Periodic Evans function analysis and stability of roll waves in inclined thin film flow

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Sponsored by NSF Grants no. DMS-0300487 and DMS-0801745

Geometric Methods for Infinite-Dimensional Dynamical Systems: Brown University, November 2011
I. Introduction: viscous roll waves

(Picture courtesy Neil Balmforth, UBC.)
Roll waves in the lab (Neil Balmforth’s)
Related models

St. Venant equations for shallow water flow on an incline

\[ \nu_t - u_x = 0 \]
\[ u_t + p(v)_x = \left( \frac{u_x}{\nu^2} \right)_x + 1 - \nu u^2. \]

Generalized Kuramoto–Sivashinsky equation (gKS)

\[ u_t + \left( \frac{u^2}{2} \right)_x + \delta u_{xx} + \epsilon u_{xxx} + \delta u_{xxxx} = 0. \]

One-dimensional viscoelasticity with surface energy

\[ \nu_t - u_x = v_{xx}, \]
\[ u_t + \sigma(v)_x = u_{xx} \]
Consider a periodic traveling wave solution

\[ u(x, t) = \bar{u}(x - st), \quad \bar{u}(X) = \bar{u}(0), \]

of a parabolic system of balance laws

\[ u_t + f(u)_x + g(u) = u_{xx}, \quad u \in \mathbb{R}^n, \quad g = \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix} \]

(equivalently stationary solution of \( u_t - su_x + f(u)_x + g(u) = u_{xx} \)).

**Extremes:** \( g = 0 \) (conservation law); \( f = 0 \) (reaction diffusion).
Goals

Understand **modulational stability**, i.e., time evolutionary stability with respect to small **localized** initial perturbations.

- Whitham averaging and averaged dispersion relation.
- Bloch decomposition and diffusive estimates.
- Numerical verification/phenomenological study.

Remarks.
1. Of interest also in optics, dispersive waves; in general, much less well understood than case of front or pulse type solutions.
2. Extends work of Schneider and others in reaction diffusion case 
   \[ u_t = u_{xx} + g(u), \ s = 0 \] (More waves/degrees of freedom, less regularity; convection as well as diffusion.)
3. Strong analogy to the viscous shock case (\( \sim \) universality of WKB, weakly nonlinear optics expansion).
II. Abstract stability framework: Traveling wave ODE

(Case $g \equiv 0$) Substituting $u = \bar{u}(x - st)$ into $u_t + f(u)_x = u_{xx}$ and integrating in $x$ yields ODE

$$\bar{u}' = f(\bar{u}) - s\bar{u} - q, \quad (u_0, q, s, X) \equiv \text{constant}. \quad (1)$$

(H1) $f \in C^5$,
(H2) The set of periodic traveling waves near $\bar{u}$ form a smooth $(n + 2)$-dimensional manifold

$$\{\bar{u}^a(x - \alpha - s(a)t)\},$$

with $\alpha \in \mathbb{R}$ corresponding to translation and $a \in \mathbb{R}^{n+1}$.

(compare reaction diffusion case: $a \in \mathbb{R}^1$; think Hopf bifurcation.)
Bloch decomposition

Linearized equations:

\[ v_t = L v := (\partial_x^2 - \partial_x A)v, \quad A := Df(\bar{u}) \text{ periodic}. \]

Bloch decomposition, \( u \in L^2(\mathbb{R}) \):

\[ u(x) = \left( \frac{1}{2\pi} \right) \int_{-\pi}^{\pi} e^{i\xi x} \hat{u}(\xi, x) d\xi, \]

\[ \hat{u}(\xi, \cdot) := \sum_{j \in \mathbb{Z}} \hat{u}(\xi + 2\pi j)e^{2\pi ij \cdot} \in L^2[0, 1]_{\text{periodic}}. \]

Inverse Bloch transform representation:

\[ e^{Lt} u_0 = \left( \frac{1}{2\pi} \right) \int_{-\pi}^{\pi} e^{i\xi x} e^{L_\xi t} \hat{u}_0(\xi, x) d\xi, \quad (2) \]

\[ L_\xi := e^{-i\xi x} L e^{i\xi x} = (\partial_x + i\xi)^2 - (\partial_x + i\xi)A. \]
Spectral stability conditions

\((D1)\) \(\sigma(L) \subset \{\text{Re}\lambda < 0\} \cup \{0\}\).

