Regularity of center manifolds via the graph transform

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Abstract

The purpose of this paper is to give a short self-contained proof of the center-manifold theorem for maps and vector fields in finite-dimensional spaces using the graph transform. In particular, regularity of the center manifold is established using a direct argument that is based on the closedness of sets of differentiable functions whose highest derivatives are Lipschitz continuous in the space of continuous functions; this argument avoids the fiber contraction theorem that is commonly used in this context.

1 Introduction

Consider the differential equation

$$\dot{u} = Au + h(u), \qquad u \in \mathbb{R}^n. \tag{1.1}$$

We assume that $A \in \mathbb{R}^{n\times n}$ and that

$$h \in C^{k,1} := \{f \in C^k(\mathbb{R}^n, \mathbb{R}^n) : \ \mathrm{D}^k f \text{ is Lipschitz continuous}\}$$

for some fixed finite integer $k \geq 1$ with $h(0) = \mathrm{D}h(0) = 0$, so that the origin is an equilibrium with linearization A. We decompose $\mathbb{R}^n = E^c \oplus E^h$ such that E^c and E^h are invariant under A with $\operatorname{spec}(A|_{E^c}) \subset \mathrm{i}\mathbb{R}$ and $\operatorname{spec}(A|_{E^h}) \cap \mathrm{i}\mathbb{R} = \emptyset$. In this situation, the dynamics of (1.1) near the equilibrium $u = 0$ can be reduced to a smooth locally invariant center manifold that is the graph of a function from E^c to E^h.

Theorem 1 (Center-manifold theorem for flows)**.** *There exists a $\rho > 0$ and a map $\sigma : \{u \in E^c : |u| \leq \rho\} \to E^h$ of class $C^{k,1}$ with $\sigma(0) = \mathrm{D}\sigma(0) = 0$ so that $W^c = \operatorname{graph}(\sigma)$ is locally invariant: if $u_0 \in W^c$ and $|u(t)| \leq \rho$ for all $0 \leq t \leq T$, then $u(t) \in W^c$ for all $0 \leq t \leq T$. Furthermore, if $|u(t)| \leq \rho$ for all $t \in \mathbb{R}$, then $u_0 \in W^c$.*

This result is well known, of course, and there are two common approaches for proving it through fixed-point arguments: the first method uses an appropriate integral equation that relies on the variation-of-constants formula, while the second approach captures center manifolds via the more geometric graph transform. To cite just two of the many papers in which center manifolds were investigated, we refer to [6] for the variation-of-constants approach and to [4] for the graph-transform approach; further references can be found in the recent survey paper [5] or in the text book [1]. Regardless of which method is used, establishing differentiability of the center manifold is much more involved than the other parts of the proof: the papers cited above utilize scales of Banach spaces for the integral-equation formulation [6], while the fiber contraction theorem is used in the approach via graph transforms [4].

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The motivation for this paper came from graduate courses in dynamical systems taught by the first author and the admittedly very subjective perceived lack of relatively short, self-contained proofs of the existence and differentiability of center manifolds: the goal of this paper is to provide such a proof. In particular, regularity is shown using an idea in [2], namely that bounded closed balls in $C^{k,1}$ are also closed in C^0; thus, if the graph transform maps such a ball into itself, the fixed point will automatically be $C^{k,1}$. We remark that this approach has been used previously in [3] to show regularity of homoclinic center manifolds. We will establish the result for flows using the following center-unstable manifold theorem for maps.

Theorem 2 (Center-unstable manifold theorem for maps). *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. We assume that there is an integer $k \geq 0$ and a T-invariant decomposition $\mathbb{R}^n = E^{cu} \oplus E^s$ such that*

$$\|T_s\| < 1, \tag{1.2}$$

$$\|T_s\| \, \|T_{cu}^{-1}\|^{k+1} < 1, \tag{1.3}$$

where $T_j = T|_{E^j} : E^j \to E^j$ for $j = cu, s$. Under this assumption, there is an $\epsilon > 0$ with the following property. If $g : \mathbb{R}^n \to \mathbb{R}^n$ is of class $C^{k,1}$ with $\mathrm{Lip}(g) \leq \epsilon$, then the map $f(u) := Tu + g(u)$ has a globally attracting invariant manifold given by the graph of a $C^{k,1}$-map $\sigma^* : E^{cu} \to E^s$.

Thus, the key assumptions in the preceding theorem are that (i) T_s is a contraction; (ii) T_{cu} cannot contract at a rate equal or higher to that of T_s; and (iii) the nonlinearity is small compared to the contraction rates that appear in (1.2)-(1.3). The remainder of this paper is organized as follows. In §2, we will prove the existence and regularity of center manifolds of maps using the graph transform, while the proof of the corresponding result for vector fields is discussed briefly in §3.

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2 Existence and regularity of center manifolds for maps

To prove Theorem 2, we will use the graph transform. Throughout the remainder of this section, we write $(x, y) \in E^{cu} \oplus E^s = \mathbb{R}^n$ with norm $|(x, y)| = \max\{|x|, |y|\}$ and consider the map $f(x, y) = T(x, y) + g(x, y)$. We will also write $f = (f_{cu}, f_s)$ to denote the components of f, and similarly for g.

