Brown/Paris Numerical Analysis: Problem set 11

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1 Numerical methods for solving PDE: Spectral Galerkin

In yesterday's lecture we learned how to derive a collocation method by selecting a subspace for our approximation functions

$$\mathbf{V}_h \subset \mathbf{C}_0^\infty([0,\pi])$$

where we constructed our subspace as that spanned by the truncated Fourier series

$$\mathbf{V}_h = span\left\{\sin mx\right\}_{m=1,\dots,M}$$

We then demonstrated how to derive an invertible system of equations by testing a function against a space of measurement functions

$$\mathbf{M}_h = span\left\{\delta(x-x_i)\right\}_{i=1,\dots,M}$$

Recall that to find an approximation to a in $\mathbf{V}_h$ of a function f, we required that for all $\phi_i \in \mathbf{M}_h$ the approximation satisfied

$$(\phi_i, u) = (\phi_i, f)$$

In today's class, we obtain the spectral Galerkin method by using the same

space $\mathbf{V}_h$, but switch our measurement space $\mathbf{M}_h = \mathbf{V}_h$. We will repeat the exercises from yesterdays assignment modified to handle this small change. Note that if you're careful you should be able to reuse yesterday's code; if you finish this assignment quickly you're welcome to use the remainder of class to work on projects.

You'll find the following identities useful:

- For $m \neq n$
$$\int_0^\pi sin(mx)sin(nx)dx = (n\sin(\pi m)\cos(\pi n) - m\cos(\pi m)\sin(\pi n))/(m^2-n^2)$$
- For $m = n$
$$\int_0^\pi sin(mx)sin(nx)dx = \pi/2 - (\sin(2m\pi))/(4m)$$

Exercises:

- Build an approximation to the functions:

$$u(x) = exp(-4(x - \pi/2)^2) - exp(-4(0 - \pi/2)^2)$$

$$u(x) = \mathbb{1}_{|x-\pi/2|<\pi/6}(x)$$

- Construct a log-log plot in the usual way comparing the maximum pointwise error to the number of gridpoints used. How does the approximation error scale differently between the two functions?

- Solve the PDE

$$u''(x) = 8(8x^2 - 8\pi x + 2\pi^2 - 1)exp(-4(x - \pi/2)^2)$$

$$u(0) = u(\pi) = 0$$

  by approximating $u \approx u_h$ and testing against each measurement function to obtain a system of equations for the polynomial coefficients in the series. How does the maximum pointwise error for this converge?

- How does this approach differ from the finite difference method we discussed earlier? Spectral collocation? How can we characterize the cost of the method? When do you think we should use one or the other?