,0}) \right) + \mathcal{O}\left( (z_{2,0}^\alpha(y_{2,0}) - z_{2,0}^\alpha(y_{2,0}))^2, e^{2/3} \right).
\]

Finally, using (5.130), (5.160), (5.166), Lemma 5.9, and the definitions of \( y_{2,0}^\alpha, y_{2,0}^\beta \), we have that
\[
\hat{X}_3^\alpha(y_{2,0}) - \hat{X}_3^\beta(y_{2,0}) = k e^{1/3} \left( (\hat{X}_3^\alpha)'(y_{2,0}^\alpha) - (\hat{X}_3^\alpha)'(y_{2,0}) \right) + \mathcal{O}(e^{2/3})
\]
\[= \mathcal{O}\left( e^{1/3} (y_{2,0}^\alpha - y_{2,0}^\beta), e^{2/3} \right).
\]
We now estimate
\[ \hat{x}_3^* - \hat{x}_3 = \hat{x}_{3,1}^*(y_{2,0}^*) - \hat{x}_3^* + O\left(e^{\beta_3^2(\rho,\delta,\epsilon) - q/\epsilon}\right) \]
\[ = \hat{x}_{3,0}^*(y_{2,0}^*) \exp\left(\beta_3^2(\rho,\delta,\epsilon) + \eta_3^2(\rho,\delta,\hat{x}_{3,0}^*(y_{2,0}^*),\hat{z}_{3,0}^*(y_{2,0}^*,\epsilon))\right) \]
\[ - \hat{x}_{3,0}^*(y_{2,0}^*) \exp\left(\beta_3^2(\rho,\delta,\epsilon) + \eta_3^2(\rho,\delta,\hat{x}_{3,0}^*(y_{2,0}^*),\hat{z}_{3,0}^*(y_{2,0}^*,\epsilon))\right) + O\left(e^{\beta_3^2(\rho,\delta,\epsilon) - q/\epsilon}\right) \]
\[ = \left(\hat{x}_{3,0}^*(y_{2,0}^*)\right)(1 + O(\epsilon^2/3)) - e^{-\frac{y_{2,0}^* - y_{2,0}^*}{4\kappa^{1/3}}} \hat{X}_3^*(y_{2,0}^*) \left(1 + O(\epsilon^2/3)\right) + O\left(e^{-q/\epsilon}\right) \]
\[ = \left(\hat{X}_3^*(y_{2,0}^*)\right)(1 + O(\epsilon^2/3)) - e^{-\frac{y_{2,0}^* - y_{2,0}^*}{4\kappa^{1/3}}} \hat{X}_3^*(y_{2,0}^*) \left(1 + O(\epsilon^2/3)\right) + O\left(e^{-q/\epsilon}\right) \]
\[ = \left(\hat{X}_3^*(y_{2,0}^*)\right) \left(1 - e^{-\frac{y_{2,0}^* - y_{2,0}^*}{4\kappa^{1/3}}} \left(1 + O(\epsilon^2/3)\right)\right) + O\left(e^{2/3}(y_{2,0}^* - y_{2,0}^*,\epsilon^2/3)\right) \]
\[ > \left(\hat{X}_3^*(y_{2,0}^*)\right) \left(-\frac{\mu k(\rho)}{\delta} + O\left(\left(\frac{\kappa(\rho)}{\delta}\right)^2\right)\right) + O\left(e^{2/3} \log \epsilon\right) \]
\[ > 0, \]

for all sufficiently small \(\epsilon > 0\). From this we deduce that \(\hat{x}_3^* > \hat{x}_3^*\) as required. The remainder of the proof follows as in the proof of Lemma 3.8. 

### 6 Discussion

In this paper, we described a phenomenon, previously observed numerically in [8, 14, 15], in which the branch of fast pulse solutions to (1.2) described in [7] turns sharply when continued numerically in the parameters \((c, a)\) for fixed \(\epsilon > 0\). This sharp turn is associated with a transition in the pulse solutions as the oscillations in the tails of the pulses grows into a secondary excursion, culminating in a double pulse solution which resembles two copies of the primary pulse.

Using geometric singular perturbation theory and blow-up techniques, we constructed this transition analytically. For each sufficiently small \(\epsilon > 0\), we proved the existence of a continuous one-parameter family of homoclinic solutions to (1.2) forming a bridge between the pulses with oscillatory tails constructed in [7] and double pulses which are \(O(\sqrt{\epsilon})\)-close to two copies of the primary singular pulse. Our construction of the double pulses was shown to break down near a Belyakov transition (1.3), which corresponds to numerical observations in which the branch of double pulses terminates near this Belyakov transition (see Figure 4 and [8]).

We showed that the sharp single-to-double-pulse transition is due to a canard-explosion-type mechanism organized by the equilibrium at the origin, which in a local center manifold takes the form of a canard point in the terminology of [23]. The homoclinic solutions were then constructed in a similar manner as in the case of a canard explosion of periodic orbits as in [24], using blow-up techniques and geometric matching.
The main complications which arise are twofold. Firstly, the problem is inherently three-dimensional: the setup requires three dimensions in order to construct homoclinic orbits with oscillatory tails. This complicates the matching procedure and also necessitates a framework in which both the hyperbolic and nonhyperbolic components of the flow can be tracked and matched accordingly in local neighborhoods of the nonhyperbolic fold points. To accomplish this, we capitalized on the exchange lemma and the techniques used in [7] to construct single pulses with oscillatory tails.

Secondly, constructing the tails of the solutions proved to be a significant technical challenge. We showed that the behavior of the tails was heavily influenced by the structure of the middle branch of the critical manifold, specifically the existence of what we refer to as an Airy point, encompassing the transition from node to focus behavior. We studied the flow near this point in great detail, utilizing blow-up techniques and invariant manifold theory, and we used this analysis to show that each transitional pulse can be matched with a tail lying in the stable manifold of the equilibrium. This analysis also explained the phenomenon previously observed numerically in which many of the transitional pulses appear to have nearly identical tails (see Figure 3).

While much of the construction is tailored to this particular problem, the techniques used are quite general and can be used to construct many different homoclinic or periodic solutions, both within the FitzHugh–Nagumo system and otherwise. A direct application within the FitzHugh–Nagumo system would be to construct $N$-pulses with oscillatory tails. On each successive excursion, exponentially small variations in the parameters $(c, a)$ allow the associated matching conditions to be solved, ending with a solution in the tail manifold.