2.1 Outline of the proof

We begin by outlining the idea behind the graph transform, which is also illustrated in Figure 1. For a given map $\sigma : E^{cu} \to E^s$, we define its graph to be

$$\mathrm{graph}(\sigma) = \{(x, \sigma(x)) : \; x \in E^{cu}\}.$$

Our goal is to find a map $\sigma^* : E^{cu} \to E^s$ such that its graph is invariant under f, which means that $\mathrm{graph}(\sigma^*) = f(\mathrm{graph}(\sigma^*))$. To find σ^*, we take any $\sigma : E^{cu} \to E^s$ and seek $\tilde{\sigma} : E^{cu} \to E^s$ so that

$$f(\mathrm{graph}(\sigma)) = \mathrm{graph}(\tilde{\sigma}).$$

If such a function exists, we define the graph transform of σ via $\Gamma_f(\sigma) := \tilde{\sigma}$ and note that the desired function σ^* is a fixed point of the graph transform so that $\Gamma_f(\sigma^*) = \sigma^*$. Figure 1 shows that $\Gamma_f(\sigma)$ is given by

$$\Gamma_f(\sigma) = f_s \circ \begin{pmatrix} \mathrm{id} \\ \sigma \end{pmatrix} \circ \left(f_{cu} \circ \begin{pmatrix} \mathrm{id} \\ \sigma \end{pmatrix} \right)^{-1} \tag{2.1}$$

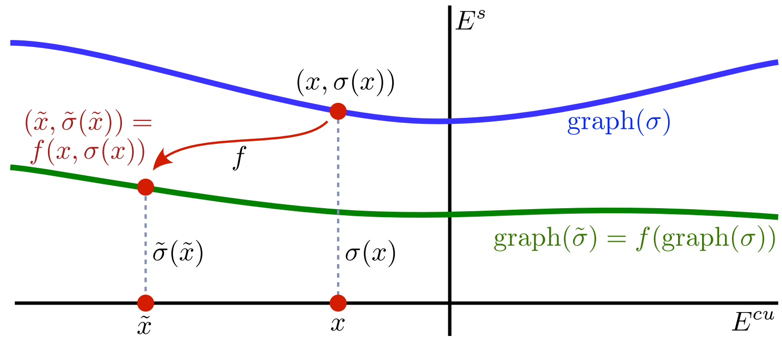

Figure 1: *Given a function* $\sigma : E^{cu} \to E^s$*, its graph transform* $\tilde{\sigma} : E^{cu} \to E^s$ *is a function for which* $\text{graph}(\tilde{\sigma}) = f(\text{graph}(\sigma))$*, if it exists. Given* $\tilde{x}$*, we see that* $\tilde{\sigma}(\tilde{x}) = f_s(x, \sigma(x))$*, where* x *is determined by the requirement that* $\tilde{x} = f_{cu}(x, \sigma(x))$*. This shows that* $x = [f_{cu} \circ (\text{id}, \sigma)]^{-1}(\tilde{x})$ *if the inverse exists.*

provided the inverse exists.

Our goal is to show that the graph transform Γ_f is well defined and a contraction on the space

$$\Sigma := \{\sigma : E^{cu} \to E^s : \ \|\sigma\| < \infty, \ \text{Lip}(\sigma) \le 1\} \subset C^0(E^{cu}, E^s) \quad \text{with norm} \quad \|\sigma\| := \sup_{x \in E^{cu}} |\sigma(x)|$$

of Lipschitz-continuous functions σ whose Lipschitz constant is bounded by one. Since Σ is a closed subset of the complete space C^0, Banach's fixed-point theorem would then imply that Γ_f has a unique fixed point σ^*, whose graph is then the desired invariant center-unstable manifold. To establish that $\sigma \in C^{k,1}$, we will show that there are constants $R = (r_2, r_3, \ldots, r_{k+1})$ with $r_j > 0$ for all j such that Γ_f maps B_R^k into itself, where B_R^k is defined by

$$B_R^k = \{\sigma \in \Sigma \cap C^{k,1} : \ \|\text{D}\sigma\| \le 1, \ \|\text{D}^2\sigma\| \le r_2, \ \ldots, \ \|\text{D}^k\sigma\| \le r_k, \ \text{Lip}(\text{D}^k\sigma) \le r_{k+1}\}. \tag{2.2}$$

By [2, Lemma 6.1.6] or Lemma A.2 below, the set B_R^k is a non-empty closed subset of Σ in the C^0-norm, and $\Gamma_f({}^k_R) \subset B_R^k$ implies that the fixed point σ^* lies in B_R^k and is therefore of class $C^{k,1}$.

2.2 Illustration: f is linear

It is illuminating to consider the case where $f = T$. In this case, the graph transform is given simply by

$$\Gamma_f(\sigma) = T_s \circ \sigma \circ T_{cu}^{-1}.$$

Even though it is clear from this expression that $\sigma^* \equiv 0$ is a fixed point of Γ_f, we will verify the properties claimed for Γ_f as an illustration of the ideas used in the general case. Since T_s and T_{cu}^{-1} are linear, we have

$$\text{Lip}(\Gamma_f(\sigma)) \le \|T_s\| \, \text{Lip}(\sigma) \, \|T_{cu}^{-1}\| \overset{(1.3)}{\le} 1$$

and

$$\|\Gamma_f(\sigma)\| \le \|T_s\| \, \|\sigma\| \, \|T_{cu}^{-1}\|.$$

Hence, Γ_f maps Σ into itself. To show that Γ_f is a contraction in Σ, note that for all $x \in E^{cu}$

$$|\Gamma_f(\sigma_1)(x) - \Gamma_f(\sigma_2)(x)| = |T_s\sigma_1(T_{cu}^{-1}x) - T_s\sigma_2(T_{cu}^{-1}x)| \le \|T_s\| \, |\sigma_1(T_{cu}^{-1}x) - \sigma_2(T_{cu}^{-1}x)| \le \|T_s\| \, \|\sigma_1 - \sigma_2\|.$$

Since $\|T_s\| < 1$ by (1.2), Γ_f is a contraction. By Banach's fixed point theorem, Γ_f has a unique fixed point σ^* in Σ. To prove smoothness, recall that

$$\text{D}^j\sigma(x) : \underbrace{E^{cu} \times E^{cu} \times \ldots \times E^{cu}}_{j \text{ copies}} \to E^s$$

is a multilinear map for each $x \in E^{cu}$. Thus, for each $x, x_1, \dots, x_j \in E^{cu}$, we have

$$\mathrm{D}^j(\Gamma_f(\sigma))(x)[x_1, x_2, \dots, x_j] = T_s \mathrm{D}^j \sigma(T_{cu}^{-1}x)[T_{cu}^{-1}x_1, \dots, T_{cu}^{-1}x_j],$$

which we write in short as $\mathrm{D}^j\Gamma_f(\sigma)(x) = T_s \mathrm{D}^j\sigma(T_{cu}^{-1}x)[T_{cu}^{-1}]^j$. We conclude that

$$\|\mathrm{D}^j\Gamma_f(\sigma)\| \le \|T_s\| \, \|\mathrm{D}^j\sigma\| \, \|T_{cu}^{-1}\|^j.$$

