Reading Group: Spatial Dynamics

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I. Introduction

What is spatial dynamics? → View one unbound space variable as evolution variable to gain insight into solution profiles

Example: Travelling waves in 1D

\[ u_t = u_{xx} + f(u) \quad (x \in \mathbb{R}) \]

\[ u(x,t) = u(x - ct) \quad \rightarrow \]

\[ -c u_g = u_{gg} + f(u) \quad (g \in \mathbb{R}) \]

\[ \frac{\epsilon}{\epsilon^2} u_{gg} + cu_g + f(u) = 0 \]

\[ \mathcal{T}: \mathcal{C}(\mathbb{R}) \times \mathbb{R} \rightarrow \mathcal{C}(\mathbb{R}) \]

Zeros of functions

- Implicit function theorems
- Multipeak

- Multi-pulse

1-pulse, 2-pulse

- Not close in function space

\[ \epsilon \ll 1 \quad \text{fast-slow} \]

\[ \text{IFT works...} \]

\[ \epsilon \gtrsim 1 \text{ does not break linearity in speed } c \]

\[ \text{not easy to analyse} \]

\[ \left( v \right)_g = \left( -[cu_g + f(v)] \right) \]

dynamical system in \( g \)

- Homoclinic orbit
- Heteroclinic orbit
- Periodic orbit

- Dynamical systems techniques

- Point-wise close
Spatial dynamics more than dimensions:

- Stationary solutions to PDEs on $\mathbb{R}^n \times \mathbb{R}$, $n \geq 2$
- Time-periodic solutions to PDEs on $\mathbb{R} \times \mathbb{R}$, $n = 1$

Example:

\[
\begin{align*}
    \dot{u} &= \Delta u + (1 + \mu) u + u^2 + \varepsilon \cos(\omega x) \quad (x, t) \in \mathbb{R} \times (0, T), \quad > 0 \\
    u(x, 0^+) &= u(x, T^+) = 0 \quad \forall (x, t)
\end{align*}
\]

Stationary solutions →

(1) \[
\begin{align*}
    \Delta u + (1 + \mu) u + u^2 + \varepsilon \cos(\omega x) &= 0 \quad (x, t) \in \mathbb{R} \times (0, T) \\
    u(x, 0) &= u(x, T) = 0 \quad \forall x
\end{align*}
\]

- $\varepsilon = 0 \rightarrow u = 0$ & a solution:

  \[
  v(x, t) = e^\lambda t \sin(\lambda x)
  \]

  Spectra: \[\lambda \in \mathbb{R}\]

  \[
  \begin{array}{c|c|c|c|c}
    \lambda < 0 & \lambda = 0 & \lambda > 0 \\
    \hline
    (-\infty, 0) & \{0\} & (0, \infty)
  \end{array}
  \]

- Fix $\mu < 0$, then there is an $\varepsilon > 0$ so that (1) has a unique small bounded solution $u(\varepsilon)$ near $u = 0$ for each $\varepsilon$ with $|\varepsilon| < \varepsilon_0$.

- What happens near $\varepsilon = 0$?

\[
\begin{align*}
    \frac{\partial u}{\partial t} &= \Delta u + (1 + \mu) u + u^2 + \varepsilon \cos(\omega x) = 0 \\
    (u, v) &= \left(\begin{array}{c}
    u \\
    v
\end{array}\right) \\
    \frac{\partial}{\partial x} \left(\begin{array}{c}
    u \\
    v
\end{array}\right) &= \left(\begin{array}{c}
    0 \\
    -\Delta u - 1 - \mu
\end{array}\right) \\
    \varepsilon \cos(\omega x)
\end{align*}
\]

where $(u, v)(x) \in Y$ appropriate space of functions in $y$ on $(0, T)$ with $u(0) = u(\varepsilon) = 0$ (Dirichlet in $y$-variable).

We write (1) as

(2) \[
\begin{align*}
    \left(\begin{array}{c}
    u \\
    v
\end{array}\right) &= \left(\begin{array}{cc}
    0 & 1 \\
    -\Delta u - 1 - \mu & 0
\end{array}\right) \left(\begin{array}{c}
    u \\
    v
\end{array}\right) + \varepsilon \cos(\omega x)
\end{align*}
\]

Spectrum of $A(\mu)$:

\[
\begin{align*}
    \lambda &= \left(\begin{array}{cc}
    0 & 1 \\
    -\Delta u - 1 - \mu & 0
\end{array}\right) \left(\begin{array}{c}
    u \\
    v
\end{array}\right)
\end{align*}
\]

- \[\lambda = \frac{1}{2} (\Delta u + 1 + \mu) u = u^2\] \[u(0) = u(\varepsilon) = 0\]

- $u = \sin(\lambda x)$ with $\sin\lambda > 0$ and we get

\[
\begin{align*}
    v_k &= \pm \sqrt{k^2 - 1 - \mu} \\
    \lambda_k &= \sqrt{k^2 - 1 - \mu}
\end{align*}
\]
two-dimensional center manifold near $U = 0$ for $(p_1, \xi_0) \neq 0$
tangent to center eigenspace, smooth in $(U, \xi, \epsilon)$.

$[\text{Kirchhoff '85}], [\text{Fisher '82}], [\text{Hsiue '85}, ...]$

1) $U = \begin{pmatrix} R \end{pmatrix} \sin \theta + O((|A|+|E|+|M|)^2)$

rescaling: $a = \sigma A, \quad \rho = -b^2, \quad R = \frac{\epsilon^2 B}{ \cos(\xi \theta)}, \quad \omega = \frac{1}{\epsilon}$

$L_{b, \epsilon} - e + \frac{\epsilon}{b} Q^2 + \frac{4 \epsilon}{b^2} \cos(\xi \theta) + O(\epsilon) = 0$

[Assume $L = \omega e^{\phi \epsilon}$]

2) Why is it non-trivial to obtain a center manifold for (3)?

$U(x) = e^{U(x)} \begin{pmatrix} 1 \\ U_0 \end{pmatrix}$ satisfy $U_0 = (\Gamma(x))U$ for $0 < x < 1$

General solutions, for $y > 0$

$U(x) = \sum_{k=2}^{\infty} a_k e^{-k^2 x} \begin{pmatrix} \sin \theta \\ \frac{\epsilon \sin \theta}{k} \end{pmatrix} + \sum_{k=2}^{\infty} b_k e^{-k^2 x} \begin{pmatrix} 1 \\ -\frac{\epsilon \sin \theta}{k} \end{pmatrix}$

No convergence for $x > 0$

(2) or (3) are not well-posed as initial-value problems / dynamical systems.

Idea for proof of existence of center manifolds:

$U_1 \in E^c = \text{center eigenspace} = \text{span} \{ U \sin \theta; \ U e^{\phi \epsilon} \}$

$U_2 \in E^h = \text{hyperbolic eigenspace} = \text{span} \{ U \sin \theta; \ U e^{\phi \epsilon}, k \geq 2 \}$

$U_1 = \mathcal{F}^c(U_1) + \mathcal{W}^c(U_0, U_2; \epsilon)$

$U_2 = \mathcal{F}^h(U_2) + \mathcal{W}^h(U_0, U_2; \epsilon)$

We think of $\mathcal{W}^c, \mathcal{W}^h$ as being truncated so that both functions are small independently of $(U_0, U_2)$.

(This can be achieved by multiplying with appropriate cut-off functions.)
any bounded or mildly growing solution to (4) must satisfy the integral equation

\[ U_1(t) = e^{R(t)} U^0 + \int_0^t e^{R(t-s)} W_1(U(s)+U_1(s)) \, ds \]

\[ U_2(t) = \int_0^t e^{R(t-s)} W_2(U(s)+U_1(s)) 
= \int_0^t G(s+U_1(s)) W_2(U(s)+U_1(s)) \, ds \]

where \( G(t) = \begin{cases} e^{R(t)} & \text{if } t < 0 \\ -e^{R(t)} & \text{if } t > 0 \end{cases} \) with \( \|G(t)\| \leq ke^{-\lambda t} \) for \( \lambda \in \mathbb{R} \)

apply contraction-mapping principle
to get unique fixed-point
which defines Lipschitz continuously on \( U^0 \).