Beyond the FitzHugh–Nagumo system, this geometric matching procedure can be used to construct a wide variety of solutions in different systems. Of particular interest is the notion of a canard explosion as being a mechanism for spike-adding [13, 28, 29, 26, 25, 10]. In this sense, one can think of the canard mechanism in the FitzHugh–Nagumo system as allowing for the addition of a second pulse or spike to the primary pulse [10]. An interesting extension would be to determine if these techniques could be transferred to the case of a slow passage through such a transition, to obtain a full analytical construction of a dynamic spike-adding mechanism.

Finally, we comment on the PDE stability of the transitional pulses. While the pulses with small oscillatory tails (corresponding to those constructed in [7]) were shown to be stable in [6], our expectation is that the pulses lose stability as the oscillations in the tail grow sufficiently large, as the bifurcation curve resembles a “global” saddle-node bifurcation. A full study of the (loss of) PDE stability of these solutions is an interesting direction for future work.

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A Layer analysis of fronts

In this section we prove the following proposition regarding heteroclinic connections between the equilibria $p_i$ for values of $w \in [0, w^\dagger]$ in the system (2.7). The results of Proposition A.1 are shown in Figure 6.

**Proposition A.1.** Consider the system (2.7) for $(c, a) = (1/\sqrt{2}, 0)$. For each $w \in (0, w^\dagger)$, there exists a front $\phi_\ell(w)$ connecting the equilibria $p_2$ and $p_1$, and a front $\phi_r(w)$, connecting the equilibria $p_2$ and $p_3$. Furthermore,

(i) For $w \in (0, w_A)$, the front $\phi_\ell$ leaves $p_2$ along a weak unstable direction and remains in \{$(u, v)$ : $u_1(w) < u < u_2(w), v < 0$\}. The front $\phi_r$ leaves $p_2$ along $\phi_\ell$, then crosses into the half space $v > 0$, where it remains until arriving at $p_3$.  


(ii) When \( w = w_A \), the fronts \( \phi_\ell, \phi_r \) leave \( p_2 \) along the line \( v = \frac{u - u_2(w)}{2\sqrt{2}} \) in the half space \( v < 0 \). There exist \( A_\ell, A_r \) and \( B_\ell, B_r > 0 \) such that \( \phi_\ell, \phi_r \) satisfy

\[
\begin{align*}
  u(t) &= u_2(w) + (A_j + B_j t)e^{\frac{t}{\sqrt{2}}} + \mathcal{O}(t^2 e^{\frac{t}{\sqrt{2}}}) \\
  v(t) &= \frac{1}{2\sqrt{2}}(A_j + B_j t)e^{\frac{t}{\sqrt{2}}} + B_j e^{\frac{t}{\sqrt{2}}} + \mathcal{O}(t^2 e^{\frac{t}{\sqrt{2}}}),
\end{align*}
\]

where \( j = \ell, r \), asymptotically as \( t \to -\infty \). There exists \( \Delta > 0 \) such that these solutions can be written as graphs \( v = v_j(u), j = \ell, r \), for \( u \in [u_2(w) - \Delta, u_2(w)] \) with \( v_r(u) > v_\ell(u) \) for all \( u \in [u_2(w) - \Delta, u_2(w)] \).

(iii) When \( w = w^\dagger - w_A \), the fronts \( \phi_\ell, \phi_r \) leave \( p_2 \) along the line \( v = \frac{u - u_2(w)}{2\sqrt{2}} \) in the half space \( v > 0 \).

(iv) For \( w \in (w^\dagger - w_A, w^\dagger) \), the front \( \phi_r \) leaves \( p_2 \) along a weak unstable direction and remains in \( \{(u,v): u_2(w) < u < u_3(w), v > 0 \} \). The front \( \phi_\ell \) leaves \( p_2 \) along \( \phi_r \), then crosses into the half space \( v < 0 \), where it remains until arriving at \( p_1 \).

Proof. We prove (i) and (ii); the remaining two assertions follow from the symmetry of the cubic nonlinearity. The claims regarding the front \( \phi_\ell \) follow from analysis of traveling fronts \([2, 16, 12]\).

It remains to show the properties of the front \( \phi_r \). We first consider the case of small \( w \). When \( w = 0 \), the equilibria \( p_1 \) and \( p_2 \) collide, and \( p_1 \) and \( p_3 \) are connected by the Nagumo front \( \phi_f \). Hence for small \( w > 0 \) property (i) follows from the fact that \( \phi_f \) breaks regularly as \( w \) increases; this can be shown in a manner similar to the proof of \([7, \text{Proposition 5.2}]\). Hence the result holds for \( w \in (0, \Delta_w) \) sufficiently small.

We next examine the linearization of (2.7) at the equilibria \( p_2, p_3 \). At \( p_1 \), the linearization of (2.7) is given by

\[
J_2 = \begin{pmatrix} 0 & 1 \\
- f'(u_\ell(w)) & c \end{pmatrix},
\]

which has eigenvalues

\[
\lambda^\pm = \frac{c \pm \sqrt{c^2 - 4f''(u_\ell(w))}}{2}.
\]

For all \( w \in (0, w_A) \) and all \( c \geq 1/\sqrt{2} \), \( p_2 \) is an unstable node (which is degenerate in the critical case of \( w = w_A, c = 1/\sqrt{2} \)) with corresponding eigenvectors

\[
c_2^\pm = \begin{pmatrix} 1 \\ c \pm \sqrt{c^2 - 4f''(u_\ell(w))} \end{pmatrix}.
\]

For \( w \in (0, w_A) \), the equilibrium \( p_2 \) has a well defined strong unstable eigenspace with nonzero \((u,v)\)-components. Hence the front \( \phi_r \) leaves the equilibrium along a trajectory tangent to this subspace with \( u \) initially either increasing or decreasing. Proving (i) amounts to showing that the former is always the case.

For all \( w \in (0, w_A) \) and all \( c \geq 1/\sqrt{2} \), \( p_3 \) is a saddle with corresponding eigenvectors

\[
c_3^\pm = \begin{pmatrix} 1 \\ c \pm \sqrt{c^2 - 4f''(u_3(w))} \end{pmatrix}.
\]

Hence for each \( w \in (\Delta_w, w_A) \) and each \( c \geq 1/\sqrt{2} \), the equilibrium \( p_2 \) has a well defined strong unstable manifold \( \mathcal{W}^{uu}(p_2) \), and the equilibrium \( p_3 \) has a well defined stable manifold \( \mathcal{W}^s(p_3) \). If a front were to exist as an intersection of \( \mathcal{W}^{uu}(p_2) \) and \( \mathcal{W}^s(p_3) \) lying in the half space \( v > 0 \) for some \( c_w \geq \sqrt{2} \), then by monotonicity of
the flow with respect to $c$, this connection will break upon varying $c$: for $c < c_w$, we must have that $W^{uu}(p_2)$ lies below $W^s(p_3)$ and vice-versa for $c > c_w$, and hence this value of $c_w$ for which a connection exists is unique among $c \geq 1/\sqrt{2}$. We show that for each $w \in (0, w_A)$, such a value $c_w > 1/\sqrt{2}$ exists by explicitly constructing the associated front.

