Linear stability analysis for traveling waves of second order in time PDE's

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Traveling waves

Consider the abstract second order in time nonlinear PDE
\[ u_{tt} + \mathcal{L}_x u + N(u) = 0, \quad (t, x) \in \mathbb{R}_+^1 \times \mathbb{R}^d \text{ or } (t, x) \in \mathbb{R}^1 \times [-L, L]^d, \]
where \( \mathcal{L}_x \) is a given linear operator, acting on the \( x \) variable and \( N(u) \) is the nonlinear term.

Our interest is in the study of the stability properties of traveling waves in the form \( \varphi(x + \tilde{c}t) \). These satisfy the stationary PDE
\[ \mathcal{L}_x \varphi + \sum_{i,j=1}^{d} c_i c_j \varphi_{x_i} \varphi_{x_j} + N(\varphi) = 0 \quad (1) \]

Introducing the operator \( H_{\tilde{c}} = \mathcal{L}_x + \sum_{j,k=1}^{d} c_j c_k \partial_{x_j} \partial_{x_k} + N'(\varphi) \),
we are lead to study the following problem
\[ v_{tt} + 2 \langle \tilde{c}, \nabla_x v_t \rangle + Hu = 0. \quad (2) \]
Traveling waves

Consider the abstract second order in time nonlinear PDE
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Our interest is in the study of the stability properties of traveling waves in the form \( \varphi(x + \vec{c}t) \). These satisfy the stationary PDE
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\[
v_{tt} + 2 \langle \vec{c}, \nabla_x v_t \rangle + Hu = 0. \tag{2}
\]
**Question:** For which values of $\bar{c}$, the corresponding traveling wave $\varphi_c$ determined by (1) is linearly/spectrally stable? More precisely, for which $c$, the equation $v_{tt} + 2cv_{tx} + Hu = 0$ has a solution in the form $e^{\lambda t} \psi$ and what is the sign of $\Re \lambda$?

- The **Evans function method** has been used to check the linear stability of such waves, more often for equations that are first order in time.
- The method of “indices counting”, used in KdV and Schrödinger type systems, also for spatially periodic waves and general Hamiltonian systems.
- To establish instability results, **direct construction of unstable modes** is also useful.

We present a complete answer to this question when $H$ is self-adjoint with at most one negative eigenvalue and $d = 1$. 
Linear stability

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**Milena Stanislavova, Atanas Stefanov**

Stability of traveling waves of second order in time PDE's
Boussinesq-type model

\[ u_{tt} + u_{xxxx} - u_{xx} + (u^p)_{xx} = 0, \quad (t, x) \in \mathbb{R}^1_+ \times \mathbb{R}^1 \]

There exists one-parameter family of traveling waves of the form \( \varphi(x - ct), \ c \in (-1, 1) \), which obey the equation

\[ c^2 \varphi + \varphi'' - \varphi + \varphi^p = 0 \]

and which have the explicit form

\[ \varphi_c(\xi) = \left[ \left( \frac{p + 1}{2} \right) (1 - c^2) \right]^{\frac{1}{p-1}} \text{sech}^{\frac{2}{p-1}} \left( \frac{\sqrt{1 - c^2(p - 1)}}{2} \xi \right). \]
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The Klein-Gordon-Zakharov system

interactions of Langmuir wave and ion acoustic wave in plasma

\[
\begin{aligned}
    u_{tt} - u_{xx} + u + nu &= 0 \quad (t, x) \in \mathbb{R}_+^1 \times \mathbb{R}^1 \\
    n_{tt} - n_{xx} - \frac{1}{2}(|u|^2)_{xx} &= 0,
\end{aligned}
\]

admits a one parameter family of traveling wave solutions in the form \( u(t, x) = \varphi(x - ct), n(t, x) = \psi(x - ct) \) for \( c \in (-1, 1) \),

where

\[
\begin{aligned}
    \varphi(y) &= 2\sqrt{1 - c^2}\sech\left(\frac{y}{\sqrt{1 - c^2}}\right) \\
    \psi(y) &= -2\sech^2\left(\frac{y}{\sqrt{1 - c^2}}\right).
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\]
The nonlinear beam equation

model of suspension bridge in McKenna, Walter in 1990

\[ u_{tt} + \Delta^2 u + u - |u|^{p-1} u = 0, \quad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^d \text{ or } (t, x) \in \mathbb{R}^1 \times [-L, L]^d, \]

where \( p > 1, L > 0 \) and either periodic boundary conditions (in the case \( x \in [-L, L] \)) or vanishing at infinity for \( x \in \mathbb{R}^d \).

Using variational methods, Levandosky has shown that traveling wave solutions exist and are orbitally unstable for small speeds, while orbitally stable solutions are observed for values of the parameter \( |\bar{c}| \sim \sqrt{2}, |\bar{c}| < \sqrt{2} \).

We are interested in the linear stability of such solutions in the solitary waves and the periodic waves case.
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Consider the equation

$$u_{tt} + 2\omega u_{tx} + Hu = 0, \ (t, x) \in \mathbb{R}^1 \times \mathbb{R}^1 \ \text{or} \ \mathbb{R}^1 \times [-L, L] \quad (5)$$

where $H = H_{c,\phi}$ is a self-adjoint operator acting on $L^2$, with domain $D(H)$ and $\omega$ is a real parameter.

