CFD - Lecture 4

- Conservation of momentum
- Burger's equation
- An introduction to compressible Euler equations
- Foam Tutorial

First, everybody log in to CCV.

Conservation of Momentum

\[ \frac{d}{dt} \left( \text{momentum in system} \right) = \sum F_i \] (Forces acting on system)

- think \( F = ma \) from high school

Let \( \rho = \text{density} \left[ \frac{m}{L^3} \right] \)
\( U = \text{velocity} \left[ \frac{L}{T} \right] \)

\[ \frac{d}{dt} \int \rho U \, dx + \int \rho (u_n) \, u \cdot n \, dA = \sum F_i \]

Ex. Drag on a cylinder in a wind tunnel

- Assume a uniform velocity at inlet \( U_0 \)
- a symmetric velocity at top and bottom
- a measured velocity profile at outlet \( u(y) \)

- Because of drag acting on obstacle, velocity at outlet is slower, causing mass flux at top and bottom
- By symmetry, assume equal & opposite
From conservation, can recover force on obstacle from
outlet vel.

\[ u(y) = \begin{cases} U_\infty & |y| > h \\ U_\infty \left(1 - \frac{1}{a} \left(\frac{y}{h}\right)^2\right) & |y| < h \end{cases} \]

\[ \int_{-\infty}^{\infty} u_n dx + \int_{-\infty}^{\infty} (eu)_n \cdot (u_n n) dA = -F \]

\[ \int_{AC} (eu) u_n dA + \int_{BD} (eu) u_n dA = -F \]

\[ e \left( \int_{-h}^{h} u_\infty (-u_\infty) dy + \int_{-h}^{h} u(y)^2 dy \right) = -F \]

\[ e \left( -u_\infty^2 \int_{-h}^{h} dy + \int_{-h}^{h} \left[U_\infty \left(1 - \frac{1}{a} \left(\frac{y}{h}\right)^2\right)\right]^2 dy \right) = -F \]

\[ e \left( -2h U_\infty^2 + \frac{43}{30} h U_\infty^2 \right) = -F \]

\[ F = \frac{17}{30} e h U_\infty^2 \]
In general, forces can take form of:
point force, surface force, body force

\[ F = F_p + \int_{A_s} \sigma \, dA + \int_{A_b} \sigma_b \, dA \]

We'll see more about this next week when we discuss Navier-Stokes derivation.

**Burgers Equation**
- Taking \( \sum F = 0 \), \( \epsilon = 1 \) gives
- Burgers equation is a model for 1D non-linear advection → exact solutions mean that it's easy to prove/demonstrate convergence
- Expand flux term to see

\[ \partial_t u + \partial_x (\frac{1}{2}u^2) = 0 \]

\[ \partial_t u + u \partial_x u = 0 \]
- 1D model problem

- Take \( u = \sin x \)

- Advection magnitude is non-uniform and depends on solution

- Solve the exact same way but we'll just be careful about picking the right upstream value

- HW3

**Compressible Euler Equations (CEE)**

- What we have now is a suite of techniques for handling a variety of physics

\[
\begin{align*}
\partial_t (en) + \nabla \cdot (eu^2) &= -\nabla p + \rho \nabla^2 u \\
\partial_t u + \nabla \cdot (eu) &= 0 \\
\rho &= \gamma p(T) \\
\end{align*}
\]

Next week we derive these, but first we'll warm up in Fam...
- In last class we did flow through a pipe.
- For easy geometry, I'll include calculation for flow between infinite plates here

- \( U_{in} = U_\infty \)

- \( U_{out} = C \ y \left( 1 - \frac{y}{L} \right) \)

- \( \oint_{\partial \Omega} \mathbf{u} \cdot \mathbf{n} \, dA = 0 \)

- \( \sum_{\Omega_1} \mathbf{u} \cdot \mathbf{n} \, dA + \sum_{\Omega_2} \mathbf{u} \cdot \mathbf{n} \, dA = 0 \quad dA = W \, dy \)

- \( - \int_0^L U_\infty W \, dy + \int_0^L C \ y \left( 1 - \frac{y}{L} \right) W \, dy = 0 \)

- \( - U_\infty L + C \int_0^L y \left( 1 - \frac{y}{L} \right) \, dy = 0 \)

- \( - U_\infty L + \frac{C \ L^2}{6} = 0 \quad \Rightarrow \quad C = \frac{6 U_\infty}{L} \)

\[ U_{out}(y) = \frac{6 U_\infty}{L^2} \ y \left( L - y \right) \]