Using the ansatz $v = b(u - u_2(w))(u - u_3(w))$, we deduce that there is a front connecting $p_2$ and $p_3$ given by

$$u(t) = \frac{u_3(w) + u_2(w)}{2} + \frac{u_3(w) - u_2(w)}{2} \tanh \left( \frac{u_3(w) - u_2(w)}{2\sqrt{2}} t \right)$$

$$v(t) = \frac{(u_3(w) - u_2(w))^2}{4\sqrt{2}} \sech \left( \frac{u_3(w) - u_2(w)}{2\sqrt{2}} t \right),$$

with wave speed

$$c = \frac{1}{\sqrt{2}} (u_2(w) + u_3(w) - 2u_1(w))$$

$$= \frac{1}{\sqrt{2}} (u_1(w) + u_2(w) + u_3(w) - 3u_1(w))$$

$$= \frac{1}{\sqrt{2}} (1 - 3u_1(w))$$

$$> \frac{1}{\sqrt{2}},$$

for all $w \in (\Delta_w, w_A)$. Hence for each $w \in (\Delta_w, w_A)$, for $c = 1/\sqrt{2}$, we must have that $W^{uu}(p_2)$ lies below $W^s(p_3)$.

Finally, we can apply the same argument as above to the case of $w = w_A$. For $c = 1/\sqrt{2}$, there is a unique trajectory decaying exponentially in backwards time along the eigenvector

$$e_2^A = \begin{pmatrix} 1 \\ 1 \\ 2\sqrt{2} \end{pmatrix},$$

with exponential rate $e^{1/\sqrt{2}}$, whereas all other trajectories decay with algebro-exponential rate $te^{1/\sqrt{2}}$. We abuse notation and refer to this trajectory as $W^{uu}(p_2)$. For $c > 1/\sqrt{2}$, $p_2$ is an unstable node, and as above we can find a front solution connecting $W^{uu}(p_2)$ and $W^s(p_3)$ at with wave speed

$$c = \frac{1}{\sqrt{2}} (1 - 3u_1(w_A))$$

$$> \frac{1}{\sqrt{2}},$$

and hence, by the above monotonicity argument, we deduce that $W^{uu}(p_2)$ lies below $W^s(p_3)$ for $c = 1/\sqrt{2}$, $w = w_A$, which completes the proof of (ii).

\[\square\]

**B Contraction/expansion rates along $M^s_\epsilon(c, a), M^u_\epsilon(c, a)$**

To construct pulses of type 4, 5, we need more explicit bounds on the rates of contraction and expansion along solutions near the slow manifolds $M^s_\epsilon(c, a), M^u_\epsilon(c, a)$. We consider the flow in neighborhoods of each of these slow manifolds in which they are normally hyperbolic, and we make coordinate transformations to put the equations in a preliminary Fenichel normal form which identifies the stable/unstable subspaces and corresponding contraction/expansion rates.

The ultimate goal is to show that the manifold $B(s; c, a)$ is well defined and exponentially close to $\gamma^s(p; c, a)$ in $\Sigma^{h,r}$ for each $(c, a) \in I_c \times I_a$. The existence of the solution $\gamma^s(p; c, a)$ for $(c, a) \in I_c \times I_a$ is clear; however,
it is not immediately obvious that the fiber of this solution in the section $\Sigma^m$ is exponentially contracted to $\gamma^s p(s; c, a)$ in backwards time to $\Sigma^{h, r}$. Along the manifold $W^s, c(a)$, this is clear as this fiber is defined by the fact that it contracts exponentially to $\gamma^s p(s; c, a)$ in backwards time. However, after passing near the fold, in backwards time, $\gamma^s p(s; c, a)$ is near the slow manifold $M^s(c, a)$ and solutions near $\gamma^s p(s; c, a)$ undergo exponential expansion. We claim that the contraction along $W^s, c(a)$ compensates for this expansion.

We proceed by determining the balance of contraction/expansion along the slow manifolds $M^s, c(a)$ in backwards time from $\Sigma^m$ to $\Sigma^{h, r}$. We break this into three pieces: first the transition from $\Sigma^m$ to $\Sigma^{\text{out}}$, where $\gamma^s p(s; c, a)$ exits a neighborhood $U_F$ of the upper right fold point along the fast jump $\phi$, second the transition from $\Sigma^{\text{out}}$ to $\Sigma^{\text{in}}$ encompassing the passage near the fold point, and finally the transition from $\Sigma^{\text{in}}$ to $\Sigma^{h, r}$ describing the passage near the right slow manifold $M^s(c, a)$.

We first follow $\gamma^s p(s; c, a)$ backwards from $\Sigma^m$ into a neighborhood of $M^s(c, a)$ at a height $w = \Delta_w$, so that we are away from the lower fold point. By construction $\gamma^s p(s; c, a)$ lies in $W^s, c(a)$ and remains in this neighborhood of $M^s(c, a)$ until some height $w = w^1 + O(\varepsilon^{2/3}, a)$ corresponding to the fast jump to $\Sigma^{\text{out}}$ in the neighborhood $U_F$ of the upper right fold point. During this entire passage, solutions corresponding to the fiber $\{(0, y^s p(s; c, a), z) : |z| \leq \Delta_z\}$ in the section $\Sigma^m$ are contracted exponentially to $\gamma^s p(s; c, a)$ in backwards time, and hence we have the following.