Summary:

\[
\begin{align*}
\begin{cases}
\partial U + (1+\mu)U + U^2 + E \cos(\omega x) &= 0 & (x, t) \in \mathbb{R} \times (0, T) \\
U(x, 0) &= 0 = U(x, T)
\end{cases}
\end{align*}
\]

Linearized problem:

\[
\begin{align*}
\partial U + (1+\mu)U &= \lambda U \\
U(x, 0) &= 0 = U(x, T)
\end{align*}
\]

seek solutions \( U(x, \lambda) = e^{\lambda x} \sin kx \) \( (k \neq 0) \)

\[
\lambda = \nu^2 - \pi^2 + \mu > 0, \quad k \neq 0
\]

\[
\lambda = -\nu^2 \quad \Rightarrow \quad U = (e^{-\nu^2 t} + 1 + \nu) \quad \text{for } \nu \in \mathbb{R}
\]

\[
\lambda = 0 \quad \Rightarrow \quad U = \pm \sqrt{-\nu^2 - 1 + \mu} \quad \nu \neq 0
\]

\[
U = \begin{pmatrix} 0 \\ -e^{-\nu^2 t} - 1 + \nu \end{pmatrix} \quad \mu \neq 0
\]

\[
\lambda = 0 \quad \text{in the essential spectrum} \quad \Rightarrow \quad \nu \in \mathbb{R} \quad \text{is in spectrum of} \quad \begin{pmatrix} 0 & 1 \\ -e^{-\nu^2 t} - 1 + \nu & 0 \end{pmatrix}
\]

a) \( U = \partial U + (1+\mu)U + U^2 + E \cos(\omega x) \quad \text{derivation amplitude equation} \)

b) \( (\nabla)_k = \begin{pmatrix} 0 & 1 \\ -e^{-\nu^2 t} - 1 + \nu & 0 \end{pmatrix} \quad \text{derivation vector field on center manifold} \)

\[ \Rightarrow \quad \text{stationary amplitude equation from a) = scaled vector field from b} \]
Remarque

\[ \Delta u + (1+i \gamma) u + u^2 \in \cos \omega x = 0 \]
\[ u(x, 0) = 0 = v(x, \pi) \]

restricted to \( \mathbb{R} \)-periodic functions in \( x \)

\[ \rightarrow u(x, y) = e^{i \omega x \sin \nu y} \quad \forall \nu \in \mathbb{Z}, \nu \geq 1 \]

\( \lambda = -u^2 - \nu^2 + 1 + \gamma \) since \( \nu \in \mathbb{Z}, \nu \geq 1 \).

→ can use FFT
Consider
\[ \dot{U} = F(U, \varepsilon) \quad U \in \mathbb{R}^n, \varepsilon \in \mathbb{R}^p \]
where \( F(0, \varepsilon) = 0 \) and \( \text{spec} (D_u F(0, \varepsilon)) \subset i\mathbb{R} \neq \emptyset \). We then reformulate \( U \) as
\[ \begin{pmatrix} U \\ \gamma \end{pmatrix} = \begin{pmatrix} F(U, \varepsilon) \\ 0 \end{pmatrix} \quad \text{or} \quad \dot{U} = f(U) \quad U = (U, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}^p = \mathbb{R}^N \]
Note that \( D_u F(0, \varepsilon) = \begin{pmatrix} D_u F(0, \varepsilon) & D_\varepsilon F(0, \varepsilon) \\ 0 & 0 \end{pmatrix} \)
it therefore suffices to construct center manifolds \( E_0 \).

We need to use a cut-off function \( \chi \) which we choose according to
\[ \chi \]
Replacing (2) by
\[ \dot{U} = D_u F(0, \varepsilon) U + \left[ \mathbb{I}(U) - D_u G(0, \varepsilon) - D_u \gamma(0) \right] X \left( \frac{\varepsilon}{\varepsilon} \right) \]
We observe that (3) and (4) coincide for \( 1 < \varepsilon < \varepsilon_0 \).

Lemma. Let \( G \in C^k \) with \( G(0) = D_0 G(0) = 0 \), then \( \gamma(0) := G(U) X \left( \frac{\varepsilon}{\varepsilon} \right) \) satisfies
\[ \gamma \in C^k \quad \text{and} \quad \| \gamma \| = 0 \quad \text{as} \quad \varepsilon \to 0. \]

(3) We consider from now on the system
\[ \dot{U} = A U + B \gamma(U) \quad U \in \mathbb{R}^n, \quad B \in \mathbb{R}^{n \times p} \quad B(0) = D_0 G(0) = 0, \quad \text{spec} (A) \subset i\mathbb{R} \neq \emptyset \]
with \( \| B \| \) sufficiently small.

\[ \begin{aligned}
\begin{pmatrix} U \end{pmatrix} & \text{stable} \\
\begin{pmatrix} \gamma \end{pmatrix} & \text{unstable}
\end{pmatrix}
\]

\[ A = \begin{pmatrix} A^R & 0 \\ 0 & A^S \end{pmatrix}, \quad B = \begin{pmatrix} B^S \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]
\[ B \begin{pmatrix} E^R & E^S \end{pmatrix} \]

Stable with projections \( P^R, P^S, P^U \) onto \( E^R, E^S, E^U \), respectively.

Theorem. Let \( \beta > 0 \) be given, then there is \( \varepsilon_0 > 0 \) such that the following is true for each \( \gamma \in C^k \) with \( \| \gamma \| \leq \beta \).

\[ W^c = \{ U = (\varepsilon U, \varepsilon\gamma(U)) : \varepsilon \to \infty \text{ where } U(0) = U_0 \text{ and } U(\varepsilon) \text{ satisfies } (*) \} \]
\[ = \text{graph} \left( \mathbb{I} \right) U = U^c + \left( \mathbb{I} - V \right) U^c \left( U^c \right)^T \gamma \]

\( B \) \( \sigma \)-opposite Lipschitz continuous function \( \mathbb{I} : E^c \to E^c \).Furthermore, \( W^c \) is invariant under (*). If \( \gamma \in C^k \) for some \( k \geq 1 \), \( \varepsilon \to 0^+ \), then \( \gamma \in C^{k+1} \).

(Note: The theorem is also true for \( \alpha = 0 \).)
Outline of the Proof:

\[ X_0 = \{ u \in C^0(\Omega) \mid \text{sup}_{\Omega} e^{-\nu t} u(0) < \infty \} \]

\[ U(0) \text{ lies in } U_{0}^{C} \]

(5) \[ U(t) = e^{\nu t} p_{\Omega} u_{0} + \int_{0}^{t} e^{\nu (t-s)} p_{\Omega} g(u(s)) \, ds + \int_{0}^{\infty} e^{\nu (t-s)} p_{\Omega} g(u(s)) \, ds + \int_{0}^{\infty} e^{\nu (t-s)} p_{\Omega} g(u(s)) \, ds \]

where \[ U_{0}^{C} = U(0) \text{ (since the integral eqn. differs from variation of constant formula by) } \]

\[ e^{-\nu t} p_{\Omega} u_{0} \text{ and } e^{-\nu t} p_{\Omega} \mu \]

Therefore, suffices to find solutions df (5) for \[ U_{0}^{C} \in U_{0}^{C} \text{ at t = 0} \]

\[ B(t) = \begin{cases} e^{\nu t} p_{\Omega} & t > 0 \\ 0 & t < 0 \end{cases} \]

so that (5) becomes

(6) \[ U(t) = e^{\nu t} p_{\Omega} u_{0} + \int_{0}^{t} e^{\nu (t-s)} p_{\Omega} g(u(s)) \, ds + \int_{0}^{\infty} B(t-s) p_{\Omega} g(u(s)) \, ds \]

we write \( u(t) \) as

(7) \[ U = G(u, u_{0}) \]

Lemma 1 \[ \mathcal{B}_{1} : X_{0} \rightarrow X_{0} \text{ \quad } U_{0}^{C} \rightarrow U_{0}^{C} \text{ \quad with } [\mathcal{B}_{1}]u(t) = \int_{0}^{t} e^{\nu (t-s)} p_{\Omega} g(u(s)) \, ds + \int_{0}^{\infty} B(t-s) p_{\Omega} g(u(s)) \, ds \]

(8) \[ \text{Linear and bounded with } [\mathcal{B}_{1}]u(t) = \left( \frac{\nu}{\gamma} + \frac{2k_{0}}{\gamma} \right) U_{0}^{C} \]

Proof \[ \text{For } \text{as } t \rightarrow \infty \]