In particular, if (1.3) holds, then

$$\|\mathrm{D}^j\Gamma_f(\sigma)\| \le \|T_s\| \, \|T_{cu}^{-1}\|^j \, \|\mathrm{D}^j\sigma\| \le \|\mathrm{D}^j\sigma\|, \quad \mathrm{Lip}(\mathrm{D}^k\Gamma_f(\sigma)) \le \|T_s\| \, \|T_{cu}^{-1}\|^{k+1} \, \mathrm{Lip}(\mathrm{D}^k\sigma) \le \mathrm{Lip}(\mathrm{D}^k\sigma)$$

for $2 \le j \le k$, and Γ_f maps B_R^k into itself for any $R = (r_2, \dots, r_{k+1})$ with $r_j > 0$, where the latter set was defined in (2.2). Since B_R^k is non-empty and closed in Σ by Lemma A.2, the unique fixed point σ^* must lie in B_R^k. In particular, σ^* is $C^{k,1}$. If $f = T + g$ with $g \neq 0$, then the preceding arguments get slightly more involved, but the essence of the arguments remains the same as we shall see.

2.3 Existence of the fixed point

Hypotheses (1.2) and (1.3) with $k = 0$ allow us to find an $\epsilon > 0$ such that

$$1 - \epsilon\|T_{cu}^{-1}\| > 0, \qquad (\epsilon + \|T_s\|)\frac{\|T_{cu}^{-1}\|}{1 - \epsilon\|T_{cu}^{-1}\|} < 1, \qquad \|T_s\| + 2\epsilon < 1. \tag{2.3}$$

From now on, we fix such an $\epsilon > 0$ and assume that $\|g\|_{C^1} \le \epsilon$. For each given $\sigma \in \Sigma$, we write $\hat{\sigma} := (\mathrm{id}, \sigma)$. We need to show that the graph transform from (2.1)

$$\Gamma_f(\sigma) = f_s \circ \hat{\sigma} \circ (f_{cu} \circ \hat{\sigma})^{-1}$$

is well defined. In particular, we need to show that

$$(f_{cu} \circ \hat{\sigma})(\tilde{x}) = f_{cu}(\tilde{x}, \sigma(\tilde{x})) = T_{cu}\tilde{x} + g_{cu}(\tilde{x}, \sigma(\tilde{x}))$$

is invertible and that its inverse is Lipschitz continuous. The first inequality in (2.3) allows us to apply the Lipschitz inverse function theorem stated in Lemma A.1 with $T := T_{cu}$ and $g := g_{cu} \circ (\mathrm{id}, \sigma)$ to conclude that f_{cu} has a Lipschitz-continuous inverse that can be written as

$$(f_{cu} \circ (\mathrm{id}, \sigma))^{-1} = T_{cu}^{-1} - h_\sigma =: w_\sigma$$

for some Lipschitz continuous function $h_\sigma : E^{cu} \to E^s$ which satisfies

$$\mathrm{Lip}(h_\sigma) \le \frac{\epsilon\|T_{cu}^{-1}\|^2}{1 - \epsilon\|T_{cu}^{-1}\|}, \qquad \mathrm{Lip}(w_\sigma) \le \|T_{cu}^{-1}\| + \mathrm{Lip}(h_\sigma) \le \frac{\|T_{cu}^{-1}\|}{1 - \epsilon\|T_{cu}^{-1}\|}. \tag{2.4}$$

Thus, we define the graph transform of σ by

$$\Gamma_f(\sigma) := f_s \circ \hat{\sigma} \circ w_\sigma \qquad \text{with} \qquad f_{cu} \circ \hat{\sigma} \circ w_\sigma = \mathrm{id}_{E^{cu}}\,. \tag{2.5}$$

We can now show that Γ_f maps Σ into itself.

Lemma 2.1. *If $\sigma \in \Sigma$, then $\Gamma_f(\sigma) \in \Sigma$.*

Proof. To prove that $\Gamma_f(\sigma) \in \Sigma$, we need to show that $\|\Gamma_f(\sigma)\| < \infty$ and $\mathrm{Lip}(\Gamma_f(\sigma)) \leq 1$. The first inequality follows from (2.5) upon using that $f_s = T_s + g_s$ where T_s is linear:

$$\|\Gamma_f(\sigma)\| \leq \|T_s \circ \sigma \circ w_\sigma\| + \|g_s \circ \hat{\sigma} \circ w_\sigma\| \leq \|T_s\| \, \|\sigma \circ w_\sigma\| + \|g_s\| \leq \|T_s\| \, \|\sigma\| + \|g\| < \infty.$$

It remains to show that $\mathrm{Lip}(\Gamma_f(\sigma)) \leq 1$. Using that $\mathrm{Lip}(f_1 \circ f_2) \leq \mathrm{Lip}(f_1)\mathrm{Lip}(f_2)$, we obtain

$$\begin{aligned} \mathrm{Lip}(\Gamma_f(\sigma)) &\leq \mathrm{Lip}(T_s \circ \sigma \circ w_\sigma) + \mathrm{Lip}(g_s \circ \hat{\sigma} \circ w_\sigma) \leq \|T_s\|\mathrm{Lip}(\sigma)\mathrm{Lip}(w_\sigma) + \mathrm{Lip}(g_s)\mathrm{Lip}(w_\sigma) \\ &\leq (\|T_s\| + \epsilon)\mathrm{Lip}(w_\sigma) \end{aligned}$$

and (2.4) implies

$$\mathrm{Lip}(\Gamma_f(\sigma)) \leq (\epsilon + \|T_s\|) \frac{\|T_{cu}^{-1}\|}{1 - \epsilon\|T_{cu}^{-1}\|} \overset{(2.3)}{<} 1. \tag{2.6}$$

Therefore, $\Gamma_f(\Sigma) \subset \Sigma$ as claimed. □

Next, we show that Γ_f is a contraction.