\((D2)\) \(\text{Re}\sigma(L_\xi) \leq -\theta|\xi|^2, \theta > 0, :or \: \xi \in [-\pi, \pi]\).

\((D3)\) \(\lambda = 0\) is an eigenvalue of \(L_0\) of (minimum) multiplicity \(n + 1\).

\((H1)-(H2)\) and \((D1)-(D3)\) \(\implies\) there exist \(n + 1\) smooth eigenvalues

\[\lambda_j(\xi) = -ia_j\xi + o(|\xi|)\]  \hspace{1cm} (3)

of \(L_\xi\) bifurcating from \(\lambda = 0\) at \(\xi = 0\), \(a_j\) constant.

\((H3)\) The coefficients \(a_j\) in (3) are distinct.

(compare reaction diffusion case: \(\lambda = 0\) simple. Here, generically Jordan block.)
Basic nonlinear stability result

Theorem (Johnson-Z2010)

Let \( \tilde{u} \) be a solution of \( u_t + f(u)_x = u_{xx} \). Assuming (H1)–(H3), (D1)–(D3), and \( E_0 := \| \tilde{u} - \bar{u} \|_{L^1 \cap H^4} \big|_{t=0} \) sufficiently small,

\[
\| \tilde{u} - \tilde{u}(\cdot - \psi) \|_{L^p(t)} \leq C(1 + t)^{-\frac{1}{2}(1-1/p)}E_0,
\]

\[
\| (\psi_t, \psi_x) \|_{W^{5,p}} \leq C(1 + t)^{-\frac{1}{2}(1-1/p)}E_0,
\]

\[
\| \tilde{u} - \bar{u} \|_{L^\infty(t)} \leq CE_0,
\]

\[
\| \psi(t) \|_{L^\infty} \leq CE_0
\]

for some \( \psi(x, t), \ C > 0, \) and all \( t \geq 0, \ p \geq 2. \)
Comments/extensions

Resolves question posed in [Oh-Z2001]..

- Very weak decay (compare Gaussian decay of reaction diffusion case).
- Exact analogy to viscous shock case [Mascia-Z]: $\psi = \psi(t)$.
- Corresponding result holds for multi-d [Johnson-Z2010], St-Venant [Johnson-Z-Noble2010], and (gKS) [Barker-J-Rodrigues-NZ2011].

**Remark.** (gKS) open since 1976...
Main tool: Inverse Bloch estimates (∼ Fourier transform analysis of constant-coefficient case).

Lemma (Generalized Hausdorff–Young ineq., OZ2010, JZ2010)

For $q \leq 2 \leq p$, $1/q + 1/p = 1$,

$$|u|_{L^p(x)} \leq |\tilde{u}|_{L^q(\xi,L^p([0,X])}. \quad (4)$$

**Proof.** Parseval ($p = 2$), Triangle inequality ($p = \infty$), plus analytic interpolation.

- |“good”| $|_{L^p(x)} \leq |e^{-\theta \xi^2 t}|_{L^q(\xi)} \sim Ct^{-\frac{1}{2}(1-\frac{1}{p})}$ (Gaussian rate).
- |“bad”| $|_{L^\infty(x)} \leq |\xi^{-1}e^{-\theta \xi^2 t}|_{L^1(\xi)} \sim C$ (all $\sim q = \bar{u}'(x)$).

(compare weighted energy estimate methods [Schneider] in reaction diffusion case.)
III. Low-frequency expansion: Whitham averaged equations

(Case $g = 0$) Rescaling $(x, t) \rightarrow (\epsilon x, \epsilon t)$: $u_t + f(u)_x = \epsilon u_{xx}$, and carrying out WKB expansion

$$u^\epsilon(x, t) = u^0(x, t, \frac{\psi(x, t)}{\epsilon}) + \epsilon u^1(x, t, \frac{\psi(x, t)}{\epsilon}) + \cdots,$$

as $\epsilon \rightarrow 0$ yields

$$M_t + F_x = 0,$$
$$k_t + \omega_x = 0,$$

where $M, F$ are averages of $u^0(x, t), f(u^0(x, t))$, and $k \sim \psi_x$, $\omega \sim -\psi_t$, and $s \sim -\psi_t/\psi_x$ are wave number, frequency, and wavespeed.