Lemma 2 \[ \mathcal{H}_{1} : E^{0} \rightarrow X_{0} \quad U_{0}^{C} \rightarrow H(0) \text{ with } [\mathcal{H}_{1}]u(t) = e^{\nu t} p_{\Omega} u_{0} \text{ is linear and bounded by } \nu U_{0}^{C}. \]

Lemma 3 \[ \text{Let } G \in E_{0}, \text{ then } G : X_{0} \rightarrow X_{0} \text{ \quad with } [G(u)](t) = G(u(t)) \text{ is well defined and bounded with } \text{lip}(G) \leq G(0)_{1}, \text{lip}(u_{0}, u_{1}) \text{lip}(u_{2}) \]

Proof \[ \text{Firstly, } G(u, u_{0}) = \sup_{t \in \mathbb{R}} e^{-\nu t} G(u(t)) \leq \sup_{t \in \mathbb{R}} e^{-\nu t} G(u_{0}) \leq G(0)_{1}, \text{ for all } u \in X_{0} \]

and \[ G(u(t)) \text{ is continuous, so } G \text{ is well defined.} \]

\[ G(u(t)) = \sup_{t \in \mathbb{R}} e^{-\nu t} |G(u(t))| \text{ \quad and } \text{lip}(G) \leq \sup_{t \in \mathbb{R}} e^{-\nu t} |G(u(t))| \leq \sup_{t \in \mathbb{R}} e^{-\nu t} |G(u_{0})| \text{ \quad for all } u \in X_{0} \]

\[ \leq \sup_{t \in \mathbb{R}} e^{-\nu t} |G(u_{0})| \text{ \quad for all } u \in X_{0} \]
In summary,

\[ g(u, v) = u + g(v) \]

\[ g : \mathbb{E} \times X \rightarrow X \]

well-defined, continuous, and
locally continuous in \( u \) with

\[ \| g(u, v) - g(u, v') \| \leq \frac{K_2}{\lambda} + \frac{2K_3}{\lambda^2} \| v - v' \| \]

uniformly in \( v \in E^2 \)

**Lemma 4**

Assume \( \| g \| \leq \frac{1}{2} \left( \frac{K_2}{\lambda} + \frac{2K_3}{\lambda^2} \right) \). Then \( u = g(u, v) \) has a unique fixed point \( \Phi(v, u) \) in \( X_u \). Let \( (u, v) \) be any point \( \in \mathbb{R}^3 \) and \( \phi : E \rightarrow X \) is Lipschitz continuous in \( u \).

**Proof**

Apply uniform contraction mapping principle:

\[ \| g(u, v) - g(u', v) \| = \| f(u, v) - f(u', v) \| \leq K_1 \| u - u' \| \]

for all \( u, u' \in X_u \).

1. **Fixed Point is Unique**: Let \( v = g(u, v) \) be an element of \( X_u \) such that \( \| u - u' \| = \| v - v' \| = 0 \).

2. **Consider** \( u = g(u, v) \) then \( \| u - u' \| = \| v - v' \| = 0 \)

3. **Let** \( \Phi(v, u) \) denote the fixed point, then

\[ \Phi(v, u) = g(\Phi(v, u)) \]

**Lemma 5**

The map \( \Pi^2(v) = \Pi^2(v) \) given by the above mapping.

**Proof**

By construction (uniqueness follows from uniqueness).

**Lemma 6**

The map \( \Gamma : \mathbb{E} \times X \rightarrow X \), \( u \rightarrow g(u) \) is \( e^K \)-like if \( g \in C^K(e^K) \).

**Proof**

Let \( \Gamma : \mathbb{E} \times X \rightarrow X \), \( u \rightarrow g(u) \) is \( e^K \)-like if \( g \in C^K(e^K) \) for any \( \varepsilon > 0 \).

\[ \| g(u) - g(u') \| \leq \varepsilon \| u - u' \| \]

\[ \varepsilon \leq \sup_{t \in [0, 1]} e^{-t \| u \|} \int_0^1 \| g'(w) \| \, dt \quad \text{for} \quad \varepsilon > 0 \]

For \( \varepsilon > 0 \), choose \( \varepsilon > 0 \) so that \( 2 \| g \| < \varepsilon \).

Next, choose \( \delta > 0 \) so small that \( \| g(u) - g(u') \| < \delta \) for all \( \| u \| < \delta \) and \( \| u \| < \delta < e^\delta \).
Smoothness of $T^c$: [Henry: Springer 1981 (66)]

\[ Y = \{ g: E^c \to E^c \}_{\text{is}}, \quad L^c \leq 1 \]  

$G: Y \to Y$, $g \mapsto G(g)$ so that $T^c = G(T^c)$.

$B = \{ g \in Y; g \in C^\infty \text{ with } 16|_{u_k} \leq b \}$ for $u_k$ with $\min\{u_1, \ldots, u_{2^n}\} \geq 0$.

(i) Show that $G: B \to B$ for an appropriate $b > 0$.

(ii) Show that $B$ is closed in $Y$.

**Construction of $G$:**

$G: Y \to Y \quad \exists! \text{ such } u^c = T^c(u_0) \quad \text{of } u^c(s) = e^{a^c} u_0^c + \int_0^s e^{a^c(t-s)} p^c g(u^c(s) + \epsilon(u^c(s))) \, ds \text{ in } X_g$

\[ [G(g)](u_0^c) = \int_0^\infty \rho(s) \, p^c g(u_0^c + \epsilon(u_0^c)) \, ds \]

1) $G: Y \to Y$ is well defined.

2) $G: Y \to Y$ is a contraction.

3) $G(B) \subseteq B$ for an appropriate $b$. 
Fronts in reaction-diffusion systems that undergo (centered) instabilities:

\[ u(x,t) = e^{i\lambda t} \]

\[ \text{Front (G > 0)} \]

Spectrum of linearized field \( U \) or \( U_t \)

Spectrum in comoving frame \( s = x - c_\lambda t \)

\[ \text{\( g \) resembles \( a \) Hopf bifurcation, but now with a continuum of modes crossing} \]

\[ U_t = DU_y + C U_y + f(U; x) \quad g \in \mathbb{R}, U \in \mathbb{R}^N \]

Critical temporal frequency \( \omega_c \). No temporal period \( \frac{2\pi}{\omega_c} \to T \)

1) Spatial dynamics:

Semi-flow of (1) \( \Phi_t(U; c, p) \) (can be done \( \Phi_t \in \mathbb{C} \))

\[ \Phi_t(0, 0, 0) = 0 \quad \forall t \]

\[ D_x \Phi_0(0, 0, 0) : \]

\[ \text{Con use amplitude equation (complex Ginzburg-Landau equation) to describe the evolution of small-amplitude patterns:} \]

\[ U(x, t) \]

\[ U(x, t) = \Phi(x, t; p) \]

2) Spatial dynamics:

Want to find solutions \( U(x, t) \) of (1) with temporal period \( \frac{2\pi}{\omega_c} \) (or period close to \( \frac{2\pi}{\omega_c} \)) which are close to \( \Phi_0(x) \) \( \forall t \):

\[ \frac{d}{dt} \left[ \frac{U_t - CV - f(U; x)}{U \omega_c} \right] = \frac{U_t}{U \omega_c} \]

with \( U(x) : \omega_c \)-periodic function in \( t \) for each fixed \( s \).

\[ \text{Temporal} \]

\[ \text{Spatial} \]

\[ \text{Impermeable} \]

\[ \text{Stable rest state} \]

\[ \text{Destabilizing Hopf} \]

\[ \text{Correct temporal period} \]

\[ \text{Destabilizing rest state} \]
bifurcation behind front

time-dependent functions

bifurcation ahead of front
\( u_t = DU_{xx} + f(u, \nu) \quad x \in \mathbb{R}, \nu \in \mathbb{R} \)

We are interested in travelling waves, \( u(x, t) = U(x-ct) \), and therefore consider eqn. (1) in the comoving frame \( s = x-ct \) in which (1) becomes

\( u_t = DU_{ss} + f(u, \nu) \quad \nu \in \mathbb{R} \).

(1.1) Assume that \( U(s, t) = \tilde{U}(s) \) is a stationary solution of (2) to \( c = C_0 > 0 \) at \( \nu = 0 \).