**Lemma B.1.** For each sufficiently small $\Delta_w$, there exists $\Delta > 0, \varepsilon_0 > 0$ and sufficiently small choice of the intervals $I_c, I_a$, such that for each $0 < \varepsilon < \varepsilon_0$, each $(c, a) \in I_c \times I_a$, and each $s \in (w^1 + \Delta_w, 2w^1 - \Delta_w)$, the following holds. The backwards evolution $B(s; c, a)$ of the fiber $\{(0, y^s p(s; c, a), z) : |z| \leq \Delta_z\}$ in $\Sigma^m$ reaches the section $\Sigma^{\text{out}}$ near the upper right fold point in a curve which is $O\left(e^{\Lambda(\Delta_w, w^1 - \Delta_w)}\right)$ close to $\gamma^s p(s; c, a)$ uniformly in $(c, a, z_B) \in I_c \times I_a \times [-\Delta_z, \Delta_z]$ where

$$\Lambda(\Delta_w, w^1 - \Delta_w) = \int_{u_1(\Delta_w)}^{u_1(w^1 - \Delta_w)} \frac{c + \sqrt{c^2 - 4f'(u)}}{2c(u - \gamma f(u))f'(u)} du,$$

$$< 0.$$  

Furthermore the derivatives of the transition map from $\Sigma^m$ to $\Sigma^{\text{out}}$ for solutions on $B(s; c, a)$ with respect to $(c, a, z_B)$ are also $O\left(e^{\Lambda(\Delta_w, w^1 - \Delta_w)}\right)$.

**Proof.** To see this, we consider the flow in a neighborhood of $M^s(c, a)$; essentially we perform coordinate transformations to explicitly determine the expansion along $W^s, c(a)$ away from the fold at the origin. Away from the origin, we can parametrize $M^s(c, a)$ by $w$, that is, the slow manifold $M^s(c, a)$ is given as a graph

$$u = H(w, \varepsilon) = f^{-1}(w) + \varepsilon h(w, \varepsilon)$$

$$v = G(w, \varepsilon) = \varepsilon g(w, \varepsilon),$$

where we take the negative root $u_1(w)$ for $f^{-1}(w)$, and the functions $H, G$ satisfy

$$\varepsilon D_u H(w, \varepsilon)(H(w, \varepsilon) - \gamma w) = G(w, \varepsilon)$$

$$\varepsilon D_u G(w, \varepsilon)(H(w, \varepsilon) - \gamma w) = c G(w, \varepsilon) - f(H(w, \varepsilon)) + w,$$

and the flow on $M^s(c, a)$ is given by

$$\dot{w} = \varepsilon(H(w, \varepsilon) - \gamma w).$$

We now write

$$u = \bar{u} + H(w, \varepsilon)$$

$$v = \bar{v} + G(w, \varepsilon),$$

where we take the negative root $u_1(w)$ for $f^{-1}(w)$, and the functions $H, G$ satisfy

$$\varepsilon D_u H(w, \varepsilon)(H(w, \varepsilon) - \gamma w) = G(w, \varepsilon)$$

$$\varepsilon D_u G(w, \varepsilon)(H(w, \varepsilon) - \gamma w) = c G(w, \varepsilon) - f(H(w, \varepsilon)) + w,$$

and the flow on $M^s(c, a)$ is given by

$$\dot{w} = \varepsilon(H(w, \varepsilon) - \gamma w).$$

$$u = \bar{u} + H(w, \varepsilon)$$

$$v = \bar{v} + G(w, \varepsilon),$$

where we take the negative root $u_1(w)$ for $f^{-1}(w)$, and the functions $H, G$ satisfy

$$\varepsilon D_u H(w, \varepsilon)(H(w, \varepsilon) - \gamma w) = G(w, \varepsilon)$$

$$\varepsilon D_u G(w, \varepsilon)(H(w, \varepsilon) - \gamma w) = c G(w, \varepsilon) - f(H(w, \varepsilon)) + w,$$

and the flow on $M^s(c, a)$ is given by

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$$u = \bar{u} + H(w, \varepsilon)$$

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where we take the negative root $u_1(w)$ for $f^{-1}(w)$, and the functions $H, G$ satisfy

$$\varepsilon D_u H(w, \varepsilon)(H(w, \varepsilon) - \gamma w) = G(w, \varepsilon)$$

$$\varepsilon D_u G(w, \varepsilon)(H(w, \varepsilon) - \gamma w) = c G(w, \varepsilon) - f(H(w, \varepsilon)) + w,$$

and the flow on $M^s(c, a)$ is given by

$$\dot{w} = \varepsilon(H(w, \varepsilon) - \gamma w).$$

We now write

$$u = \bar{u} + H(w, \varepsilon)$$

$$v = \bar{v} + G(w, \varepsilon),$$

where we take the negative root $u_1(w)$ for $f^{-1}(w)$, and the functions $H, G$ satisfy

$$\varepsilon D_u H(w, \varepsilon)(H(w, \varepsilon) - \gamma w) = G(w, \varepsilon)$$

$$\varepsilon D_u G(w, \varepsilon)(H(w, \varepsilon) - \gamma w) = c G(w, \varepsilon) - f(H(w, \varepsilon)) + w,$$

and the flow on $M^s(c, a)$ is given by

$$\dot{w} = \varepsilon(H(w, \varepsilon) - \gamma w).$$
and compute the flow nearby for small \( \tilde{u}, \tilde{v} \) as
\[
\begin{align*}
\dot{\tilde{u}} &= \tilde{v} - \epsilon \tilde{u} D_w H(w, \epsilon) \\
\dot{\tilde{v}} &= c \tilde{v} - \tilde{u} f'(H(w, \epsilon)) - \epsilon \tilde{u} D_w G(w, \epsilon) + \mathcal{O}(\tilde{u}^2) \\
\dot{w} &= \epsilon (\tilde{u} + H(w, \epsilon) - \gamma w).
\end{align*}
\] (B.6)

We consider the linearization of the two dimensional \((\tilde{u}, \tilde{v})\) system about \((\tilde{u}, \tilde{v}, \epsilon) = (0, 0, 0)\) for each \(w\). There is one stable and one unstable eigenvalue
\[
\lambda^{\pm} = \frac{c \pm \sqrt{c^2 - 4f'(f^{-1}(w))}}{2},
\] (B.7)
with corresponding eigenvectors
\[
e^{\pm} = \begin{pmatrix} 1 \\ \lambda^{\pm} \end{pmatrix}.
\] (B.8)

We now introduce the coordinates
\[
U = \tilde{v} - \lambda^+ \tilde{u} \\
V = \tilde{v} - \lambda^- \tilde{u},
\] (B.9)
which, using the identities
\[
\lambda^+ \lambda^- = f'(f^{-1}(w)) \\
\lambda^{\pm} = c - \lambda^{\mp},
\] (B.10)
results in the system
\[
\begin{align*}
\dot{U} &= \lambda^- U + F^-(U, V, w, \epsilon) \\
\dot{V} &= \lambda^+ V + F^+(U, V, w, \epsilon) \\
\dot{w} &= \epsilon (f^{-1}(w) - \gamma w + F^s(U, V, w, \epsilon)),
\end{align*}
\] (B.11)
where
\[
\begin{align*}
F^{\pm}(U, V, w, \epsilon) &= \mathcal{O}(\epsilon U, \epsilon V, U^2, U V, V^2) \\
F^s(U, V, w, \epsilon) &= \mathcal{O}(U, V, \epsilon).
\end{align*}
\] (B.12)