Formally, stability of $\bar{u} \sim$ stability of $(M, k) \equiv (\bar{M}, \bar{k})$, hyperbolicity of (6)
Relation to spectral stability

Linearized dispersion relation

\[ \hat{\Delta}(\xi, \lambda) := \det \left( \lambda \text{Id} + i \xi \frac{\partial (F, \omega)}{\partial (\bar{M}, \bar{k})} (\bar{M}, \bar{k}) \right) = 0, \]

gives first-order expansion of eigenvalues \( \lambda_j(\xi) = 0 \) of \( L_\xi, |\xi| << 1. \)

• Remarkable since Jordan block (⇒ expect square root splitting).
• There exists a second-order Whitham expansion, likewise matching spectral expansion, yielding additional diffusive stability condition (D3).
• Second-order Whitham captures asymptotic behavior ([JNRZ], work in progress).
IV. Study of St. Venant: Typical phase portrait

Figure: Limit cycle between Hopf and Homoclinic limits ($\tau'$ vs. $\tau$)
Figure: Hopf bifurcation to homoclinic limit ($\tau'$ vs. $\tau$)
Speed vs. period

Figure: $c$ vs. $X$; minimum (Hopf) period $\approx 3.9$
Figure: Period $\approx 6.2$ (green orbit), $h = \tau^{-1}$ vs. $\chi$
A general feature of thin film flow

CENTRAL FACT: IN THE REGIME OF EXISTENCE (FROUDE NUMBER $> 4$), ALL CONSTANT (HENCE ALSO HOMOCLINIC) SOLUTIONS ARE UNSTABLE!

Hence, “band” of stability... Also, stability if it occurs is due to dynamic (spatial variation) effects.
Spectra of periodic roll waves I

**Figure:** Constant solution ($\sim$ Hopf, unstable), $\Im \lambda$ vs. $\Re \lambda$
Figure: Homoclinic limit (unstable), $\Im \lambda$ vs. $\Re \lambda$
Spectra of periodic roll waves III

Figure: Intermediate-amplitude (green) solution (stable!), \( \Im \lambda \) vs. \( \Re \lambda \).
Evolution of the spectra

Change in spectrum as period is increased, $\Im \lambda$ vs. $\Re \lambda$. 
V. Study of gKS: Typical phase portrait (near KS)

Recall gKS: \( u_t + (u^2/2)_x + \delta u_{xx} + \epsilon u_{xxx} + \delta u_{xxxx} = 0. \)

Figure: Some trajectories for \( \epsilon = 0.2, \delta = 1 \)
Family of periodic orbits

Figure: One-parameter family of periodic orbits.
Stability boundaries across parameters

Figure: Stability boundaries, period $X$ vs. $\epsilon$, $\delta = \sqrt{1 - \epsilon^2}$ (shaded region $\sim$ stability). Excellent agreement with results of [Frisch-She-Thual76], [Chang-Demekhin-Kopelevich93], [Bar-Nepomnyashchyy95] by different algorithms. “Spikes” correspond to bifurcations.
Analytical verification: the small-amplitude (KdV) limit

Canonical example, singular perturbation analysis analogous to Goodman’s study for viscous shock waves—more precisely, later studies of [Gardner-Jones], [Freistühler-Szmolyan], [Plaza-Z]. (In preparation, [JNRZ].)

Fixing $\epsilon = 1$, singular perturbation

$$u_t + (u^2/2)_x + u_{xxx} = -\delta(u_{xx} + u_{xxxx}).$$

Second-order Whitham is relaxation system (see also [NR10]):

$$k_t - \omega(k, M, E)_x = 0$$
$$M_t - E_x = 0$$
$$E_t + \Sigma(k, M, E)_x = -\delta Q(k, M, E).$$

Relates low-frequency to high-frequency stability through a simple subcharacteristic condition, reduces to “intermediate-frequency” computations carried out by Bar-Nepomnyashchyi.
Thanks and final comments

• Numerical approximation of spectrum done with STABLAB (Evans function based) and SpectrUW (Galerkin based) packages. Warm thanks to Jeff Humpherys and Bernard Deconink.
• Rigorous convergence bounds. Interesting issues connected with convergence of Galerkin or discretization methods (new in generality considered here).
AND...

HAPPY 60TH BIRTHDAY TO CHRIS!