We also assume that

(i) \( \tilde{U}(s) \rightarrow U_2^0 \) as \( s \rightarrow \pm \infty \)

(ii) \( \det \tilde{f}_u(U_2^0, \nu) \neq 0 \).

In particular, the homogeneous stationary \( U(x, t) = U_2^0 \) of (1) (or U) at \( \nu = 0 \) persist for all \( \nu \) close to zero \( U_2 \) and is the only \( f(U_2^\nu, \nu) = 0 \) which we can solve (locally uniquely) for \( U = U_2(\nu) \) with \( U_2(0) = U_2^0 \).

Consider the linearisation

\( u_t = DU_{xx} + f_u(U_2(\nu), \nu) u \)

at \( U^2 \). We seek solutions of the form \( e^{\lambda t + i kx} U_0 \) with \( U_0 \neq 0 \) which exist iff

\[ d^0_\nu(\lambda, ik \nu) = \det (-D^2 + f_u(U_2(\nu), \nu) - \lambda I) \neq 0. \]

(1.2) [Diagnosis: We assume that there are \( \lambda, \tilde{\nu} \neq 0 \) so that the following is true for \( \nu = 0 \):

(i) \( d^0_\nu(\lambda, ik \nu) = 0 \) for some \( \nu \in \mathbb{R} \), \( \lambda \in \mathbb{C} \) implies \( Re \lambda < -\tilde{\nu} \).

(ii) \( \exists \lambda > 0 \) such that

* \( d^0_\nu(\lambda, ik \nu) = 0 \) for some \( \nu \in \mathbb{R} \setminus \{ \nu_0 \} \), \( \lambda \in \mathbb{C} \) implies \( Re \lambda < -\tilde{\nu} \).

* \( d^0_\nu(\lambda, ik \nu) = C_1 \left[ -\lambda + ik \nu - \tilde{\nu} \right] + C_2 \left( \nu^2 + \lambda^2 + \lambda(1+1) \nu + (1+1) \nu^2 \right) \)

for some \( C_1, C_2, \tilde{\nu} > 0 \), for \( \nu \in \mathbb{R} \setminus \{ \nu_0 \} \).

In particular, we have

\[ \lambda = \lambda_0^0(ik \nu) = -\tilde{\nu} + i k \nu^2 + \nu + o(\nu). \]

satisfies \( d^0_\nu(\lambda, ik \nu) = 0 \).

These assumptions on \( U_2 \) in the laboratory frame carry over immediately to the comoving frame.

Conclude

\( u_t = DU_{xx} + f_u(U_2(\nu), \nu) u \)

then non-trivial solutions of the form \( e^{\lambda t + i k x} U_0 \) exist, \( \lambda < 0 \) for \( \nu = 0 \).

\[ d^0_\nu(\lambda, ik \nu) = \det (-D^2 + f_u(U_2(\nu), \nu) - \lambda I) \neq 0 \rightarrow \lambda < 0. \]

Thus, \( \lambda = \lambda_0^0(ik \nu) = \nu^2 + i k \nu^2 - \tilde{\nu} + o(\nu). \)

Hence, \( \lambda^0 \) becomes

\[ \lambda = \lambda_0^0(ik \nu) = \nu^2 + i k \nu^2 - \tilde{\nu} + o(\nu). \]

in the frame moving with speed \( c \).

\[ \nu \near \nu_0 \]

\[ k \]

\[ \mathbb{R} \]
\[ X(x) = \rho + \text{inc.} - \text{sc.} = \rho + \text{inc.} + d(x) \]

Corresponds to: \( u_0 \rightarrow x_0, \text{inc.} \rightarrow \text{finc.} \) (Fourier transform), so
\[ u_0 = x_0 + c \text{inc.} + \mu \nu \]
\[ c > 0 \] \quad \text{transport to the left.}

Group velocity:
\[ c_{\text{group}} = \frac{\partial \nu}{\partial \omega} \text{ at } \omega = 0 \quad \text{in the direction of transport.} \]

Remark: \( \lambda^2(\omega_0) = \omega_0 c_{\text{group}} = \omega_0^2 + 0 \) \quad anticipated temporal frequency of bifurcating patterns

\[ \omega_0 = \omega_0 c_{\text{group}} \quad \text{frequency.} \]

Return to \( \omega_0 = \omega_0 c_{\text{group}} \). Linearizing \( (3) \) about \( y = 0 \) \quad with \( c = c_0 \), we get
\[ \dot{y}_0 = D \dot{y}_0 + c \ddot{y} + f(y) \]

(4) We assume that \( X \) has an isolated eigenvalue \( \lambda = 0 \) \quad with eigenfunction \( \phi_1 \) with
\[ \text{simple, and that any other isolated eigenvalue } \lambda \text{ has } \text{Re} \lambda < -\delta. \]

This implies

\[ \text{Spec}(X) \text{ on } L^2(\mathbb{R}) \]

\[ \text{Re} \lambda \]

\[ \lambda \]

\[ \text{im} \lambda \]

\[ \text{simple} \]

We need to exclude that \( \lambda = \pm \text{im} \delta \) is an embedded eigenvalue but will state this assumption later.

Special dynamics

We have \( \dot{y}_0 = \omega_0 c_{\text{group}} \) \quad and seek solutions of \( (3) \),
\[ u_0 = D^2 u_0 + \text{inc.} + f(u, y) \]

that have temporal period \( 2\pi / \omega_0 \) \quad for \( c = c_0, \mu = \nu \) \quad and we close, in an appropriate sense, to \( X(\delta) \).

Thus, we consider

\[ (U) = \begin{pmatrix} V \\ u_0 \end{pmatrix} \quad \text{on } X = H^1(S') \times H^1(S'), \quad S' = [0, \frac{\pi}{\omega_0}] / \omega \]

where
\[ H^1(S') = \left\{ u(x) = \sum_{\omega \in \Omega} a_n e^{i\lambda_n} \right\} \quad \text{with norm} \quad \| u \|_{H^1} = \left( \sum_{\omega \in \Omega} (1 + \lambda_n^2) a_n^2 \right)^{1/2} \]

Note: \( H^1(S') = C^2(S') \) and \( f(u, y) \) makes sense for \( u \in H^1(S') \).
Eqn. (7) is equivalent under the action
\[ T_\theta : Y \to Y, \quad u(x) \mapsto u(x - \theta) \] 
shift in time

Define \( \gamma_0 = \{ u \in Y : u(0) = 0 \in \mathbb{R}^N \} \), \( \gamma_1 = \{ u \in Y : u \text{ does not depend on time} \} \) then (7) leaves \( \gamma_0 \) invariant and reduces to the travelling wave case

\[ V_k = \begin{pmatrix} \frac{\partial}{\partial y} \left[ F(u; y) + CV \right] \end{pmatrix} \]

We begin by studying (8) with \( \nu = 0 \) one case:

\[ \begin{pmatrix} \nu \end{pmatrix} = \begin{pmatrix} 0 \\ -D^{-1} \left[ F(u; 0) + CV \right] \end{pmatrix} \quad (\nu, u) \in \mathbb{R}^N \times \mathbb{R}^N + \gamma_0 \]

By assumption, this equation has the equation \( u_\nu = (u_\nu; 0) \) and a heteroclinic \( q_\nu \)
\[ q_\nu = (q, q(\theta)) \] 
that connects \( u_0 \) at \( \xi = 0 \) to \( u_\nu \) at \( \xi = \phi \). Since we assumed that \( \det F(u_\nu; 0) \neq 0 \), we now that \( u_\nu \) has both hyperbolic and the isochronation

of (9) about \( u_\nu \) is given by

\[ \begin{pmatrix} \nu_\nu \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -D^{-1} F(u_\nu; 0) & -\nu \end{pmatrix} \]

Lemma: \( \nu_\nu \) has \( N \) eigenvalues \( \nu \) with \( \Re \nu > 0 \) and \( N \) eigenvalues \( \nu \) with \( \Re \nu < 0 \).

Before proving this lemma, we write hypothesis (H2) using the dispersion relation

\[ \gamma_0^2(\lambda, u; \nu) = \det \left( D\nu^2 + u \nu + F(u; 0; \nu) - \lambda I \right) \]

which gives

(i) \( \gamma_0^2(\lambda, u; \nu) = 0 \) or some \( R \) implies \( \Re \lambda < -R \)

(ii) \( \gamma_0^2(\lambda, u; \nu) = 0 \) for some \( R \) implies either \( \Re \lambda = 0 \) or \( \nu = 0 \) or \( \nu = u \nu \nu \) and \( \lambda = \gamma_0(\nu) = \mu + \nu \nu = \nu^0(0) \) or \( \nu = \nu \nu \nu \).