**Definition**

We say that the periodic wave $\varphi$ is linearly unstable if there exists $\lambda : \Re \lambda > 0$, and a function $\psi$, such that the following equation is satisfied

$$\lambda^2 \psi + 2c\lambda \psi_x + H\psi = 0 \quad (6)$$

Otherwise, we say that the traveling wave $\varphi$ is linearly stable.
Consider the equation

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Otherwise, we say that the traveling wave \( \varphi \) is linearly stable.
Shkalikov’s theory for the pencil

\[ A(\lambda) = \lambda^2 F + (D + iG)\lambda + T \]
\[ T = T^*, \quad F = F^*, \quad G = G^*, \quad D \geq 0 \]

for the case \( F = \text{Id} \), \( D = 0 \), \( G = -2i\omega \partial_x \), \( T = H_c \),

\[ \#\{\lambda - \text{unstable}\} \leq \#\{\sigma(H_c) \cap (-\infty, 0)\} \]

Assumption

- \( H_c \) has one simple negative eigenvalue
- \( 0 \) is simple eigenvalue for \( H_c \)
Stability/instability results for quadratic pencils

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- \( H_c \) has one simple negative eigenvalue
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Theorem

Let $\psi_0 : \|\psi_0\| = 1, H\psi_0 = 0$.

- If $\langle H^{-1}[\psi'_0], \psi'_0 \rangle \geq 0$, - instability for all $\omega \in \mathbb{R}^1$.
  Otherwise,

  - the problem (5) is unstable if

    \[
    0 \leq |\omega| < \frac{1}{2\sqrt{-\langle H^{-1}[\psi'_0], \psi'_0 \rangle}} =: \omega^*(H)
    \]  \hspace{1cm} (7)

  - the problem (5) is stable, if

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   Otherwise,
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   \[ |\omega| \geq \omega^*(H) \]
Corollary

We can write (5) in the form

\[
\begin{pmatrix} u \\ ut \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ -H & -2\omega \partial_x \end{pmatrix} \begin{pmatrix} u \\ ut \end{pmatrix} =: \mathcal{T} \begin{pmatrix} u \\ ut \end{pmatrix}
\]

Corollary

In the statement of Theorem 2, assume in addition that 
\([Hh(\cdot)](x) = (Hh)(-x)\). Then, in the cases of instability, there is \(\lambda > 0\), so that \(\lambda, -\lambda\) are both eigenvalues of \(\mathcal{T}\) and moreover

\[
\sigma(\mathcal{T}) \subset \{\lambda\} \cup \{-\lambda\} \cup i\mathbb{R}^1.
\]

If on the other hand, there is stability, we have \(\sigma(\mathcal{T}) \subset i\mathbb{R}^1\).
Construction of a special function

\[ \lambda^2 \psi + 2\omega \lambda \psi' + H\psi = 0 \]

Take \( \psi := \phi + \nu \), where \( H\phi = -\delta^2 \phi; \nu \perp \phi \).

\[
(\lambda^2 + 2\omega \lambda \partial_x + H)\nu = (\delta^2 - \lambda^2)\phi - 2\omega \lambda \phi' \quad \tag{9}
\]

Thus,

\[
\langle \nu, \phi' \rangle = \frac{\lambda^2 - \delta^2}{2\omega \lambda} \quad \text{and} \quad \nu = -2\omega \lambda [H + \lambda^2 + 2\omega \lambda P_0 \partial_x P_0]^{-1} [\phi'] \in \{\phi\}^\perp
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Construction of a special function

\[ \lambda^2 \psi + 2\omega \lambda \psi' + H \psi = 0 \]

Take \( \psi := \phi + v \), where \( H\phi = -\delta^2 \phi \); \( v \perp \phi \).

\[ (\lambda^2 + 2\omega \lambda \partial_x + H)v = (\delta^2 - \lambda^2)\phi - 2\omega \lambda \phi' \]  \hspace{1cm} (9)

Thus,

\[ \langle v, \phi' \rangle = \frac{\lambda^2 - \delta^2}{2\omega \lambda} \]

\[ v = -2\omega \lambda [H + \lambda^2 + 2\omega \lambda P_0 \partial_x P_0]^{-1}[\phi'] \in \{ \phi \}^\perp \]
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Construction of a special function - “Evans function like behavior”

Proposition

\[
\mathcal{G}(\omega; \lambda) := \left\langle [H + \lambda^2 + 2\omega \lambda P_0 \partial_x P_0]^{-1} [\phi'], \phi' \right\rangle + \frac{\lambda^2 - \delta^2}{4\omega^2 \lambda^2} \quad (10)
\]

