1D Euler Equations

\[ \frac{\partial e}{\partial t} + \frac{\partial}{\partial x}(en) = 0 \]

\[ \frac{\partial (en)}{\partial t} + \frac{\partial}{\partial x}(enu + p) = 0 \]

\[ \frac{\partial (ee)}{\partial t} + \frac{\partial}{\partial x}(enuE + np) = 0 \]

We'll consider an equivalent EOS for gases to obtain a simpler derivation.

\[ p = \rho (E - \frac{1}{2} u^2) (\gamma - 1) \quad \text{where } \gamma \text{ is ratio of specific heats} \]

Air \( \rightarrow \gamma = 1.4 \)

Showed last week that \( \Theta \) can be put into conservative form

\[ U_t + \left[ F(u) \right]_x = 0 \]

\[ U = \begin{bmatrix} e \\ eu \\ ee \end{bmatrix}, \quad F(u) = \begin{bmatrix} en \\ enu + p \\ enuE + np \end{bmatrix} \]

Alternatively, we can write in terms of primitive variables

\[ g_t + A(g) g_x = 0 \]

\[ g = \begin{bmatrix} e \\ u \\ p \end{bmatrix}, \quad A = \begin{pmatrix} 0 & e & 0 \\ 0 & u & 0 \\ 0 & 0 & p \end{pmatrix} \]

\[ \Rightarrow \begin{pmatrix} \frac{\partial g}{\partial t} \\ \frac{\partial g}{\partial x} \end{pmatrix} = - \begin{pmatrix} (u e o) \\ 0 u p \\ 0 0 p \end{pmatrix} \]
- We'll spell out the first two equations

\[ \begin{align*}
0 &= \partial_t e + \partial_x \eta n = \partial_t e + u \partial_x e + \varepsilon \partial_x n \\
0 &= \partial_t \eta e + \partial_x \eta n + \varepsilon \partial_x \eta = \varepsilon \partial_t e + u \partial_x e + u \partial_x \eta + \varepsilon \partial_x n + \varepsilon \partial_x \eta
\end{align*} \]

(continuity \( \varepsilon \partial_n \rightarrow 0 \))

\[ \begin{align*}
0 &= \varepsilon \partial_t n + u \varepsilon \partial_x n + \varepsilon \partial_x \eta \\
&= \partial_t n + u \partial_x n + \frac{1}{\varepsilon} \partial_x \eta
\end{align*} \]

(-the third is similar but more tedious)

- We can study the characteristics of \( \ast \ast \) by finding the eigenvalues of \( \mathbf{A} \)

\[ \text{Solve for } \lambda \text{ such that } \det(A - \lambda I) = 0 \]

\[ (-\varepsilon p^2 u + u^3) + (\varepsilon \rho^2 - 3u^2) \lambda + 3u \lambda^2 - \lambda^3 = 0 \]

\[ \lambda = \left\{ u, u + \sqrt{\frac{3u^2}{\varepsilon}}, u - \sqrt{\frac{3u^2}{\varepsilon}} \right\} \]

\( C = \text{Speed of sound} \)

- For those without a PDE background, I'll sketch a brief picture of what this tells us. The reference I pulled these notes from is on the website, if people want the technical details.
- Eigenvalues let us diagonalize A

\[ I = L \Lambda R \]

where \( \Lambda = \text{diag} (\lambda_1, \lambda_2, \lambda_3) \)

left/right eigenvectors \( LR = RL = I \)

So multiplying on the left, right, and by I

\[ q_t = -A q_x \]

\[ L q_t R = -LA I q_x R \]

\[ = -LAR L q_x R \]

Let \( w = L q_x R \)

\[ \begin{vmatrix} \hat{w}_t = -\Lambda \hat{w}_x \end{vmatrix} \]

and we have a set of 3 decoupled PDEs

\[ \partial_t w_i + \lambda_i \partial_x w_i = 0 \]

So we obtain a collection of transport equations where information propagates at a speed \( \lambda_i \in \{u, u+c, u-c\} \)
Shallow water equations

- Most people don't have experience with supersonic jet, but most know what water waves in a bathtub look like.

Compressible CNavier-Stokes (1D, primitive, isothermal)
\[ \frac{\partial}{\partial t} e + \frac{\partial}{\partial x} (\rho u) = 0 \]
\[ \frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho u) = -\frac{\partial}{\partial x} p \]
\[ \rho = \frac{e}{c} = \frac{RT}{\gamma} \]

Shallow water equations
\[ \frac{\partial}{\partial t} \eta + \frac{\partial}{\partial x} (\eta u) = 0 \]
\[ \frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u = -g \frac{\partial}{\partial x} \eta \]

For \( h = 0 \) (think bathtub), we can identify
\[ \rho \leftrightarrow \eta \]
\[ u \leftrightarrow u \]
and it is "equivalent" to talk about water waves or density waves

- For the shallow water equations, we could apply the same eigenvalue analysis to see that
\[ \lambda = \left\{ \begin{array}{l} u, \quad \gamma + \sqrt{g(\eta \rho)} \end{array} \right. \]

Easier to interpret characteristics of water waves.

capillary speed of sound
- For water waves, wave height holds potential energy

Take $h = 0$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\rho \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right)$$

\[\text{advection}\] \[\text{squeezing}\]

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\rho \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right)$$

\[\text{squeezing cartoon}\]

- Euler analogy

Same idea but disturbances propagate as plane waves

\[\text{High } p \rightarrow \leftarrow \text{ } \]
- Consider disturbance tapping water every $\Delta t$

- Each disturbance will propagate at a rate $c$

- Now consider disturbance with speed $u < c$. Define $Ma^2$.

- If $u = c$, wave fronts all coincide at same point

\[ \text{Diagram with circles representing wave fronts and arrows indicating propagation directions.} \]
If \( u > c \), waves can't "keep up" with disturbance.

For water waves, \( c \) is slow - think about the speed of surfers.

The propagation cone sketched above is what's commonly referred to as a wake for a ship (pictures on website).

For compressible gas dynamics, several differences:
- Not so easy to visualize density compression
- Schlieren photography (pictures on website)
- Waves propagate differently through \( \rho, P, u \)

Today, we'll look at a test problem for studying how a code can track these discontinuities.
Shock tube

- 1D model for flow in a shock tube (Sod 1978, see website)

\[ p_L = 1.0 \quad p_R = 0.125 \]
\[ p_L = 1.0 \quad p_R = 0.1 \]

thin membrane that "bursts" at \( t = 0 \)

- While a water wave would propagate only in \( h \),
for us we'll get different types of shocks that
will propagate from initial discontinuity.

Today

Implement in 1D code and in OpenFoam