Proof of Lemma

Consider \( \gamma_0^2(\lambda, u; \nu) = \det \left( D\nu^2 + u \nu + F(u; 0; \nu) - \lambda I \right) = 0 \) for \( \mu > 1 \).

For fixed \( \lambda \), \( \gamma_0^2(\lambda, u; \nu) = 0 \) polynomial in \( u \) of degree \( 2N \) which therefore

has precisely \( 2N \) roots. For \( \mu > 1 \), we seek \( u = \xi \) to get
\[ \det \left( Dv^2 + \nabla c_0 + f_0 (U^2, 0) - \lambda \right) = \det \left( Dv^2 + \nabla c_0 + f_0(U^2, 0) - \lambda \right) \]

which vanishes if, and only if,

\[ \det \left( Dv^2 + \nabla c_0 + f_0(U^2, 0) - 1 \right) = 0. \]

Setting \( E = \frac{1}{\lambda} \) with \( \epsilon > 0 \) close to \( \lambda_0 \), we obtain

\[ \det \left( Dv^2 + \nabla c_0 + f_0(U^2, 0) - 1 \right) = \left( \delta_0(U^2) - 1 \right) + O(\epsilon) = 0 \]

where \( D = \text{diag}(d_j) > 0 \). For \( E = 0 \), we obtain 2N roots

\[ \delta_0 = \pm \sqrt{d_j} \quad j = 1, \ldots, N \]

(counted with multiplicity). Rouche's theorem (see any complex variables textbook) implies that \( \delta_0 \) has 2N solutions which are close to \( \delta_0 \) for all \( E > 0 \) sufficiently small. Hence, we see that

\[ \delta_0(U^2, 0) = 0 \]

has precisely N solutions \( \lambda \) with \( Re(\lambda) > 0 \) and \( N \) solutions with \( Re(\lambda) < 0 \) for all \( \lambda \neq 1 \). We now move along the positive axis from \( \lambda_0 \) towards \( \lambda_0 \), while monitoring the 2N roots of \( \delta_0(U^2, 0) = 0 \). Rouche's theorem implies that the number of roots with positive or negative real part can only change if one of these roots crosses the imaginary axis. Thus, if the roots of \( \delta_0(0, U, 0) \) do not split, \( N \) \( N \), then there is a \( \lambda > 0 \) and a \( u \) \( u \) such that \( \delta_0(u, U, 0) = 0 \); this, however, cannot happen due to (i) and (ii) on the previous page. Thus, the roots of \( \det \left( Dv^2 + \nabla c_0 + f_0(U^2, 0) - \lambda \right) = 0 \) split \( N \) \( N \) at \( \lambda = 0 \). For \( \lambda = 0 \), this is the characteristic polynomial of \( \mathbb{R}^2 \), and the lemma is proved.

The proof of Lemma 1 is very useful: it shows that spatial roots \( \lambda \) of \( \delta_0(U, x, y) \) are very rigid for \( x \neq 1 \) and that they depend only on the leading-order term \( V_0 = Dv_{xx} \) of the underlying PDE. We will use this property again later.

As a consequence of Lemma 1, the stable and unstable manifolds of the hyperbolic equilibrium \( U_0 \) are \( (9) \) have dimension \( N \). We also know that

\[ \mathcal{R}(u) \in \mathcal{W}^s(U_0) \cup \mathcal{W}^u(U_0), \]

and therefore

\[ \mathcal{P}(u) \in \mathcal{T}(u) \mathcal{W}^s(U_0) \cup \mathcal{T}(u) \mathcal{W}^u(U_0). \]
I claim that \( \dim \left[ T(0, V^0(U_0)) \cap T(0, W^0(U_0)) \right] = 1 \)

which corresponds to the statement that \( V^*(0) = Q_s^*(0) \) is the only bounded solution (up to scalar multiples) of the variational equation

\[
V_0^* = \left( \begin{array}{c} 0 \\ -D^*F_u(Q(0), 0) \\ -c_0D^* \\ \end{array} \right) V \\
V \in X_0
\]

of the travelling-wave ODE (6) associated with the heteroclinic orbit \( Q(0) \). Any solution \( V(0) = (U, V)(0) \) of (11) corresponds to a solution \( U \) of

\[
x_v U = D[U^0] + c_0 U_0 + F_u(Q(0), 0) U = 0.
\]

and vice versa. If \( v(s) \) is a bounded solution of (11), then \( V(s) \) decays exponentially as \( s \to \pm \infty \) and corresponds therefore to an element \( U(s) \) in the null space of \( x_v \) in \( L^2 \).

We assumed in (H3) that the null space contains only multiples of \( Q_s^*(0) \) which proves that

\[
\dim \left[ T(0, V^0(U_0)) \cap T(0, W^0(U_0)) \right] = 1
\]

as claimed.

(More precisely, the intersection is one-dimensional since the geometric multiplicity of \( Q \) is assumed to be one, which gives \( \dim V^0(U_0) = 1 \).

Next, we need to understand how \( W^0(U_0) \) and \( W^0(U_+0) \) behave when \( c \) is varied near \( c_0 \). We expect that these manifolds no longer intersect when \( c = c_0 \) but need to make this expectation more precise.
Consider

(12) \[ u_t = A(x)u \quad u \in \mathbb{R}^n \]

where

\[ |A(x) - A_\infty| \leq \alpha e^{\lambda |x|} \quad x \to \infty \]

for appropriate hyperbolic matrices \( A_\infty \). The solution operator \( \Phi(x,y) \) of (12) maps \( u(y) \) to \( u(x) \) for any solution \( u \) of (12). We have

\[ \Phi(x,x) = 1, \quad \Phi(x,y) \Phi(y,z) = \Phi(x,z) \quad \forall x,y,z \in \mathbb{R} \]

We need to separate initial data of solutions that decay as \( x \) increases from data whose solution decay as \( x \) decreases.

**Proposition** Let \( u_0 \in \mathbb{R} \) be an unbounded interval (i.e., \( \mathbb{R} \) or \( \mathbb{R}^+ \)). Eqn. (12) is said to have an exponential dichotomy on \( u_0 \) if there are operators \( \Phi^+(x,y) \) and \( \Phi^-(x,y) \) defined for \( x,y \in \mathbb{R} \) with \( \Phi^+(x,y) \) and \( \Phi^-(x,y) \), respectively, and numbers \( \beta, \gamma \geq 0 \) so that

\[ \begin{align*}
\Phi^+(x,y) u_0 & \leq \eta e^{-\beta |x-y|} u_0 & \text{for } x > y \\
\Phi^-(x,y) u_0 & \leq \eta e^{-\gamma |x-y|} u_0 & \text{for } x < y
\end{align*} \]

\[ \begin{align*}
\Phi^+(x,y) \Phi^+(y,z) & = \Phi^+(x,z) & x > y > z \\
\Phi^-(x,y) \Phi^-(y,z) & = \Phi^-(x,z) & x < y < z \end{align*} \]

\[ \Phi^+(x,y) \Phi^-(y,z) = 0 \quad x > y > z \]

\[ \Phi^-(x,y) \Phi^+(y,z) = 0 \quad x < y < z \]

\[ \Phi^+(x,y) + \Phi^-(x,y) = 2 \quad \forall x \]

**Proof of Proposition** → following typed notes.
\[
\begin{align*}
\int p(x) \phi(x) \mathbb{E} \left[ \int (x) \phi(x) \mathbb{E} \left[ \int (x) \phi(x) \right] \right] dx + \\
\int p(x) \phi(x) \mathbb{E} \left[ \int (x) \phi(x) \right] dx + \\
\int p(x) \phi(x) \mathbb{E} \left[ \int (x) \phi(x) \right] dx = (1) \phi
\end{align*}
\]

Simplifying these expressions, we obtain:

\[
\begin{align*}
(\phi \geq s) & \quad \int p(x) \phi(x) \mathbb{E} \left[ \int (x) \phi(x) \right] dx + \\
& \quad \int p(x) \phi(x) \mathbb{E} \left[ \int (x) \phi(x) \right] dx + \\
& \quad \int p(x) \phi(x) \mathbb{E} \left[ \int (x) \phi(x) \right] dx = (1) \phi
\end{align*}
\]

and therefore:

\[
\begin{align*}
(\phi \geq s) & \quad \int p(x) \phi(x) \mathbb{E} \left[ \int (x) \phi(x) \right] dx = (1) \phi
\end{align*}
\]

Next, we have:

\[
\begin{align*}
(\phi \leq s) & \quad \int p(x) \phi(x) \mathbb{E} \left[ \int (x) \phi(x) \right] dx + \\
& \quad \int p(x) \phi(x) \mathbb{E} \left[ \int (x) \phi(x) \right] dx + \\
& \quad \int p(x) \phi(x) \mathbb{E} \left[ \int (x) \phi(x) \right] dx = (1) \phi
\end{align*}
\]

Integrating these expressions, we obtain:

\[
\begin{align*}
(\phi \leq s) & \quad \int p(x) \phi(x) \mathbb{E} \left[ \int (x) \phi(x) \right] dx = (1) \phi
\end{align*}
\]

Thus, we have proven the expected equation.