We now identify the part of \( W_{\epsilon,a}^s(c) \) which intersects this neighborhood as a graph \( V = V^*(U, w, \epsilon) \). This manifold is foliated by strong unstable fibers tangent to lines \((U, w) = \text{const}\) for \(\epsilon = 0\). Setting \(\tilde{V} = V - V^*(U, w, \epsilon)\) and performing a coordinate change
\[
\begin{pmatrix} U \\ w \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{U} \\ \tilde{W} \end{pmatrix} = \begin{pmatrix} U \\ w \end{pmatrix} + \mathcal{O}(\tilde{V}),
\] (B.13)
to straighten out the unstable fibers, we arrive at the system
\[
\begin{align*}
\dot{\tilde{U}} &= \lambda^- \tilde{U} + \tilde{F}^-(\tilde{U}, \tilde{W}, \epsilon) \\
\dot{\tilde{V}} &= \lambda^+ \tilde{V} + \tilde{F}^+(\tilde{U}, \tilde{V}, \tilde{W}, \epsilon) \tilde{V} \\
\dot{\tilde{W}} &= \epsilon (f^{-1}(\tilde{W}) - \gamma \tilde{W} + \tilde{F}^s(\tilde{U}, \tilde{W}, \epsilon)),
\end{align*}
\] (B.14)
where
\[
\begin{align*}
\tilde{F}^-(\tilde{U}, \tilde{W}, \epsilon) &= \mathcal{O}(\tilde{U}^2, \epsilon \tilde{U}) \\
\tilde{F}^+(\tilde{U}, \tilde{V}, \tilde{W}, \epsilon) &= \mathcal{O}(\tilde{U}, \tilde{V}, \epsilon) \\
\tilde{F}^s(\tilde{U}, \tilde{W}, \epsilon) &= \mathcal{O}(\tilde{U}, \epsilon).
\end{align*}
\] (B.15)
We can now estimate the contraction rate $\Lambda^\ell(\hat{W}_1, \hat{W}_2)$ in backwards time along the fiber of a given trajectory lying on $\mathcal{M}^s_{\epsilon}(c, a)$ between heights $\hat{W}_1$ and $\hat{W}_2$, under the assumption that this trajectory remains in a small neighborhood of $\mathcal{M}^s_{\epsilon}(c, a)$, say $|\hat{U}|, |\hat{V}| \leq \Delta \ll 1$, for $\hat{W} \in [\hat{W}_1, \hat{W}_2]$. We compute

$$
\Lambda^\ell(\hat{W}_1, \hat{W}_2) = \int_{\hat{W}_1}^{\hat{W}_2} \frac{\lambda^+ + \hat{F}^+(\hat{U}, \hat{V}, \hat{W}, \epsilon)}{\epsilon(f^{-1}(\hat{W}) - \gamma W + \hat{F}^+(\hat{U}, \hat{V}, \epsilon))} d\hat{W} \\
= \int_{\hat{W}_1}^{\hat{W}_2} \frac{\lambda^+}{\epsilon(f^{-1}(\hat{W}) - \gamma W)} (1 + O(\epsilon, \Delta)) d\hat{W} \\
= \int_{\hat{W}_1}^{\hat{W}_2} \frac{c + \sqrt{\epsilon^2 - 4f'(f^{-1}(\hat{W}))}}{2\epsilon(f^{-1}(\hat{W}) - \gamma W)} (1 + O(\epsilon, \Delta)) d\hat{W} \\
= \int_{u_1(\hat{W}_1)}^{u_2(\hat{W}_2)} \frac{c + \sqrt{\epsilon^2 - 4f'(u)}}{2(u - \gamma f(u))} f'(u) (1 + O(\epsilon, \Delta)) du.
$$

Hence, by fixing $\Delta_W > 0$ small, and taking $\Delta, \epsilon > 0$ sufficiently small, we obtain the result.

We proceed by considering the flow near the upper right fold point. Using the analysis in [23, 7], it is clear that the transition in backwards time from $\Sigma^\text{out}$ to $\Sigma^{\text{h.r.}}$ in the neighborhood $U_F$ of the upper right fold point can be bounded by $e^{\eta/\epsilon}$ for each $\eta > 0$ by taking the neighborhood $U_F$ sufficiently small, that is, by shrinking $\Delta_W$. The derivatives of the transition map also satisfy the same bounds.

Finally, we consider the transition from $\Sigma^{\text{h.r.}}$ to $\Sigma^{\text{s-r}}$. We first prove the following technical lemma.

**Lemma B.2.** For each sufficiently small $\Delta_W$ and $(c, a) \in I_c \times I_a$, we have that

$$
\int_{u_1(w^\dagger - \Delta_W)}^{u_1(\Delta_W)} \frac{c + \sqrt{\epsilon^2 - 4f'(u)}}{u - \gamma f(u)} f'(u)du + \int_{u_3(-\Delta_W)}^{u_3(w^\dagger)} \frac{c - \sqrt{\epsilon^2 - 4f'(u)}}{u - \gamma f(u)} f'(u)du > 0. \quad (B.17)
$$

**Proof.** We first write

$$
\int_{u_3(-\Delta_W)}^{u_3(w^\dagger)} \frac{c - \sqrt{\epsilon^2 - 4f'(u)}}{u - \gamma f(u)} f'(u)du = \int_{u_3(-\Delta_W)}^{u_3(w^\dagger - \Delta_W)} \frac{c - \sqrt{\epsilon^2 - 4f'(u)}}{u - \gamma f(u)} f'(u)du + O(\Delta_W). \quad (B.18)
$$

Hence it suffices to show that there exists $C > 0$ such that

$$
\int_{u_1(w^\dagger - \Delta_W)}^{u_1(\Delta_W)} \frac{c + \sqrt{\epsilon^2 - 4f'(u)}}{u - \gamma f(u)} f'(u)du + \int_{u_3(-\Delta_W)}^{u_3(w^\dagger - \Delta_W)} \frac{c - \sqrt{\epsilon^2 - 4f'(u)}}{u - \gamma f(u)} f'(u)du > C, \quad (B.19)
$$

for $(c, a) = (1/\sqrt{2}, 0)$ uniformly in $\Delta_W > 0$ sufficiently small; the result then follows by continuity provided $\Delta_W$ and the intervals $I_c, I_a$ are sufficiently small. For $(c, a) = (1/\sqrt{2}, 0)$, we have that $w^\dagger = \frac{4}{27}$, and the following identities hold for each $w \in (0, w^\dagger)$ and $u < 0$.