Lemma 2.2. *Γ_f is a contraction in Σ equipped with the C^0-norm.*

Proof. Recall that $f_j(x, y) = T_j x + g_j(x, y)$ for $j = cu, s$. For each $\sigma_1 \in \Sigma$ and $(x, y) \in E^{cu} \oplus E^s$, we have

$$\begin{aligned} &|f_s(x, y) - \Gamma_f(\sigma_1)(f_{cu}(x, y))| \\ &\quad \leq |f_s(x, y) - f_s(x, \sigma_1(x))| + |f_s(x, \sigma_1(x)) - \Gamma_f(\sigma_1)(f_{cu}(x, y))| \\ &\quad \leq |f_s(x, y) - f_s(x, \sigma_1(x))| + \|\Gamma_f(\sigma_1)(f_{cu}(x, \sigma_1(x))) - \Gamma_f(\sigma_1)(f_{cu}(x, y))\| \\ &\quad \leq |T_s y - T_s \sigma_1(x)| + |g_s(x, y) - g_s(x, \sigma_1(x))| + \mathrm{Lip}(\Gamma_f(\sigma_1))|f_{cu}(x, \sigma_1(x)) - f_{cu}(x, y)| \\ &\quad \leq (\|T_s\| + \epsilon)|y - \sigma_1(x)| + |g_{cu}(x, \sigma_1(x)) - g_{cu}(x, y)| \\ &\quad \leq (\|T_s\| + 2\epsilon)|y - \sigma_1(x)|. \end{aligned} \tag{2.7}$$

Next, given $\sigma_1, \sigma_2 \in \Sigma$ and $z \in E^{cu}$, we set $x = w_{\sigma_2}(z)$ and $y = \sigma_2(w_{\sigma_2}(z))$ and obtain

$$\begin{aligned} |\Gamma_f(\sigma_2)(z) - \Gamma_f(\sigma_1)(z)| &= |(f_s \circ \hat{\sigma_2} \circ w_{\sigma_2})(z) - \Gamma_f(\sigma_1)((f_{cu} \circ \hat{\sigma_2} \circ w_{\sigma_2})(z))| \\ &= |f_s(x, \sigma_2(x)) - \Gamma_f(\sigma_1)(f_{cu}(x, \sigma_2(x)))| \overset{(2.7)}{\leq} (\|T_s\| + 2\epsilon)|\sigma_2(x) - \sigma_1(x)| \\ &\leq (\|T_s\| + 2\epsilon)\|\sigma_2 - \sigma_1\|. \end{aligned}$$

Since this holds for each $z \in E^{cu}$, we conclude that

$$\|\Gamma_f(\sigma_2) - \Gamma_f(\sigma_1)\| \leq (\|T_s\| + 2\epsilon)\|\sigma_2 - \sigma_1\|.$$

From (2.3), we have $\|T_s\| + 2\epsilon < 1$, and Γ_f is a contraction as claimed. □

Applying Banach's fixed point theorem, we obtain the following result.

Proposition 1. *Assume (2.3) holds, then there exists a unique $\sigma^* \in \Sigma$ such that $\Gamma_f(\sigma^*) = \sigma^*$.*

2.4 Regularity of the fixed point

We assume $k \geq 1$. Hypotheses (1.2) and (1.3) imply that there is an $\epsilon > 0$ so that (2.3) holds and

$$\frac{(\|T_s\| + \epsilon)\|T_{cu}^{-1}\|^{k+1}}{\left(1 - \epsilon\|T_{cu}^{-1}\|\right)^{k+2}} < 1. \tag{2.8}$$

Our goal is to establish the following result.

Proposition 2. *If (2.3) and (2.8) hold, then the fixed point σ^* is $C^{k,1}$.*

Recall that

$$B_R^k = \{\sigma \in \Sigma \cap C^{k,1} :\ \|\mathrm{D}\sigma\| \leq 1,\ \|\mathrm{D}^2\sigma\| \leq r_2,\ ...,\ \|\mathrm{D}^k\sigma\| \leq r_k,\ \mathrm{Lip}(\mathrm{D}^k\sigma) \leq r_{k+1}\}.$$

Lemma A.2 shows that B_R^k is closed in Σ. Hence, Proposition 2 is proved provided we can show that there exists an $R = (r_2, \ldots, r_{k+1})$ with $r_j > 0$ such that $\Gamma_f(B_R^k) \subset B_R^k$, since this non-empty set must then contain the fixed point σ^*. We start with the following abstract result for the derivatives of compositions.

Lemma 2.3. *For each $\ell \geq 1$ and any functions G_1, G_2 and G_3 in C^ℓ, we have*

$$\|\mathrm{D}^\ell(G_1 \circ G_2 \circ G_3)\| \leq \|\mathrm{D}^\ell G_1\|\, \|\mathrm{D}G_2\|^\ell \|\mathrm{D}G_3\|^\ell + \|\mathrm{D}G_1\|\, \|\mathrm{D}^\ell G_2\|\, \|\mathrm{D}G_3\|^\ell + \|\mathrm{D}G_1\|\, \|\mathrm{D}G_2\|\, \|\mathrm{D}^\ell G_3\| + \mathcal{R}_\ell \quad (2.9)$$

where $\mathcal{R}_\ell$ depends only on $\|\mathrm{D}^j G_m\|$ for $1 \leq j \leq \ell - 1$ and $m = 1, 2, 3$.