(1)

\[
\begin{align*}
(\phi \geq s) & \quad \int p(x) \phi(x) \mathbb{E} \left[ \int (x) \phi(x) \right] dx + \\
& \quad \int p(x) \phi(x) \mathbb{E} \left[ \int (x) \phi(x) \right] dx + \\
& \quad \int p(x) \phi(x) \mathbb{E} \left[ \int (x) \phi(x) \right] dx = (1) \phi
\end{align*}
\]
\[- \int_s^t e^{A(t-r)} P_0^r B(r) \phi^0(r) \, dr, \quad (t \geq s)\]

\[\phi^0(t) = \int_s^t e^{A(t-r)} P_0^r B(r) \phi^0(r) \, dr + \int_s^t e^{A(t-r)} P_0^r B(r) \phi^1(r) \, dr + \int_s^t e^{A(t-r)} P_0^r B(r) \phi^2(r) \, dr, \quad (t \leq s)\]

We had shown that this equation has the unique solution

\[\phi^0(t) = \phi^0(t, s) \phi^0(s, r), \quad (t \geq s)\]

\[\phi^1(t) = \phi^1(t, s) \phi^0(s, r), \quad (t \leq s)\]

On the other hand, we can substitute

\[\phi^0(t) = \Phi(t, r), \quad (t \geq s)\]

\[\phi^1(t) = 0 \quad (t \leq s)\]

and obtain

\[\Phi^0(t, r) = e^{A(t-s)} P_0^s B(r) \phi^0(r) \, dr - \int_s^t e^{A(t-r)} P_0^r B(r) \phi^0(r) \, dr + \int_s^t e^{A(t-r)} P_0^r B(r) \phi^1(r) \, dr + \int_s^t e^{A(t-r)} P_0^r B(r) \phi^2(r) \, dr, \quad (t \geq s)\]

\[0 = \int_s^t e^{A(t-r)} P_0^r B(r) \phi^0(r) \, dr + \int_s^t e^{A(t-r)} P_0^r B(r) \phi^1(r) \, dr + \int_s^t e^{A(t-r)} P_0^r B(r) \phi^2(r) \, dr, \quad (t \leq s)\]

The second equation is obviously satisfied, while the first equation can be simplified to

\[\Phi^0(t, r) = e^{A(t-r)} P_0^s B(r) \phi^0(r) \, dr - \int_s^t e^{A(t-r)} P_0^r B(r) \phi^0(r) \, dr + \int_s^t e^{A(t-r)} P_0^r B(r) \phi^1(r) \, dr + \int_s^t e^{A(t-r)} P_0^r B(r) \phi^2(r) \, dr, \quad (t \geq s)\]

This equation, however, is also met: it is the first equation in the fixed-point equation that is satisfied by \((\Phi^0, \Phi^1)\). We conclude that

\[\phi^0(t, s) \phi^0(s, r) = \phi^0(t, r)\]

\[\phi^1(t, s) \phi^0(s, r) = 0\]
Consider now the full linearization about the equilibrium $U^0_\nu$ in $Y$:

$$A_2 = \begin{pmatrix} 0 & 1 \\ D^{-1} [ \text{i} \omega - \mathbf{F}_U(U^0_\nu, 0)] & -\mathbf{C}_D^{-1} \end{pmatrix}$$

$Y = H^1(S^1) \times H^2(S^1)$

Since $U^0_\nu \in \mathbb{R}^n$ does not depend on $\nu$, the operator decouples on each Fourier space:

$$\psi_\nu = \begin{pmatrix} \text{e}^{i \text{w}_\nu \mathbf{e}_\nu} (\nu \phi_\nu) \\ (\nu \phi_\nu) \in \mathbb{R}^n \times \mathbb{R}^n \end{pmatrix} \quad \forall \nu \in \mathbb{Z}$$

$$A_2 \circ \psi^{\nu}_\nu = \begin{pmatrix} 0 \\ D^{-1} [ \text{i} \omega \nu - \mathbf{F}_U(U^0_\nu, 0)] \end{pmatrix}$$

$$= \begin{pmatrix} \nu \psi_\nu \\ \nu \psi_\nu \end{pmatrix}$$

has nontrivial solutions $(\psi^{\nu}_\nu)$ if:

$$0 = \text{det} \left[ \nu (\nu \mathbf{C}_D^{-1}) - D^{-1} [ \text{i} \omega \nu - \mathbf{F}_U(U^0_\nu, 0)] \right]$$

$$= \left( \text{det} D^{-1} \right) \left( \text{det} \left[ \nu \mathbf{D}^2 + \nu \mathbf{C}_U + \mathbf{F}_U(U^0_\nu, 0) - \text{i} \omega \nu \right] \right)$$

$$\ast \left( \text{det} D^{-1} \right) \text{det} \left( \text{i} \omega \nu, \nu, 0 \right)$$

We therefore conclude that:

1. $A_2 |_{\psi^{\nu}_\nu}$ is hyperbolic with an $N\nu$ splitting at its eigenvalues $\nu$.
2. $A_2 |_{\psi^{\nu}_\nu}$ is hyperbolic with an $N\nu$ splitting.
3. $A_2 |_{\psi^{\nu}_\nu}$ is hyperbolic except for a simple pair of spatial eigenvalues $\mu_0$.

Given by

$$\nu \mu = \frac{\text{i} \omega \nu - \text{i} \nu \omega}{\nu} + O(\nu^2) \quad \text{for} \ \nu \to 0$$

Furthermore, $A_2 |_{\psi^{\nu}_\nu}$ is hyperbolic for $\nu < 0$ with an $N\nu$ splitting.

$$\text{Spec} \ A_2^- \quad \text{for} \ \nu < 0$$

**Proof:** Homotopy from $\lambda = \nu \omega + \text{i} \nu \omega$ to $\lambda = \nu \omega$ and use (1)-(11) on p. 6.

The spatial Herd eigenvalues $\lambda_s$ arise for $\lambda \to 0$ at $\nu = 0$ by solving

$$\lambda = \nu \omega = \nu + i \nu \omega = \nu ( \nu \omega - \nu \omega)^2 + 0(\nu^3)$$

for $\nu$ and setting $\nu = \omega$.
We need to solve the variational equation

\[
\begin{pmatrix}
U \\
V
\end{pmatrix}_\varepsilon = \begin{pmatrix}
0 & 1 \\
D^{-1}(\partial_x - \mathcal{F}_0(Q(\varepsilon,0)) & -cD^{-1}
\end{pmatrix}
\begin{pmatrix}
U \\
V
\end{pmatrix}
\quad \text{in } \mathbb{R}^2
\]

Recall

\[
\begin{pmatrix}
U \\
V
\end{pmatrix}_\varepsilon = \begin{pmatrix}
\varepsilon \mathcal{W}(\varepsilon U) \\
\varepsilon \mathcal{W}(\varepsilon V)
\end{pmatrix},
\begin{pmatrix}
U \\
V
\end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}^2
\]

For \( \varepsilon \neq 0 \), the norm on \( \varepsilon U \) is

\[
\| \varepsilon U \|_{L^2}^2 = \| U \|_{L^2}^2 + \varepsilon \| V \|_{L^2}^2
\]

So that

\[
\sum_{\varepsilon \neq 0} (\| U \|_{L^2}^2 + \varepsilon \| V \|_{L^2}^2) < \infty.
\]

Thus, we set

\[
\begin{pmatrix}
U_e \\
V_e
\end{pmatrix} = \begin{pmatrix}
0 \\
\varepsilon
\end{pmatrix},
\begin{pmatrix}
U_e \\
V_e
\end{pmatrix}_\varepsilon = \begin{pmatrix}
\mathcal{W}(\varepsilon U) \\
\mathcal{W}(\varepsilon V)
\end{pmatrix}
\]

with the ordinary \( \varepsilon \)-norm for \( \begin{pmatrix}
U_e \\
V_e
\end{pmatrix} \).