$$
\begin{align*}
    u_3(w) &= \frac{2}{3} - u_1(w^\dagger - w) \\
    f(u) &= \frac{4}{27} - f\left(\frac{2}{3} - u\right) \\
    f'(u) &= f'\left(\frac{2}{3} - u\right).
\end{align*}
$$

(B.20)
Hence
\[
\int_{u_3(\Delta_w)}^{u_3(w^t - \Delta_w)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) \, du = \int_{\frac{2}{3} - u_1(\Delta_w)}^{\frac{2}{3} - u_1(w^t - \Delta_w)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) \, du
\]
\[
= - \int_{u_1(\Delta_w)}^{u_1(w^t - \Delta_w)} \frac{c - \sqrt{c^2 - 4f'(\frac{2}{3} - u)}}{\frac{2}{3} - u - \gamma f\left(\frac{2}{3} - u\right)} f'(\frac{2}{3} - u) \, du
\]
\[
= \int_{u_1(\Delta_w)}^{u_1(w^t - \Delta_w)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) \, du
\]
\[
> \int_{u_1(\Delta_w)}^{u_1(w^t - \Delta_w)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) \, du,
\]
since \(0 < \gamma < 4\). We therefore have that
\[
\int_{u_1(w^t - \Delta_w)}^{u_1(\Delta_w)} \frac{c + \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) \, du + \int_{u_3(w^t - \Delta_w)}^{u_3(\Delta_w)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) \, du
\]
\[
> \int_{u_1(\Delta_w)}^{u_1(w^t - \Delta_w)} \frac{c + \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) \, du + \int_{u_1(w^t - \Delta_w)}^{u_1(\Delta_w)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) \, du
\]
\[
= \int_{u_1(\Delta_w)}^{u_1(w^t - \Delta_w)} \frac{c}{u - \gamma f(u)} f'(u) \, du
\]
\[
> C,
\]
uniformly in \(\Delta_w > 0\) sufficiently small, which completes the proof.

In combination with the above results, we can now prove Lemma 3.9.

**Proof of Lemma 3.9.** In a neighborhood of \(\mathcal{M}^\gamma(r, a)\), we can put the flow into the Fenichel normal form
\[
\dot{U} = -\lambda^- U + F^-(U, V, \tilde{w}, \epsilon)U
\]
\[
\dot{V} = -\lambda^+ V + F^+(U, V, \tilde{w}, \epsilon)V
\]
\[
\dot{\tilde{w}} = \epsilon(-f^{-1}(\tilde{w}) + \gamma \tilde{w} + F^s(U, V, \tilde{w}, \epsilon)),
\]
where
\[
F^-(U, V, \tilde{w}, \epsilon) = \mathcal{O}(U, V, \epsilon)
\]
\[
F^+(U, V, \tilde{w}, \epsilon) = \mathcal{O}(U, V, \epsilon)
\]
\[
F^s(U, V, \tilde{w}, \epsilon) = \mathcal{O}(UV, \epsilon),
\]
\[
\lambda^\pm = \frac{c \pm \sqrt{c^2 - 4f'(f^{-1}(\tilde{w}))}}{2},
\]
and \(f^{-1}(\tilde{w})\) refers to the largest root \(u_3(\tilde{w})\) of \(f(u) = \tilde{w}\). We note that the flow is now in backwards time. By construction, up to a reparameterization of \(\gamma^s(s; c, a)\) according to the smooth coordinate transformation \((u, v, w) \rightarrow (U, V, \bar{w})\), in backwards time \(\gamma^s(s; c, a)\) exits at height \(\bar{w} = 2w^t - s\). Between \(\bar{w} = 2w^t - s\) and \(\bar{w} = w^t - \Delta_w\), \(\gamma^s(s; c, a)\) is given as a solution
\[
U = U^s(t; c, a, \epsilon)
\]
\[
V = V^s(t; c, a, \epsilon)
\]
\[
\bar{w} = \bar{w}^s(t; c, a, \epsilon)
\]
where \(|U^s|, |V^s| \leq \Delta\) for \(\bar{w} \in (2w^t - s, w^t - \Delta_w)\). We now obtain estimates on this solution and its derivatives. We first recall/comment on how the solution \(\gamma^s(s; c, a)\) is constructed.
For a given value of \((s, c, a)\), \(\gamma^{sp}(s; c, a)\) is defined as the unique transverse intersection of the forward evolution of the line \(\{u = 2/3, w = 2w^{1} - s\}\) with the manifold \(W^{s,tf}_{c}(s, c, a)\). Equivalently, for the same effect we could have worked in this Fenichel neighborhood of \(\mathcal{M}_{c}(c, a)\) and considered constructing \(\gamma^{sp}(s; c, a)\) as the unique transverse intersection of the forward evolution of the line \(\{U = \Delta, |V| \leq \Delta, \tilde{w} = 2w^{1} - s\}\) with the manifold \(W^{s,tf}_{c}(c, a)\). Using arguments similar to those in [7, §5] in the proof of Proposition 2.6, we obtain the solution \(\gamma^{sp}(s; c, a) = (U^{sp}, V^{sp}, \tilde{w}^{sp})\) which satisfies \((U^{sp}, V^{sp}) = (O(e^{-q/\epsilon}), O(e^{-\eta/\epsilon}))\) at \(\tilde{w} = w^{1} - \Delta w\) and \((U^{sp}, V^{sp}) = (\Delta, O(e^{-q/\epsilon}))\) at \(\tilde{w} = 2w^{1} - s\), where \(q > \eta > 0\). Furthermore, the derivatives with respect to \((c, a)\) of these boundary values satisfy similar bounds, where \(q, \eta\) may need to be taken slightly smaller.

