Quenching Solutions of a fourth order nonlinear parabolic PDE modeling a MEMS capacitor

Geometric Methods for Infinite-Dimensional Dynamical Systems

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**Background**

- MEMS ≡ (Electronics + Machinery) × ε
- Resonant Gate Transistor - Nathanson et al (1964)

**Figure:** Source: mems.sandia.gov

**Challenges.**

- Devices well understood on macro scales need to be redesigned.
- New physical processes need to be understood to effectively design MEMS.

- Beam deflecting in the presence of an electric field.
- Touchdown possible for large voltage $V > V^*$. 
- Device can act as a switch, valve or just capacitor.
- Loss of a stable solution known as the *pull-in* instability.
PDE Model.

- Incorporate elements of linear elasticity and electrostatics to capture geometry of deflecting plate.
- Implement small aspect ratio $d/L \ll 1$.

$$\frac{\partial u}{\partial t} = -\Delta^2 u - \frac{\lambda}{(1 + u)^2}, \quad x \in \Omega;$$
$$u(x, 0) = 0, \quad x \in \Omega,$$
$$u = \partial_n u = 0, \quad x \in \partial\Omega.$$

- $\lambda \sim V^2$ quantifies relative strengths of electrostatic and elastic forces.
- Clamped Boundary conditions - Zero deflection and gradient.
- Pelesko (2001)
Structure of talk.

Under what conditions does touchdown occur?

- Where does touchdown occur?

- What is the local structure of singular solutions near touchdown?

- Future Work
Theorem (Existence of Pull-In Voltage & Touchdown): No equilibrium solutions for $\lambda > \bar{\lambda} = \frac{4\mu_0}{27}$ when $\Omega = [-1, 1]$ or $\Omega = \{x^2 + y^2 \leq 1\}$ where $(\phi_0, \mu_0)$ first eigenpair of

$$\Delta^2 \phi_0 = \mu_0 \phi_0, \quad x \in \Omega; \quad \phi_0 = \partial_n \phi_0 = 0; \quad x \in \partial \Omega$$

When $\lambda > \bar{\lambda} = \frac{4\mu_0}{27}$, touchdown occurs in finite time.

Problem: Proof requires positivity of $\phi_0$ which is not true in general.
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**Problem:** Proof requires positivity of \( \phi_0 \) which is not true in general.
Time dependent problem, $\Omega = (0, 1)$

(a) $\lambda < \lambda^*$

(b) $\lambda > \lambda^*$

(c) $\lambda \gg \lambda^*$
✓ Under what conditions does touchdown occur?

- When $\lambda > \lambda^*$ for some $\lambda^* < \frac{4\mu_0}{27}$ when $\Omega$ 1D and unit disc in 2D.

⇒ Where does touchdown occur?

- What is the local structure of singular solutions near touchdown?

- Future Work
Rescale Equation

Reduction

- Rescale with $t \to \lambda^{-1}t$ and $\epsilon^2 = \lambda^{-1}$.
- Large $\lambda$ is now small $\epsilon$. Analyze in the limit as $\epsilon \to 0$.

$$u_t = -\epsilon^2 \Delta^2 u - \frac{1}{(1 + u)^2}; \quad x \in \Omega.$$  

$$\Delta^2 u = u''' + \frac{2(N - 1)}{r} u''' - \frac{(N - 1)}{r^2} u'' + \frac{(N - 1)}{r^3} u'.$$

Numerics (MOVCOL4, Williams et al. (2007)), (Budd et al. 2009).