In these variables, equation (13) becomes

\[
\begin{pmatrix}
\dot{U}_e \\
\dot{V}_e
\end{pmatrix} = \begin{pmatrix}
0 & \varepsilon l_1^\nu \\
-D^{-2}(\mathcal{W}(\varepsilon U) - \mathcal{F}_0(Q(\varepsilon,0)) & -cD^{-1}
\end{pmatrix}
\begin{pmatrix}
U_e \\
V_e
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 \\
-D^{-2}(\mathcal{W}(\varepsilon U) - \mathcal{F}_0(Q(\varepsilon,0)) & -cD^{-1}
\end{pmatrix}
\begin{pmatrix}
U_e \\
V_e
\end{pmatrix}
\]

Rescaling \( \varepsilon = \varepsilon l_1^\nu \), we obtain

\[
\begin{pmatrix}
\dot{U}_e \\
\dot{V}_e
\end{pmatrix} = \begin{pmatrix}
0 & \varepsilon l_1^\nu \\
D^{-2}(\mathcal{W}(\varepsilon U) - \mathcal{F}_0(Q(\varepsilon,0)) & -cD^{-1}
\end{pmatrix}
\begin{pmatrix}
U_e \\
V_e
\end{pmatrix}
\]

For \( \varepsilon \neq 0 \), we obtain

\[
\dot{U}_e = \begin{pmatrix}
0 & \varepsilon l_1^\nu \\
D^{-2}(\mathcal{W}(\varepsilon U) - \mathcal{F}_0(Q(\varepsilon,0)) & -cD^{-1}
\end{pmatrix} U,
\quad U \in \mathbb{C}^2 \times \mathbb{C}^2
\]

while, for \( \varepsilon = 0 \), we get

\[
\dot{U}_e = \begin{pmatrix}
0 & \varepsilon l_1^\nu \\
D^{-2}(\mathcal{W}(\varepsilon U) - \mathcal{F}_0(Q(\varepsilon,0)) & -cD^{-1}
\end{pmatrix} U
\quad \text{with } U \in \mathbb{C}^2 \times \mathbb{C}^2
\]

Equation (14) has an exponential dichotomy on \( \mathbb{R} \). Thus, (14) has exponential dichotomies on \( \mathbb{R}^2 \) for all \( \varepsilon \in \mathbb{R} \) with \( \varepsilon l_1 \rightarrow 0 \) to some \( \varepsilon_0 \in \mathbb{N} \), with exponential rate and constants that do not depend on \( \varepsilon \).
We study the spatial dynamical system

\[
\begin{pmatrix}
U \\
V \\
\end{pmatrix}_t = 
\begin{pmatrix}
D^0 (U_t - f(U,V)) - CV \\

\end{pmatrix}
\begin{pmatrix}
U \\
V \\
\end{pmatrix} \in H^1(\Omega) \times H^2(\Omega) = : Y
\]

The linearization

\[
\begin{pmatrix}
U \\
V \\
\end{pmatrix}_t = 
\begin{pmatrix}
0 \\

\end{pmatrix}
\begin{pmatrix}
D^0 \left[ \partial_t - \partial U_U(U,V) \right] - CD^{-1} \\

\end{pmatrix}
\begin{pmatrix}
U \\
V \\
\end{pmatrix}
\]

about any time-independent solution \((U_0, V_0) (t) \in Y\) decouples on each Fourier subspace

\[
X_e = \{ e^{i \omega t} \begin{pmatrix} U \\ V \end{pmatrix} : (U, V) \in \mathbb{C}^N \}_{\omega \in \mathbb{C}^N} \subseteq \mathbb{C}^N
eq 0
\]

where it becomes

\[
\begin{pmatrix}
U_e \\
V_e \\
\end{pmatrix}_t = 
\begin{pmatrix}
0 \\

\end{pmatrix}
\begin{pmatrix}
D^0 \left[ i \omega U - \partial U_U(U, \nu) \right] - CD^{-1} \\

\end{pmatrix}
\begin{pmatrix}
U_e \\
V_e \\
\end{pmatrix}
\]

2N-dimensional ODE

The two geometric solutions we are interested in are:

1) Tuning bifurcation ahead of fronts:

2) Tuning bifurcation behind fronts:

---

\( \lambda = \partial \partial_{\lambda} + \partial_{\lambda} + U(\lambda, 0) \\)
We concentrate on Turing bifurcation ahead of fronts:

- Center manifold

\[ W^c(u^c) \]

The bifurcating solution \( W^*(u^*; \mu) \) is continuous in \( \mu \), and \( W^*(u^*(0); \mu) = u^0 \).

We consider the (strong) stable slice \( W^s(u^*(0); \mu) \) which consists of all initial data \( u(0) \) such that \( |u(0) - W^*(u^*(0); \mu)| \leq \varepsilon \) for some \( \varepsilon > 0 \).

\( W^s(u^*(0); \mu) \) is a smooth manifold that depends continuously on \( \mu \) in the \( C^1 \)-topology with \( W^s(u^*(0); \mu) = W^s(u^*(0); \mu) \) (inject) stable manifold of \( u^* \).

(Note that these manifolds are constructed as solutions to appropriate integral equations which can be used to deduce the above properties — see below for more details)

- We set \( \mu = 0 \) and discuss the geometry of \( W^0(u^0) \) and \( W^s(u^0) \):

Since \( G(\delta) \in \mathcal{X} \), the linearization of the spatial dynamical system depends on each \( \chi \).

We then obtain that

\[
\begin{align*}
E^u &= T_{\mu(0)} W^0(u^0) \ni E^e \quad \text{and} \quad E^s \ni E^e \quad \text{in } N - \text{dimensional } \mathcal{X}.
\end{align*}
\]

and

\[
E^u \cap E^s = R q(0) \subset \mathcal{X}
\]

which implies \( (E^u + E^s)^\perp = 1 R q_e \in \mathcal{X} \)

Let \( \Sigma = \{ q(0) \} \), then

\[
\begin{align*}
E^u &= T_{\mu(0)} W^0(u^0) \cap \Sigma \\
E^s &= T_{\mu(0)} W^s(u^0) \cap \Sigma
\end{align*}
\]

satisfy

\[
E^u \cap E^s = 0
\]

\( (E^u + E^s)^\perp = R q_e \)

\( \Sigma = E^s \times E^u \times R q_e \)

- Therefore, we can compute \( W^0(u^*(\mu)) \) and \( W^s(u^*(0); \mu) \) by

\[
W^0(u^*(\mu)) \cap q(0) + \Sigma = q(0) + \{ \mathcal{W}(u^0, u^*(\mu), u^0, u^*(\mu)) \} \quad \text{with } E^u \in \mathcal{W}^\perp
\]

\[
W^s(u^*(0); \mu) \cap q(0) + \Sigma = q(0) + \{ \mathcal{W}_s, \mathcal{W}_c, u^0, u^*(0); \mu, u^0, u^*(0); \mu \} \quad \text{with } E^s \in \mathcal{W}^\perp
\]

where

\[
\mathcal{W}^\perp, \mathcal{W}_s^\perp = 0 \quad \text{at } (0,0,0) \quad \text{for } \delta = 0
\]

\[
\mathcal{W}(0,0,0) - \mathcal{W}(0,0,0) = M(c-c_0) + O((c-c_0)^2)
\]

for some \( M > 0 \)

\( \mathcal{W} \) is smooth in \( (\mathcal{X}, \mathcal{X}) \) for each fixed \( \mu \), and in \( \mathcal{X}, \mathcal{W} \) and its derivatives with respect to \( (\mathcal{X}, \mathcal{X}) \) are continuous in \( \mu \).
I had argued previously that \(x = 0\) is equivalent to the condition that 
\(x = 0\) is algebraically simple as an eigenvalue of \(D\).