Proof. We will show by induction on ℓ that

$$\begin{aligned} \mathrm{D}^\ell[(G_1 \circ G_2 \circ G_3)(x)] &= \mathrm{D}^\ell G_1(G_2(G_3(x)))[\mathrm{D}G_2(G_3(x)) \cdot \mathrm{D}G_3(x)]^\ell \\ &\quad + \mathrm{D}G_1(G_2(G_3(x))) \cdot \mathrm{D}^\ell G_2(G_3(x))[\mathrm{D}G_3(x)]^\ell + \mathrm{D}G_1(G_2(G_3(x))) \cdot \mathrm{D}G_2(G_3(x)) \cdot \mathrm{D}^\ell G_3(x) + \tilde{\mathcal{R}}_\ell(x), \end{aligned} \quad (2.10)$$

where $\tilde{\mathcal{R}}_\ell$ depends only on $\mathrm{D}^j G_m$ for $1 \leq j \leq \ell - 1$ and $m = 1, 2, 3$. The chain rules gives

$$\begin{aligned} \mathrm{D}[(G_1 \circ G_2 \circ G_3)(x)] &= \mathrm{D}G_1(G_2(G_3(x))) \cdot \mathrm{D}G_2(G_3(x)) \cdot \mathrm{D}G_3(x) \\ \mathrm{D}^2[(G_1 \circ G_2 \circ G_3)(x)] &= \mathrm{D}^2 G_1(G_2(G_3(x)))[\mathrm{D}G_2(G_3(x)) \cdot \mathrm{D}G_3(x)]^2 \\ &\quad + \mathrm{D}G_1(G_2(G_3(x))) \cdot \mathrm{D}^2 G_2(G_3(x))[\mathrm{D}G_3(x)]^2 \\ &\quad + \mathrm{D}G_1(G_2(G_3(x))) \cdot \mathrm{D}G_2(G_3(x)) \cdot \mathrm{D}^2 G_3(x), \end{aligned}$$

and (2.10) is true for $\ell = 2$. Suppose (2.10) holds, and differentiate it once more to get $D^{\ell+1}[(G_1 \circ G_2 \circ G_3)(x)]$. Since the only terms that contribute to the $\ell + 1$-derivatives are the first three terms of (2.10), it is not hard to see that (2.10) is true for $\ell + 1$. Taking norms in (2.10), we obtain (2.9). □

Lemma 2.4. *For each $\ell \geq 2$, we have*

$$\|\mathrm{D}^\ell w_\sigma\| = \|\mathrm{D}^\ell h_\sigma\| \leq \epsilon \|\mathrm{D}^\ell \sigma\| \left(\frac{\|T_{cu}^{-1}\|}{1 - \epsilon \|T_{cu}^{-1}\|} \right)^{\ell+1} + C_\ell$$

for some constant C_ℓ that depends only on T, $\|g\|_{C^\ell}$, and $\|\mathrm{D}^j \sigma\|$ for $1 \leq j \leq \ell - 1$.

Proof. Consider $h_\sigma = T_{cu}^{-1}(g_{cu} \circ \hat{\sigma} \circ w_\sigma)$. The estimate (2.9) shows that

$$\|\mathrm{D}^\ell h_\sigma(x)\| \leq \|T_{cu}^{-1}\| (\|\mathrm{D}^\ell g_{cu}\|\, \|\mathrm{D}\hat{\sigma}\|^\ell \|\mathrm{D}w_\sigma\|^\ell + \|\mathrm{D}g_{cu}\|\, \|\mathrm{D}^\ell \hat{\sigma}\|\, \|\mathrm{D}w_\sigma\|^\ell + \|\mathrm{D}g_{cu}\|\, \|\mathrm{D}\hat{\sigma}\|\, \|\mathrm{D}^\ell w_\sigma\| + \mathcal{R}_\ell).$$

We know that $\|\mathrm{D}g_{cu}\| < \epsilon$, $\|\mathrm{D}\hat{\sigma}\| \leq 1$, and $\|\mathrm{D}^\ell \hat{\sigma}\| = \|\mathrm{D}^\ell \sigma\|$. Thus, (2.4) implies that

$$\|\mathrm{D}^\ell h_\sigma\| \leq \|T_{cu}^{-1}\|\, \|\mathrm{D}^\ell g_{cu}\| \left(\frac{\|T_{cu}^{-1}\|}{1 - \epsilon \|T_{cu}^{-1}\|} \right)^\ell + \epsilon \|T_{cu}^{-1}\|\, \|\mathrm{D}^\ell \sigma\| \left(\frac{\|T_{cu}^{-1}\|}{1 - \epsilon \|T_{cu}^{-1}\|} \right)^\ell + \epsilon \|T_{cu}^{-1}\|\, \|\mathrm{D}^\ell h_\sigma\| + \mathcal{R}_\ell.$$

Since $1 - \epsilon \|T_{cu}^{-1}\| > 0$, we obtain

$$\begin{aligned} \|\mathrm{D}^\ell h_\sigma\| &\leq \frac{1}{1 - \epsilon \|T_{cu}^{-1}\|} \left(\|T_{cu}^{-1}\|\, \|\mathrm{D}^\ell g_{cu}\| \left(\frac{\|T_{cu}^{-1}\|}{1 - \epsilon \|T_{cu}^{-1}\|} \right)^\ell + \epsilon \|T_{cu}^{-1}\|\, \|\mathrm{D}^\ell \sigma\| \left(\frac{\|T_{cu}^{-1}\|}{1 - \epsilon \|T_{cu}^{-1}\|} \right)^\ell + \mathcal{R}_\ell \right) \\ &= \epsilon \|\mathrm{D}^\ell \sigma\| \left(\frac{\|T_{cu}^{-1}\|}{1 - \epsilon \|T_{cu}^{-1}\|} \right)^{\ell+1} + C_\ell \end{aligned}$$

where C_ℓ depends only on T, $\|g\|_{C^\ell}$, and $\|\mathrm{D}^j \sigma\|$ for $1 \leq j \leq \ell - 1$, but not on $\|\mathrm{D}^\ell \sigma\|$. Since $w_\sigma = T - h_\sigma$ and $\mathrm{D}^\ell T = 0$ for $\ell \geq 2$, we have $\|\mathrm{D}^\ell w_\sigma\| = \|\mathrm{D}^\ell h_\sigma\|$. □

We are now in a position to estimate derivatives of the graph transform.