\(\mathcal{G}\) has a positive root.
We show the continuity of the function $G(\omega; \lambda)$ in $\mathbb{R}^1_+ \times \mathbb{R}^1_+$, then analyze its behavior close to $\lambda = \infty$ and close to $\lambda = 0$. We have that

$$
\lim_{\lambda \to \infty} G(\lambda) = \frac{1}{4\omega^2} > 0.
$$

Regarding the behavior close to $\lambda = \varepsilon \sim 0$, we compute the sign of $G(\omega; \varepsilon)$. If $G(\omega; \varepsilon) < 0$, we have instability. If $G(\omega; \varepsilon) > 0$, we have stability, two zeroes or a double zero. One can exclude the case of two zeroes by Schkalikov’s theory. The double zero needs a bit more work but can be excluded as well. Thus we can prove stability in this case.
We show the continuity of the function $\mathcal{G}(\omega; \lambda)$ in $\mathbb{R}_+^1 \times \mathbb{R}_+^1$, then analyze its behavior close to $\lambda = \infty$ and close to $\lambda = 0$. We have that

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"good" Boussinesq model $u_{tt} + u_{xxxx} - u_{xx} + (u^p)_{xx} = 0$

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\begin{align*}
    v_{tt} + 2cv_{tx} + Tv &= 0 \\
    Tv &= \partial_x^4 v - (1 - c^2)\partial_x^2 v + p(\varphi_c^{p-1} v)_{xx},
\end{align*}
\]

**Note:** $T$ is not self-adjoint. In the variable $z : z_x = v$ it becomes

\[
\begin{align*}
    z_{tt} + 2cz_{tx} + Hz &= 0 \\
    H = H^* &= \partial_x^4 - (1 - c^2)\partial_x^2 + p(\varphi_c^{p-1} (\cdot)_{xx}),
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**Note:** $H$ has one simple eigenvalue at zero and one simple negative eigenvalue.
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\begin{align*}
\nu_{tt} + 2c\nu_{tx} + T\nu &= 0 \\
TV &= \partial_x^4 \nu - (1 - c^2)\partial_x^2 \nu + p(\varphi^{-1}_c \nu)_{xx},
\end{align*}
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**Note:** $T$ is not self-adjoint. In the variable $z : z_x = \nu$ it becomes

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\begin{align*}
z_{tt} + 2cz_{tx} + Hz &= 0 \\
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**Note:** $H$ has one simple eigenvalue at zero and one simple negative eigenvalue.
The traveling wave $\varphi_c$ of the Boussinesq equation is linearly unstable, if $p \geq 5$. If $2 \leq p < 5$, then it is linearly unstable if $0 \leq |c| < \frac{\sqrt{p-1}}{2}$ and linearly stable, when $\frac{\sqrt{p-1}}{2} \leq |c| < 1$.

- Bona-Sachs’88 - orbital stability for $1 > |c| > \frac{\sqrt{p-1}}{2}, p < 5$
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Recall $H = -\partial_x \mathcal{L} \partial_x$. In our notations, we need to compute
\[
\langle H^{-1} \psi_0', \psi_0' \rangle = \frac{1}{\|\varphi_c\|^2} \langle H^{-1} \varphi_c', \varphi_c' \rangle = -\frac{1}{4c} \frac{\partial_c [\|\varphi_c\|^2]}{\|\varphi_c\|^2} = \frac{5-p}{4(p-1)(1-c^2)}.
\]
We set up the inequality to find the intervals, in which the speeds yield stable traveling waves.

\[
1 > |c| \geq \omega^*(H) = \frac{\sqrt{(p-1)(1-c^2)}}{\sqrt{5-p}}.
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The solution to this inequality is $1 > |c| \geq \frac{\sqrt{p-1}}{2}$ and we have stability here.

In the complementary set, $0 \leq |c| < \frac{\sqrt{p-1}}{2}$, we have instability.
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Linearized system with
\[(\varphi, \psi) = (2\sqrt{1 - c^2} \text{sech}(y / \sqrt{1 - c^2}), -2\text{sech}^2(y / \sqrt{1 - c^2})\]

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\ddot{\Phi}_{tt} - 2c\dot{\Phi}_{tx} + H\dot{\Phi} = 0, \quad H := \begin{pmatrix} H_1 & A \\ A^* & H_2 \end{pmatrix},
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H_1 = -(1 - c^2)\partial_{xx} + 1 + \psi = -\mu^2\partial_{xx} + 1 - \frac{\varphi^2}{2\mu^2}
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The operator $\mathcal{H}$ is self-adjoint and it satisfies the requirements of Theorem 2.

**Theorem**

Let $c \in (-1, 1)$. Then, the traveling wave solution $(\varphi(x - ct), \psi(x - ct))$ described in (4) is linearly stable for $|c| \in [\sqrt{\frac{2}{3}}, 1)$ and linearly unstable for $|c| \in [0, \sqrt{\frac{2}{3}})$.

The linear stability results match precisely the orbital stability results by Chen, except at the endpoints $|c| = \sqrt{\frac{2}{3}}$. At this point, we have linear stability, according to the Theorem, but it is unclear whether the wave is orbitally stable or not.

**Coming up:**
- whole line waves for KGZ- any $p$
- periodic waves for Boussinesq and KGZ, $p = 2, 3$
- Klein-Gordon models

Milena Stanislavova, Atanas Stefanov
Stability of traveling waves of second order in time PDE's
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Thank you for your attention.