Intersections of \(W^u(x,0,0)\) and \(W^s(\omega,0,0)\) in \(\mathbb{R}^3 + \mathbb{R}^3\) are therefore round as solutions of the system

\[ T : E^s \times E^u \times \mathbb{R} \times \mathbb{R} \rightarrow E^s \times E^u \times \mathbb{R} \times \mathbb{R} \]

\((\omega, \omega^*, c, \mu) \rightarrow (\omega^*, \omega^*, c, \mu), \omega^* = \omega^*(\omega, c, \mu), \omega^* = \omega^*(\omega, c, \mu)\)

We have that

\[ T \] is smooth in \((\omega^*, \omega, c)\) for fixed \(\mu\) and \(T\), \(D(w^*, w^*, c) = 0\) in \(\mu\)

\[ T(0,0,0,0) = 0 \]

\[ \begin{pmatrix} -1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & \mu \end{pmatrix} \] is invertible

\[ \Rightarrow \] The implicit function theorem shows the existence of a unique solution of \(T(\omega^*, \omega, c, \mu) = 0\) near \((0,0,0,0)\) for each fixed \(\mu \neq 0\),

and the solution is continuous in \(\mu\).

2) For turning bifurcations behind fronts, the following manifolds are considered:

\[ \begin{matrix} W^s & W^u \end{matrix} \]

Any pattern that converges to \(W^u(\mu, r)\) as \(r \to -\infty\)

and to \(W^s(\mu, r)\) as \(r \to +\infty\) lies in the intersection of

\[ W^u(\mu, r) \] and \[ W^s(\mu, r) \]

Note that \(W^u(\mu, r)\) for \(r \to 0\) breaks \(W^u(\mu, r)\) at \(r \to 0\), and extends smoothly

in \(\mu\) into \(r \to 0\) as \(W^u(\mu, r)\). The reason is that it is a unique solution from right

(unequal) into left (stable) half plane.

Proceeding as in 1), we find that \(W^u(\mu, r)\) and \(W^s(\mu, r)\) intersect

in \(\mathbb{R}^3 + \mathbb{R}^3\) for a unique \(c \neq 0\) and at a unique intersection point \(p\). Each 

\(p\) near \(z_0\). However, the same construction works also for the restriction of the

spatial dynamical system to the invariant subspace \(Y_0\), where it gives the continuation

of the front \(Q(\mu)\) for \(\mu \neq 0\), uniqueness in either case implies that the

unique intersection points of \(W^u(\mu, r)\) and \(W^s(\mu, r)\) in \(Y\) actually lies in \(Y_0\)

and is given by the passing front.

\[ \Rightarrow \] Genuinely three-dimensional solutions near the front do not bifurcate.
Exponential dichotomies for general spatial dynamical systems:

1) \( U \subset \{ \mathbf{A} + \mathbf{B}(x) \} u \quad \text{\( u \in X \)} \quad (X \text{ finite-dimensional space})

We need the following assumptions:

1. \( U \subset \mathbf{A}u \) has an exponential dichotomy with rate \( \gamma \) on \( \mathbb{R}^+ \):

   \[
   \mathbb{R}^+ \times \mathbb{R}^+ \subset E^u \oplus E^s, \quad E^u \oplus E^s = X
   \]

   \[
   e^{\mathbf{A}t} : \text{semigroup on } E^s
   \]

   \[
   e^{-\mathbf{A}t} : \text{semigroup on } E^u
   \]

2. \( \mathbb{B} \subset \mathbb{C}(\mathbb{R}^+, L(X;X)) \) is a family of \( \gamma \)-continuous \( \mathbb{C}^0 \) semigroups such that \( \gamma \mathbb{B} \subset \mathbb{C}(\mathbb{R}^+, L(X;X)) \) is a family of \( \gamma \)-continuous \( \mathbb{C}^0 \) semigroups such that:

   \[
   \mathbb{B}(x) = \mathbb{S}(x) + \mathbb{K}(x) \quad \forall x \in \mathbb{R}^+
   \]

3. Compactness: \( \mathbb{R}^+ \) is compact as operator on \( X \)

4. Backward uniqueness: If \( \mathbb{B} \) satisfies (1) for its adjoint on \( \mathbb{R}^+ \) with \( \mathbb{B}(0) = 0 \), then \( \mathbb{B} = 0 \).

Theorem: Assume 1. is met, then for each \( \gamma \) with \( 0 < \gamma < 2\mu \), \( \exists \epsilon > 0 \) and \( \forall \gamma > 1 \) so that the following is true when 2-4 are satisfied: Equation (1) has an exponential dichotomy on \( \mathbb{R}^+ \) with rate \( \gamma \) and constant \( \mu \).

Outline of the proof:

\[
\begin{aligned}
\mathbb{e}^{\mathbf{A}x \cdot X} \cdot \mathbb{P} \cdot \mathbb{u} = \mathbb{P}^0 (x,0) + \mathbb{P}^0 (0,0) + \int \mathbb{Q}^0 (x,2) \mathbb{P}^0 (2,0) \mathbb{d}x & \\
- \int \mathbb{Q}^0 (x,2) \mathbb{P}^0 (2,0) \mathbb{d}x & \\
\mathbb{e}^{-\mathbf{A}x \cdot X} \cdot \mathbb{P} \cdot \mathbb{u} = \mathbb{P}^0 (x,0) - \mathbb{P}^0 (0,0) + \int \mathbb{Q}^0 (x,2) \mathbb{P}^0 (2,0) \mathbb{d}x & \\
- \int \mathbb{Q}^0 (x,2) \mathbb{P}^0 (2,0) \mathbb{d}x & \\
\end{aligned}
\]

(1) Find the stable subspace of \( \gamma \cdot 0 \), is all initial data leading to stable evolving solutions of (1).

(2) Set \( \gamma = 0 \) and \( \mathbb{Q}^0 (0,0) = 0 \) in (1) to get

\[
\begin{aligned}
\mathbb{e}^{\mathbf{A}x \cdot X} \cdot \mathbb{P} \cdot \mathbb{u} & = \mathbb{Q}^0 (x,0) + \int \mathbb{Q}^0 (x,2) \mathbb{P}^0 (2,0) \mathbb{d}x & - \int \mathbb{Q}^0 (x,2) \mathbb{P}^0 (2,0) \mathbb{d}x & \\
\mathbb{P}^0 \cdot \mathbb{u} & = - \mathbb{Q}^0 (x,2) \mathbb{P}^0 (2,0) \mathbb{d}x & \\
\end{aligned}
\]

which we write as \( \mathbb{u} = \mathbb{P}^0 \mathbb{Q}^0 \) which is given.

Lemma: \( \mathbb{P}^0 \) is Fredholm with index \( 2\mu \).

Proof: \( \mathbb{R}^+ \) is closed + small = compact \( \rightarrow \) Fredholm index \( 0 \)

\[
\begin{aligned}
\mathbb{E}^s & = \left( \mathbb{P}^0 \mathbb{Q}^0 \right)^{-1} \mathbb{K} (x) \mathbb{K} (x) \mathbb{e}^{\mathbf{A}(x) \cdot X} \mathbb{u} \mid \mathbb{R}^+ \mathbb{K} (x) \mathbb{e}^{\mathbf{A}(x) \cdot X} \mathbb{u} \mid \mathbb{R}^+ \mathbb{E}^s = \left( \mathbb{P}^0 \mathbb{Q}^0 \right)^{-1} \mathbb{K} (x) \mathbb{e}^{\mathbf{A}(x) \cdot X} \mathbb{u} \mid \mathbb{R}^+ \mathbb{K} (x) \mathbb{e}^{\mathbf{A}(x) \cdot X} \mathbb{u} \mid \mathbb{R}^+
\end{aligned}
\]
(ii) Fix \( y \neq 0 \) and consider again (i).

We choose a closed complement \( E^0 \) of \( E^\perp \) in \( X^* \) (note: the existence of closed complements)

Then we prove:

\[
T : \{ (\phi, \phi') | \phi'(0, 1) \in E^0 \} \rightarrow \{ (\phi, \phi') | \phi'(0, 1) \in E^\perp \}
\]

Lemma \( T \) is an isomorphism.

Proof

- \( T \) is Fredholm with index for \( B \neq 0 \).

- \( T \) has trivial null space. \( \Box \)