Use moving mesh methods to accurately resolve touchdown event. Grid evolution:

$$-\gamma \dot{X}_{\xi\xi} = (M(u)X_{\xi})_{\xi}, \quad \gamma \ll 1; \quad M(u) = \frac{1}{(1 + u)^3} + \int_{\Omega} \frac{1}{(1 + u)^3}$$
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Numerics

Strip: \( \Omega = [-1, 1] \)

(d) Clamped Strip and \( \epsilon = 0.02 \)

(e) \( \epsilon = 0.02 \) and \( N = 24 \) Zoomed.
Unit Disc: \( \Omega = \{ |x| \leq 1 \} \)

(f) Unit Disc and \( \epsilon = 0.02 \)

(g) \( \epsilon = 0.02 \) and \( N = 16 \) Zoomed.
Small $t$ behaviour:

- Flat central region coupled to a propagating boundary effect.
- In the flat central region, $u(x, t) \sim f(t)$;
  \[
  f_t = -\frac{1}{(1 + f)^2}, \quad f = -1 + (1 - 3t)^{1/3}
  \]

- Propagating boundary effect (at $x = 1$) in stretching coordinates:
  \[
  u(x, t) \sim -f(t) v_0(\eta) \quad \eta = \frac{1 - x}{\epsilon^{1/2} f^{1/4}}
  \]

- $t \to 0$ corresponds to $f \to 0$ so the $(1 + u)^{-2}$ term is linearized.

Touchdown Behaviour: Small $(t_c - t)$

- $u = -1$ is a global attractor
Stretching Boundary Layer:

After analysis:

\[
\frac{d^4 v_0}{d\eta^4} - \frac{\eta}{4} \frac{dv_0}{d\eta} + v_0 = -1, \quad \eta > 0; \quad v_0 \sim -1 \quad \text{as} \quad \eta \to \infty
\]

Solution:

\[
u_0(\eta) \sim -f(t) v_0(\eta) \quad \eta = \frac{1 - x}{\epsilon^{1/2} f^{1/4}}
\]
Comparison to full numerics

(h) Strip and $\epsilon = 0.02$

(i) Unit Disc $\epsilon = 0.02$
Critical Point Location:

Follow the first peak of the asymptotic solution:

- Critical Point $\eta_c$:

  \[ \eta_c(t) = \eta_0 + f(t)\eta_1 + f^2(t)\eta_2 + f^3(t)\eta_3 + \cdots \]

- Solve $\nu_\eta(\eta_c) = 0$ to highest possible order for above expansion:

- Prediction of touchdown point(s) is $x_c = \pm \left(1 - \epsilon^{1/2}f(t)^{1/4}\eta_c(t_c)\right)$.

**Figure**: Solid Line: Numerics, Dashed Line: Formula using Exact touchdown term, Dotted Line: Curve from $t_c = 1/3$. 
✓ Under what conditions does touchdown occur?
✓ Where does touchdown occur?
  - At multiple points if $\epsilon < \epsilon_c$ due to a non-monotone profile $v_0(\eta)$ in a moving boundary layer.
  - Multiple touchdown on two distinct points symmetric about the origin when $\Omega = [-1, 1]$ and on a ring of points when $\Omega = \{|x| \leq 1\}$.

What is the local structure of singular solutions near touchdown?

Future Work
Quenching solutions appear to be self-similar.

Seeking a touchdown profile of form:

\[ u(x, t) = (t_c - t)^{1/3} v(\eta), \quad \eta = \frac{|x - x_c|}{\epsilon^{1/2}(t_c - t)^{1/4}} \]

\[-v_{\eta\eta\eta} - \frac{\eta}{4} v_{\eta} + \frac{v}{3} = \frac{1}{v^2}, \quad \eta \in \mathbb{R}; \quad v \sim c_0 \eta^{4/3}, \quad \eta \to \pm \infty.\]

Figure: Full numerical solutions (dashed lines) converging to \( \bar{v}(\eta) \) (solid line).

(a) \( \epsilon = 0.2 \) with \( x_c = 0 \).

(b) \( \epsilon = 0.02 \) with \( x_c \neq 0 \).
Future Work.

- Stability of ring-like touchdown sets.
- What is the touchdown set for more general geometries?
- Is asymmetric touchdown possible?
- What happens to solutions beyond touchdown?
Square Geometry.

(a) Square
The End.

Thank You! Questions?