We now obtain more precise bounds for this solution and its derivatives. We write \(\tilde{w} = w^{*} + W\) where \(w^{*}(t)\) is the solution to

\[
\dot{\tilde{w}} = \epsilon(-f^{-1}(\tilde{w}) + \gamma\tilde{w} + F^{sl}(0, 0, \tilde{w}, \epsilon)),
\]

(B.27)
satisfying \(\tilde{w}(0) = w^{1} - \Delta w, \tilde{w}(T) = 2w^{1} - s\), where we note that \(\epsilon/C < T < C/\epsilon\) for some \(C > 0\). This results in the equations

\[
\dot{U} = \Lambda^{u}(t)U + G^{-}(U, V, W, \epsilon)U
\]
\[
\dot{V} = -\Lambda^{s}(t)V + G^{+}(U, V, W, \epsilon)V
\]
\[
\dot{W} = \epsilon(-(f^{-1})'(w^{*})W + \gamma W + F^{sl}_{w}(0, w^{*}, \epsilon)W + G^{sl}(U, V, W, \epsilon)),
\]

(B.28)

where

\[
\Lambda^{u}(t) = -\lambda^{-}(w^{*}(t)) + O(\epsilon)
\]
\[
\Lambda^{s}(t) = \lambda^{+}(w^{*}(t)) + O(\epsilon)
\]
\[
G^{-}(U, V, W, \epsilon) = O(U, V, W)
\]
\[
G^{+}(U, V, W, \epsilon) = O(U, V, W)
\]
\[
G^{sl}(U, V, W, \epsilon) = O(UV, W^{2}).
\]

(B.29)

We now define for each sufficiently small \(\delta > 0\) the functions

\[
\beta_{\delta}^{-}(t, s) = \int_{s}^{t} \Lambda^{u}(\tau) - \delta d\tau
\]
\[
\beta_{\delta}^{+}(t, s) = \int_{s}^{t} -\Lambda^{s}(\tau) + \delta d\tau
\]
\[
\beta^{sl}(t, s) = \epsilon \int_{s}^{t} -(f^{-1})'(w^{*}(\tau)) + \gamma + F^{sl}_{w}(0, w^{*}(\tau), \epsilon)d\tau.
\]

(B.30)

Hence the solution \(\gamma^{sp}(s; c, a)\) given by \((U^{sp}, V^{sp}, W^{sp})\), \(W^{sp} = w^{sp} - w^{*}\), solves

\[
U(t) = e^{\beta_{\delta}^{-}(t, T)} \Delta + \int_{T}^{t} e^{\beta_{\delta}^{-}(t, \tau)} G^{-}(U(s), V(s), W(s), \epsilon) U(s) ds
\]
\[
:= F^{-}(U, V, W, \Delta, 0; c, a)(t)
\]
\[
V(t) = e^{\beta_{\delta}^{+}(t, 0)} V_{0} + \int_{0}^{t} e^{\beta_{\delta}^{+}(t, \tau)} G^{+}(U(s), V(s), W(s), \epsilon) V(s) ds
\]
\[
:= F^{+}(U, V, W, \Delta, V_{0}; c, a)(t)
\]
\[
W(t) = \int_{T}^{t} e^{\beta^{sl}(t, \tau)} G^{sl}(U(s), V(s), W(s), \epsilon) ds
\]
\[
:= F^{sl}(U, V, W, \Delta, 0; c, a)(t).
\]

(B.31)
We define the spaces
\[ V_{\delta}^{-} = \left\{ U : [0, T] \to \mathbb{R}^2 : \|U\|_{\delta}^{-} = \sup_{t \in [0, T]} e^{\beta \delta(T, t)}|U(t)| < \infty \right\} \]
\[ V_{\delta}^{+} = \left\{ V : [0, T] \to \mathbb{R} : \|V\|_{\delta}^{+} = \sup_{t \in [0, T]} e^{\beta \delta(0, t)}|V(t)| < \infty \right\} \]
\[ V^{sl} = \left\{ W : [0, T] \to \mathbb{R} : \|W\|^{sl} = \sup_{t \in [0, T]} |W(t)| < \infty \right\}, \]
and for each fixed small \( \delta > 0 \) we have that
\[ \|F^{-}(U, V, W, \Delta, V_0; c, a)\|_{\delta}^{-} = \Delta + O \left( \|U\|_{\delta}^{-} \left( \|U\|_{\delta}^{-} + \|V\|_{\delta}^{+} + \|W\|^{sl} \right) \right) \]
\[ \|F^{+}(U, V, W, \Delta, V_0; c, a)\|_{\delta}^{+} = \eta + O \left( \|V\|_{\delta}^{+} \left( \|U\|_{\delta}^{-} + \|V\|_{\delta}^{+} + \|W\|^{sl} \right) \right) \]
\[ \|F^{sl}(U, V, W, \Delta, V_0; c, a)\|^{sl} = O \left( \left(\|W\|^{sl}\right)^2, \|U\|_{\delta}^{-}\|V\|_{\delta}^{+} \right), \]
and hence \( \gamma^{sp}(s; c, a) \) satisfies
\[ \|U^{sp}\|_{\delta}^{-} = O(\Delta) \]
\[ \|V^{sp}\|_{\delta}^{+} = O(\eta) \]
\[ \|W^{sp}\|^{sl} = O \left( e^{-\eta/\epsilon} \right). \]

Taking derivatives of (B.31) with respect to the parameters \( (c, a) \) and taking \( \delta \) slightly larger and \( \eta \) slightly smaller if necessary, we can bound the derivatives
\[ \|D_{\nu} U^{sp}\|_{\delta}^{-} = O(\Delta) \]
\[ \|D_{\nu} V^{sp}\|_{\delta}^{+} = O(e^{-\eta/\epsilon}) \]
\[ \|D_{\nu} W^{sp}\|^{sl} = O \left( e^{-\eta/\epsilon} \right), \]
for \( \nu = (c, a) \).

To determine the contraction/expansion of solutions along \( \gamma^{sp}(s; c, a) \), we write
\[ U = U^{sp}(t; c, a, \epsilon) + \tilde{U} \]
\[ V = V^{sp}(t; c, a, \epsilon) + \tilde{V} \]
\[ W = W^{sp}(t; c, a, \epsilon) + \tilde{W} \]
and obtain the equations
\[ \dot{U} = \Lambda^u \tilde{U} + \tilde{G}_1^u(\tilde{U}, \tilde{V}, \tilde{W}, \epsilon) + \tilde{G}_2^u(\tilde{U}, \tilde{V}, \tilde{W}, \epsilon)U^{sp} \]
\[ \dot{V} = -\Lambda^s \tilde{V} + \tilde{G}_1^v(\tilde{U}, \tilde{V}, \tilde{W}, \epsilon)\tilde{V} + \tilde{G}_2^v(\tilde{U}, \tilde{V}, \tilde{W}, \epsilon)V^{sp} \]
\[ \dot{W} = \epsilon(-f^{-1})(w^*)\tilde{W} + \gamma \tilde{W} + F_{\omega}^{sl}(0, w^*, \epsilon)\tilde{W} + \tilde{G}^{sl}(\tilde{U}, \tilde{V}, \tilde{W}, \epsilon), \]
where
\[ \tilde{G}_1^u(\tilde{U}, \tilde{V}, \tilde{W}, \epsilon) = O \left( U^{sp}, V^{sp}, W^{sp}, \tilde{U}, \tilde{V}, \tilde{W} \right) \]
\[ \tilde{G}_2^u(\tilde{U}, \tilde{V}, \tilde{W}, \epsilon) = O \left( \tilde{U}, \tilde{V}, \tilde{W} \right) \]
\[ \tilde{G}_1^v(\tilde{U}, \tilde{V}, \tilde{W}, \epsilon) = O \left( U^{sp}, V^{sp}, W^{sp}, \tilde{U}, \tilde{V}, \tilde{W} \right) \]
\[ \tilde{G}_2^v(\tilde{U}, \tilde{V}, \tilde{W}, \epsilon) = O \left( \tilde{U}, \tilde{V}, \tilde{W} \right) \]
\[ \tilde{G}^{sl}(\tilde{U}, \tilde{V}, \tilde{W}, \epsilon) = O(\tilde{U}V^{sp}, U^{sp}\tilde{V}, \tilde{U}\tilde{V}, \tilde{W}W^{sp}, \tilde{W}^2). \]
We can write this as the integral equation