Lemma 2.5. *For $\sigma \in \Sigma \cap C^\ell$, we have*

$$\|\mathrm{D}^\ell \Gamma_f(\sigma)\| \le \frac{(\|T_s\| + \epsilon)\|T_{cu}^{-1}\|^\ell}{\left(1-\epsilon\|T_{cu}^{-1}\|\right)^{\ell+1}} \|\mathrm{D}^\ell \sigma\| + \tilde{C}_\ell$$

for some constant $\tilde{C}_\ell$ that depends only on T, $\|g\|_{C^\ell}$, and $\|\mathrm{D}^j\sigma\|$ for $1 \le j \le \ell - 1$.

Proof. Since $\Gamma_f(\sigma) = f_s \circ \hat{\sigma} \circ w_\sigma$, the estimate (2.9) gives

$$\|\mathrm{D}^\ell(f_s \circ \hat{\sigma} \circ w_\sigma)\| \le \|\mathrm{D}^\ell f_s\| \, \|\mathrm{D}\hat{\sigma}\|^\ell \|\mathrm{D}w_\sigma\|^\ell + \|\mathrm{D}f_s\| \, \|\mathrm{D}^\ell\hat{\sigma}\| \, \|\mathrm{D}w_\sigma\|^\ell + \|\mathrm{D}f_s\| \, \|\mathrm{D}\hat{\sigma}\| \, \|\mathrm{D}^\ell w_\sigma\| + \mathcal{R}_\ell.$$

Using $\|\mathrm{D}\hat{\sigma}\| \le \|\mathrm{D}\sigma\| \le 1$, the estimates (2.4) and $\|\mathrm{D}f_s\| \le \|T_s\| + \epsilon$, and Lemma 2.4, we obtain

$$\begin{aligned}
\|\mathrm{D}^\ell(f_s \circ \hat{\sigma} \circ w_\sigma)\| \quad &\le \quad \|\mathrm{D}^\ell f_s\| \left(\frac{\|T_{cu}^{-1}\|}{1-\epsilon\|T_{cu}^{-1}\|} \right)^\ell + \|\mathrm{D}f_s\| \, \|\mathrm{D}^\ell\sigma\| \left(\frac{\|T_{cu}^{-1}\|}{1-\epsilon\|T_{cu}^{-1}\|} \right)^\ell \\
&\quad + \|\mathrm{D}f_s\| \left(\frac{\|T_{cu}^{-1}\|}{1-\epsilon\|T_{cu}^{-1}\|} \right)^{\ell+1} \epsilon\|\mathrm{D}^\ell\sigma\| + C_\ell + \mathcal{R}_\ell \\
&= \quad \frac{\|\mathrm{D}f_s\| \, \|T_{cu}^{-1}\|^\ell}{\left(1-\epsilon\|T_{cu}^{-1}\|\right)^{\ell+1}} \|\mathrm{D}^\ell\sigma\| + \tilde{C}_\ell \le \frac{(\|T_s\| + \epsilon)\|T_{cu}^{-1}\|^\ell}{\left(1-\epsilon\|T_{cu}^{-1}\|\right)^{\ell+1}} \|\mathrm{D}^\ell\sigma\| + \tilde{C}_\ell,
\end{aligned}$$

where $\tilde{C}_\ell$ depends only on T, $\|g\|_{C^\ell}$, and $\|\mathrm{D}^j\sigma\|$ for $1 \le j \le \ell - 1$. □

The next lemma shows that $\Gamma_f(B_R^k) \subset B_R^k$ for an appropriate R, thus finishing the proof of Proposition 2, and hence of Theorem 2.

Lemma 2.6. *There exists an $R = (r_2, r_3, \ldots, r_{k+1})$ with $r_j > 0$ such that $\Gamma_f(B_R^k) \subset B_R^k$.*

Proof. We proceed by induction. For $k = 1$, we have

$$B_{r_2}^1 = \{\sigma \in \Sigma \cap C^{1,1} : \ \|\mathrm{D}\sigma\| \le 1, \, \mathrm{Lip}(\mathrm{D}\sigma) \le r_2\}.$$

For each $\sigma \in B_{r_2}^1 \cap C^2$, Lemma 2.5 implies that

$$\mathrm{Lip}(\mathrm{D}\Gamma_f(\sigma)) \le \frac{(\|T_s\| + \epsilon)\|T_{cu}^{-1}\|^2}{\left(1-\epsilon\|T_{cu}^{-1}\|\right)^3} \mathrm{Lip}(\mathrm{D}\sigma) + \tilde{C}_2 \le \frac{(\|T_s\| + \epsilon)\|T_{cu}^{-1}\|^2}{\left(1-\epsilon\|T_{cu}^{-1}\|\right)^3} r_2 + \tilde{C}_2.$$

Using (2.8) and observing that $\tilde{C}_2$ does not depend on r_2, we see that the right-hand side is less than or equal to r_2 provided we choose r_2 such that

$$r_2 \ge \tilde{C}_2 \left[1 - \frac{(\|T_s\| + \epsilon)\|T_{cu}^{-1}\|^2}{\left(1-\epsilon\|T_{cu}^{-1}\|\right)^3} \right]^{-1}.$$

Since $B_{r_2}^1 \cap C^2$ is dense in $B_{r_2}^1$, this choice of r_2 ensures that $\Gamma_f(B_{r_2}^1) \subset B_{r_2}^1$. The same argument can now be used to show that we can recursively pick constants $r_j > 0$ so that $\Gamma_f(B_R^k) \subset B_R^k$. □

