AM119: Assignment 2 - (Due beginning of class 3/7)

Prof. Trask

March 1, 2016

1 Finite volume solution of the 1D periodic advection-diffusion equation

In our last homework assignment we implemented the finite difference method to solve the 1D advection diffusion equation. Today, we've introduced the finite volume method, which we will use at length for the remainder of the class. For this assignment, we will be repeating the previous homework but swapping in a finite volume discretization. This should require minimal changes to the previous homework assignment. A solution to the previous assignment will be put on the website tomorrow (3/1) - if you had difficulty completing the previous assignment feel free to use the solution as a starting point.

Goal: We seek a numerical solution of the advection-diffusion equation.

$$\partial_t u + a\partial_x u - \nu \partial_{xx} u = 0$$

with periodic boundary conditions

$$u(0,t) = u(2\pi, t)$$

and initial condition

$$u(x,0) = \sin(x)$$

- We first write the PDE in conservation form and discretize in time to obtain the semi-discrete problem:

$$\frac{u^{n+1} - u^n}{\Delta t} + \nabla \cdot \mathbf{F}(u^n) = 0$$

$$\mathbf{F}(u^n) = \mathbf{a}u^n - \nu \nabla u^n$$

- For each cell C_i, we integrate to obtain

$$\frac{\overline{u}_i^{n+1} - \overline{u}_i^n}{\Delta t} = \int_{C_i} \frac{u^{n+1} - u^n}{\Delta t} dx = -\int_{C_i} \nabla \cdot \mathbf{F}(u^n) dx = -\left(F(u^n)_{i+\frac{1}{2}} - F(u^n)_{i-\frac{1}{2}} \right)$$

  where we define $\overline{u}_i = \int_{C_i} u dx$.

- To simplify notation, we will associate the mean value with the function evaluated at the cell centroid

$$u_i = \frac{1}{V_i}\int_{C_i} u dx = \frac{\bar{u}_i}{V_i}$$

to obtain the following update formula

$$V_i \frac{u_i^{n+1} - u_i^n}{\Delta t} = -\left(F(u^n)_{i+\frac{1}{2}} - F(u^n)_{i-\frac{1}{2}}\right)$$

- To approximate the fluxes through the faces of each cell, we split the flux into an advection component and a diffusion component

$$F(u) = F_a(u) + F_d(u)$$

- We discretize the advection term with the upwind cell value and the diffusion term with a centered finite difference as follows:

$$F_a(u^n)_{i+\frac{1}{2}} = a u_i^n$$

$$F_d(u^n)_{i+\frac{1}{2}} = -\nu \frac{u_{i+1}^n - u_i^n}{\Delta x}$$

- Modify your code from HW1 to solve the 1D periodic problem using this finite volume scheme. For parameters, again take $a = \nu = 1$, and choose Δx and Δt so that $\frac{a\Delta t}{\Delta x} < \frac{1}{2}$ and $\frac{\nu \Delta t}{\Delta x^2} < \frac{1}{4}$. For an initial condition, take $u(x, 0) = sin(x)$ and compare your final result to the exact solution $u(x, t) = e^{-t}\sin(x - at)$.

- Convince me that your code works - provide evidence that for more points and for smaller timesteps your code converges to the exact solution.

- At each timestep, calculate the quantity $I^n = \int_\Omega u dx = \sum_{i=1}^N V_i u_i^n$ and write to screen. In addition to a plot showing that you get the correct solution, provide a plot showing how I^n evolves over time. Explain this behavior in your writeup.

When turning in this, and other assignments, make sure to leave a functioning copy of the code that you've run in your data directory. Prepare a 1-2 page short report demonstrating your results with properly labeled plots/tables, and submit to me via email with the subject line APMA119-YourName-HWXX. It is not necessary to print/email me any code - I will look into your data directory and make sure that your code backs up your claim. Please make sure to include in your report the directory and filename where I can find your code.