\[
\begin{align*}
\dot{U}(t) &= e^{\beta^{-}_0(t,T)} \dot{U}_T + \int_T^t e^{\beta^{-}_0(t,s)} \dot{G}_1^{-}(\bar{U}(s), \bar{V}(s), \bar{W}(s), e) \dot{U}(s) + \dot{G}_2^{-}(\bar{U}(s), \bar{V}(s), \bar{W}(s), e) U^{sp}(s) ds \\
\dot{V}(t) &= e^{\beta^{-}_0(t,0)} \dot{V}_0 + \int_0^t e^{\beta^{-}_0(t,s)} \dot{G}_1^{+}(\bar{U}(s), \bar{V}(s), \bar{W}(s), e) \dot{V}(s) + \dot{G}_2^{+}(\bar{U}(s), \bar{V}(s), \bar{W}(s), e) V^{sp}(s) ds \\
\dot{W}(t) &= e^{\beta^{-}_0(t,0)} \dot{W}_0 + \int_0^t e^{\beta^{-}_0(t,s)} \dot{G}^{sl}(\bar{U}(s), \bar{V}(s), \bar{W}(s), e) ds.
\end{align*}
\]

(B.39)

Provided \(|\dot{U}_T|, |\dot{V}_0|, \) and \(|\dot{W}_0|\) are sufficiently small, we can solve this by the implicit function theorem and obtain a solution satisfying

\[
\begin{align*}
\|\dot{U}\|_0^- &= O \left( |\dot{U}_T| + \Delta(0,0) \right) \\
\|\dot{V}\|_0^+ &= O \left( |\dot{V}_0| + e^{-\eta/\epsilon}(0,0) \right) \\
\|\dot{W}\|_0^{sl} &= O \left( |\dot{W}_0| + e^{-\eta/\epsilon}0 + \Delta(0,0) \right).
\end{align*}
\]

Taking derivatives of (B.39) with respect to the parameters \((c,a)\) and taking \(\delta\) slightly larger if necessary, we can bound the derivatives

\[
\begin{align*}
\|D_c \dot{U}\|_0^- &= O \left( |\dot{U}_T| + |D_c \dot{U}_T| + \Delta(0,0) \right) \\
\|D_c \dot{V}\|_0^+ &= O \left( |\dot{V}_0| + |D_c \dot{V}_0| + e^{-\eta/\epsilon}(0,0) \right) \\
\|D_c \dot{W}\|_0^{sl} &= O \left( |\dot{W}_0| + |D_c \dot{W}_0| + e^{-\eta/\epsilon}(0,0) \right),
\end{align*}
\]

(B.41)

for \(\nu = (z_B, c, a)\).

By Lemma B.1, at \(t = 0\) the manifold \(\mathcal{B}(s; c, a)\) is \(O(e^{\Lambda'(\Delta_w,w^t - \Delta_w)})\)-close to \(\gamma^{sp}(s; c, a)\). That is, the solutions on \(\mathcal{B}(s; c, a)\) can be represented in the above coordinates by solutions \((\tilde{U}, \tilde{V}, \tilde{W})\) satisfying

\[
(\tilde{U}, \tilde{V}, \tilde{W})(0) = (\tilde{U}_0^B, \tilde{V}_0^B, \tilde{W}_0^B) = O(e^{\Lambda'(\Delta_w,w^t - \Delta_w)}),
\]

uniformly along with their derivatives with respect to \((z_B, c, a)\). We now solve for the solution to (B.39) which satisfies \((\tilde{V}_0, \tilde{W}_0) = (\tilde{V}_0^B, \tilde{W}_0^B)\) and

\[
\tilde{U}_0^B = e^{\beta_0^{-}(0,T)} \tilde{U}_T + \int_T^0 e^{\beta_0^{-}(0,s)} \tilde{G}_1^{-}(\tilde{U}(s), \tilde{V}(s), \tilde{W}(s), e) \tilde{U}(s) + \tilde{G}_2^{-}(\tilde{U}(s), \tilde{V}(s), \tilde{W}(s), e) U^{sp}(s) ds.
\]

(B.42)

Provided \(e^{\beta_0^{-}(T,0)} \tilde{G}_0^B\) is sufficiently small, we can find a solution \(\tilde{U}_T\) satisfying (B.42). Performing a similar computation as in the proof of Lemma B.1 shows that the expansion \(\beta^{-}_{0}(T,0)\) in backwards time from \(w = w^t - \Delta_w\) to \(w = 2w^t - s\) can be estimated by

\[
\Lambda'(w^t - \Delta_w, 2w^t - s) = \int_{w^t - \Delta_w}^{2w^t - s} \frac{c - \sqrt{c^2 - 4f(u)}}{2c(u - \gamma f(u))} (1 + O(\epsilon, \Delta)) f(u) du.
\]

(B.43)

Using this in combination with the \(O(e^{\Lambda'(\Delta_w,w^t - \Delta_w)})\) bounds on \(\tilde{U}_0^B, \tilde{V}_0^B, \tilde{W}_0^B\) for the contraction/expansion from \(\Sigma^{out}_r\) to \(\Sigma^{out}_l\) and \(\Sigma^{out}_r\) to \(\Sigma^{in}_l\), and Lemma B.2, we deduce that \(\mathcal{B}(s; c, a)\) is \(O(e^{-\eta/\epsilon})\)-close to \(\gamma^{sp}(s; c, a)\) in \(\Sigma^{h,r}\).

\[\Box\]

**References**