3 Center manifolds for vector fields

It remains to deduce Theorem 1 from Theorem 2. Recall the system (1.1)

$$\dot{u} = Au + h(u), \qquad u \in \mathbb{R}^n \tag{3.1}$$

where $h \in C^{k,1}$ for some fixed finite $k \geq 1$ with $h(0) = \mathrm{D}h(0) = 0$. We write $\mathbb{R}^n = E^{cu} \oplus E^s$, where E^{cu} and E^s are invariant under A with $\mathrm{Re}\,\mathrm{spec}(A|_{E^{cu}}) \geq 0$ and $\mathrm{Re}\,\mathrm{spec}(A|_{E^s}) < 0$. Using the Jordan normal form of A, we see that there is a $\tau > 0$ such that $T = \mathrm{e}^{A\tau}$ and $T_j := T|_{E^j} : E^j \to E^j$ with $j = cu, s$ satisfy

$$\|T_s\| < 1, \qquad \|T_s\|\,\|T_{cu}^{-1}\|^{k+1} < 1.$$

We fix this τ from now on, and let $\epsilon > 0$ be as in Theorem 2. Next, pick a C^∞-cutoff function $\chi : \mathbb{R} \to \mathbb{R}$ such that

$$\chi(r) = \begin{cases} 1 & 0 \leq r \leq 1 \\ 0 & r \geq 2 \end{cases} \qquad\qquad \chi' \leq 0,$$

and define $h_\rho(u) := \chi(\frac{|u|}{\rho})h(u)$ for each $\rho > 0$, then it is not difficult to see that $h_\rho(u) = h(u)$ for $|u| \leq \rho$ and that

$$L_\rho := \|h_\rho\|_{C^1(\mathbb{R}^n,\mathbb{R}^n)} \to 0 \quad \text{as} \quad \rho \to 0. \tag{3.2}$$

We now consider the system

$$\dot{u} = Au + h_\rho(u), \qquad u(0) = u_0 \in \mathbb{R}^n \tag{3.3}$$

and denote its solutions by $\Phi_t^\rho(u_0)$. Due to (3.2), solutions of (3.3) exist for all times, and we have

$$\Phi_\tau^\rho(u) = \mathrm{e}^{A\tau}u + g_\rho(u), \qquad g_\rho(u) := \int_0^\tau \mathrm{e}^{A(\tau-s)} h_\rho(\Phi_s^\rho(u_0))\,\mathrm{d}s.$$

Using again (3.2), it is not difficult to show that $\|g_\rho\|_{C^1} \to 0$ as $\rho \to 0$ for each fixed τ. In particular, there is a $\rho_0 > 0$ such that $\|g_{\rho_0}\|_{C^1} \leq \epsilon$, and Theorem 2 implies that $\Phi_\tau^{\rho_0}$ admits a center-unstable manifold of the form $W^{cu} = \mathrm{graph}(\sigma^*)$ of class $C^{k,1}$.

We now argue that W^{cu} is invariant not just under $\Phi_\tau^{\rho_0}$ but in fact under $\Phi_t^{\rho_0}$ for all $t \in \mathbb{R}$. Indeed, since $\mathrm{Lip}(\sigma^*) < 1$ by (2.6), it follows as in §2.3 that $\Phi_t^{\rho_0}(\mathrm{graph}(\sigma^*))$ is given by $\mathrm{graph}(\sigma_t^*)$ for some $\sigma_t^* \in \Sigma$ for each t sufficiently close to zero. Furthermore,

$$\Phi_\tau^{\rho_0}(\mathrm{graph}(\sigma_t^*)) = \Phi_\tau^{\rho_0}(\Phi_t^{\rho_0}(\mathrm{graph}(\sigma^*))) = \Phi_{\tau+t}^{\rho_0}(\mathrm{graph}(\sigma^*)) = \Phi_t^{\rho_0}(\mathrm{graph}(\sigma^*)) = \mathrm{graph}(\sigma_t^*),$$

and uniqueness of the fixed point of the graph transform implies that $\sigma_t^* = \sigma^*$ for all t close to zero, and hence for all $t \in \mathbb{R}$, as claimed.

Since the flows of (3.1) and (3.3) coincide for solutions that stay inside the ball of radius ρ, the manifold W^{cu} constructed above is a locally invariant center-unstable manifold of (3.1). Upon reversing the time direction in (3.1), we can construct similarly a local center-stable manifold of (3.1), and the desired local center manifold of (3.1) is then obtained as the intersection of the local center-stable and center-unstable manifolds. This completes the proof of Theorem 1.

4 Discussion

We end this paper with a few remarks. First, the regularity proof can be extended easily to the case where $h \in C^{k,\alpha}$ for some $\alpha \in (0, 1]$, that is, where $\mathrm{D}^k h(u)$ is only Hölder continuous in u for some positive Hölder exponent. Indeed, Lemma A.2 remains true in this more general situation, and we refer to [2, Lemma 6.1.6] for the proof. Second, we did not discuss the existence and regularity of invariant fibrations associated with the center, center-stable, and center-unstable manifolds. It might be possible to use the ideas in [2, Lemma 6.1.6] also for some of the regularity properties of these fibrations, but we did not check any details.

A Two technical lemmas

In this appendix, we give a proof of the global Lipschitz inverse function theorem and show that bounded closed balls in $C^{k,1}$ are also closed in C^0.

