Brown/Paris Numerical Analysis: Problem set 10

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July 21, 2015

1 Numerical methods for solving PDE: Spectral collocation

This week we will be learning several methods for numerically solving boundary value problems in 1D. Today we will learn how to approximate smooth functions with Fourier series. Let

$$u \in C_0^\infty([0,\pi]) := \{\text{infinitely continuous functions s.t. } u(0) = u(\pi) = 0\}$$

. We seek an an approximation $u_h \approx u$ such that $u_h \in C_0^\infty([0,\pi])$. One possible choice of such functions is the Fourier sine series

$$u_h(x) = \sum_n^\infty a_n \sin nx$$

Let $\mathbf{X} = \{x_1, \cdots, x_N\}$ be a set of grid points such that $x_i = i\Delta$ for $\Delta = \pi/(N+1)$. We will build an approximation to our function u by truncating the sine series to N terms and requiring interpolation at N point, i.e. for $i = 1, \ldots, N$

$$u(x_i) = u_h(x_i)$$

- Build an approximation to the functions:

$$u(x) = exp(-4(x-\pi/2)^2) - exp(-4(0-\pi/2)^2)$$

$$u(x) = \mathbb{1}_{|x-\pi/2|<\pi/6}(x)$$

- Construct a log-log plot in the usual way comparing the maximum pointwise error to the number of gridpoints used. How does the approximation error scale differently between the two functions?

- Solve the PDE

$$u''(x) = 8(8x^2 - 8\pi x + 2\pi^2 - 1)exp(-4(x-\pi/2)^2)$$

$$u(0) = u(\pi) = 0$$

by approximating $u \approx u_h$ and testing against each grid point to obtain a system of equations for the polynomial coefficients in the series. How does the maximum pointwise error for this converge?

- How does this approach differ from the finite difference method we discussed earlier? How can we characterize the cost of the method? When do you think we should use one or the other?