Lemma A.1 (Global Lipschitz inverse function theorem)**.** *Assume that $g, T : \mathbb{R}^n \to \mathbb{R}^n$ are such that T is linear and invertible, while g is Lipschitz with Lipschitz constant $\mathrm{Lip}(g) < \|T^{-1}\|^{-1}$ and $\|g\| < \infty$. The function $T + g$ is then invertible, and $(T + g)^{-1} =: T^{-1} - h$ is bounded and Lipschitz continuous with $\mathrm{Lip}(h) \leq \mathrm{Lip}(g)\|T^{-1}\|^2/(1 - \mathrm{Lip}(g)\|T^{-1}\|)$.*

Proof. For $x \in \mathbb{R}^n$, write $y = Tx + g(x)$, then there exists a function $h : \mathbb{R}^n \to \mathbb{R}^n$ with $x = T^{-1}y - h(y)$ if, and only if,

$$y = y - Th(y) + g(T^{-1}y - h(y)),$$

which is equivalent to

$$h(y) = T^{-1}g(T^{-1}y - h(y)).$$

We prove the existence of a solution h by Banach's fixed point theorem. Consider the closed set

$$H = \left\{ h : \mathbb{R}^n \to \mathbb{R}^n : \ \|h\| < \infty, \ \mathrm{Lip}(h) \leq \frac{\mathrm{Lip}(g)\|T^{-1}\|^2}{1 - \mathrm{Lip}(g)\|T^{-1}\|} \right\} \tag{A.1}$$

equipped with the C^0-norm and let $\mathcal{G}(h) := T^{-1} \circ g \circ (T^{-1} - h)$. We need to show that $\mathcal{G}(H) \subset H$ and that $\mathcal{G}$ is a contraction. For $h \in H$, and using the estimate for $\mathrm{Lip}(h)$ in (A.1), we have

$$\mathrm{Lip}(\mathcal{G}(h)) \leq \|T^{-1}\|\mathrm{Lip}(g)(\|T^{-1}\| + \mathrm{Lip}(h)) \leq \ldots \leq \frac{\mathrm{Lip}(g)\|T^{-1}\|^2}{1 - \mathrm{Lip}(g)\|T^{-1}\|}$$

and $\|\mathcal{G}(h)\| \leq \|T^{-1}\| \, \|g\| < \infty$, which shows that $\mathcal{G}(H) \subset H$. For any $h_1, h_2 \in H$ and $x \in \mathbb{R}^n$, we have

$$|\mathcal{G}(h_1)(x) - \mathcal{G}(h_2)(x)| \leq \|T^{-1}\| \, |g(T^{-1}x - h_1(x)) - g(T^{-1}x - h_2(x))| \leq \|T^{-1}\| \, \mathrm{Lip}(g)\|h_1 - h_2\| < \|h_1 - h_2\|.$$

Thus, $\mathcal{G}$ is a contraction in the complete metric space H, and consequently there exists a unique $h^* \in H$ with $\mathcal{G}(h^*) = h^*$. In particular, $T^{-1} + g$ is invertible , and its bounded and Lipschitz-continuous inverse is given by $T^{-1} - h^*$. □

Lemma A.2 ([2, Lemma 6.1.6])**.** *A bounded closed ball in $C^{k,1}(\mathbb{R}^n, \mathbb{R}^m)$ is also closed in $C^0(\mathbb{R}^n, \mathbb{R}^m)$.*

Proof. The result is easy to show for $k = 0$. We prove the result for the first derivative as the general case then follows easily by induction. It suffices to choose the ball

$$B = \{f \in C^1 : \ \|f\| < \infty, \ \|\mathrm{D}f\| \leq 1, \ \mathrm{Lip}(\mathrm{D}f) \leq r\}$$

where $r > 0$. Let $u_n \in B$ be a sequence such that $f_n \to f$ in C^0 as $n \to \infty$. We need to show that $f \in B$, that is, $f \in C^1$, $\|\mathrm{D}f\| \leq 1$ and $\mathrm{Lip}(\mathrm{D}f) \leq r$. For each $g \in C^{1,1}$ and $x, y \in \mathbb{R}^n$ we have

$$g(x + y) - g(x) = \int_0^1 \mathrm{D}g(x + \tau y)\mathrm{d}\tau y$$

and therefore

$$|g(x+y) - g(x) - \mathrm{D}g(x)y| = \left| \int_0^1 [\mathrm{D}g(x + \tau y) - \mathrm{D}g(x)]\mathrm{d}\tau \, y \right| \leq |y| \int_0^1 \|\mathrm{D}g(x+\tau y) - \mathrm{D}g(x)\|\mathrm{d}\tau \leq \mathrm{Lip}(\mathrm{D}g)|y|^2.$$

Hence,

$$|\mathrm{D}g(x)y| \leq |g(x + y) - g(x)| + |g(x + y) - g(x) - \mathrm{D}g(x)y| \leq 2\|g\| + \mathrm{Lip}(\mathrm{D}g)|y|^2.$$

With $g = f_n - f_m$, we obtain

$$\|\mathrm{D}f_n(x) - \mathrm{D}f_m(x)\| \le \frac{2\|f_n - f_m\|}{|y|} + 2r|y|$$

and, since this inequality holds for all y,

$$\|\mathrm{D}f_n(x) - \mathrm{D}f_m(x)\| \le \min_y \left\{ \frac{2\|f_n - f_m\|}{|y|} + 2r|y| \right\} = 2\sqrt{r}\|f_n - f_m\|^{1/2}.$$

Thus, since f_n is a Cauchy sequence, so is $\mathrm{D}f_n$, and we conclude that f is differentiable with $\mathrm{D}f_n \to \mathrm{D}f$ in C^0. In particular, $\|\mathrm{D}f\| \le 1$. Finally, for each $\epsilon > 0$, there is an N such that

$$\|\mathrm{D}f(x) - \mathrm{D}f(y)\| \le \|\mathrm{D}f(x) - \mathrm{D}f_n(x)\| + \|\mathrm{D}f_n(x) - \mathrm{D}f_n(y)\| + \|\mathrm{D}f_n(y) - \mathrm{D}f(y)\| < \epsilon + r|x-y|$$

for all $n \ge N$ and all $x, y \in \mathbb{R}^n$. Since $\epsilon > 0$ is arbitrary, we conclude that $\|\mathrm{D}f(x) - \mathrm{D}f(y)\| \le r|x-y|$, which means that $\mathrm{Lip}(\mathrm{D}f) < r$. This shows that B is closed in $C^0(\mathbb{R}^n, \mathbb{R}^m)$ as claimed. □

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