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Foreword: The Synergy of Pure and Applied Mathematics, of the Abstract and the Concrete

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All of us mathematicians have discovered a sad truth about our passion: It is pretty hard to tell anyone outside your field what you are so excited about! We all know the sinking feeling you get at a party when an attractive person of the opposite sex looks you in the eyes and asks—"What is it you do?" Oh, for a simple answer that moves the conversation along.

Now Mircea Pitici has stepped up to the plate and for the third year running has assembled a terrific collection of answers to this query. He ranges over many aspects of mathematics, including interesting pieces on the history of mathematics, the philosophy of mathematics, mathematics education, recreational mathematics, and even *actual presentations of mathematical ideas*! This volume, for example, has accessible discussions of n-dimensional balls, the intricacies of the distribution of prime numbers, and even of octonions (a strange type of algebra in which the "numbers" are 8-tuples of the ordinary sort of number)—none of which are easy to convey to the layperson. In addition—and I am equally pleased with this—several pieces explain in depth how mathematics can be used in science and in our lives—in dancing, for the traveling salesman, in search of marriage, and for full-surround photography, for instance.

To the average layperson, mathematics is a mass of abstruse formulae and bizarre technical terms (e.g., perverse sheaves, the monster group, barreled spaces, inaccessible cardinals), usually discussed by academics in white coats in front of a blackboard covered with peculiar symbols. The distinction between mathematics and physics is blurred and that between pure and applied mathematics is unknown.

But to the professional, these are three different worlds, different sets of colleagues, with different goals, different standards, and different customs.

The layperson has a point, though. Throughout history many practitioners have crossed seamlessly between one or another of these fields. Archimedes not only calculated the volume of a ball of radius r (a pure mathematics problem with the answer $4\pi r^3/3$) but also studied the lever (a physics problem) and used it both in warfare (applied mathematics: hurling fiery balls at Roman ships) and in mind experiments ("Give me a place to stand and I will move the earth"). Newton was both a brilliant mathematician (inventing calculus) and physicist (discovering the law of gravity).

Today it is different: The three fields no longer form a single space in which scientists can move easily back and forth. Starting in the mid-twentieth century, mathematicians were blindsided by the creation of quantum field theory and even more by string theory. Here physicists, combining their physical intuition with all the latest and fanciest mathematical theories, began to use mathematics in ways mathematicians could not understand. They abandoned rigorous reasoning in favor of physical intuition and played wildly with heuristics and extrapolations from well-known mathematics to "explain" the world of high energy. At about the same time (during the '50s and '60s), mathematics split into pure and applied camps. One group fell in love with the dream of a mathematics that lived in and for itself, in a Platonic world of blinding beauty. The English mathematician G. H. Hardy even boasted that his work could never be used for practical purposes. On the other side, another group wanted a mathematics that could solve real-world problems, such as defeating the Nazis. John von Neumann went to Los Alamos and devised a radical new type of mathematics based on gambling, the Monte Carlo technique, for designing the atom bomb. A few years later, this applied group developed a marvelous new tool, the computer—and with it applied mathematics was off and running in its own directions.

I have been deeply involved with both pure mathematics and applied mathematics. My first contact with real mathematical problems was during a summer job in 1953, when I used an analog computer to simulate the neutron flux in the core of an atomic reactor. I was learning the basics of calculus at the time, just getting used to writing Greek letters for numbers and operations—and the idea of connecting resistors in a grid to simulate Δ (technically, the Laplace differential operator) struck me as profoundly beautiful. I was struggling to get my mind around the abstract notions, but luckily I was well acquainted with the use of a soldering iron. I was delighted that I could construct simple electrical circuits that made calculus so tangible.

Later, in college, I found that I could not understand what quantum field theory was all about; *ergo*, I was not a physicist but a mathematician! I went all the way and immersed myself in one of the purest areas of pure mathematics. (One can get carried away: At one time the math department at Cambridge University advertised an opening and a misprint stated that the position was in the Department of "Purer" Mathematics!) I "constructed" something called "moduli schemes." I do not expect the reader to have ever heard of moduli schemes or have a clue what they are. But here is the remarkable thing: To mathematicians who study them, moduli schemes are just as real as the regular objects in the world.

I can explain at least the first steps of the mental gymnastics that led to moduli schemes. The key idea is that an ordinary object can be studied using *the set of functions on the object*. For example, if you have a pot of water, the water at each precise location, at each spatial point inside the pot, has a temperature. So temperature defines a function, a rule that associates to each point in the pot the real number that is the temperature at that exact point. Or you can measure the coordinates of each point, for instance, how many centimeters the point is above the stove. Secondly, you can do algebra with these functions—that is, you can add or multiply two such functions and get a third function. This step makes the set of these functions into a *ring*. I have no idea why, but when you have any set of things that can be added and multiplied, consistent with the usual rules (for instance, the distributive law a [times] $(b + c) = a$ [times] $b + a$ [times] c), mathematicians call this set a ring. You see, ordinary words are used in specialized ways. In our case, the ring contains all the information needed to describe the geometry of the pot because the points in the pot can be described by the map carrying each function to its value at that point.

Then the big leap comes: If you start with any ring—that is, any set of entities that can be added and multiplied subject to the usual rules, you simply and brashly declare that this creates a new kind of geometric object. The points of the object can be given by maps from the ring to the real numbers, as in the example of the pot. But they may also be given by maps to other *fields*. A field is a special sort of ring in which division is possible. To see how strange the situation becomes, the set consisting of just the numbers 0 and 1 with the rule $1 + 1 = 0$ is a field. As you see, pure mathematics revels in creating variations on the algebra and geometry you learned in high school. I have sometimes described the world that opens up to devotees as a secret garden for which you have to work hard before you get a key.

Applied mathematics is different. It is driven by real-world problems. You may want a mathematical model that accurately describes and predicts the fission of uranium in a nuclear reactor—my summer job in 1953—but there is no limit to the important practical problems to which mathematics can be applied, such as global warming, tornadoes, or tsunamis. Modeling these physical effects requires state-of-the-art mathematical tools known as partial differential equations (or PDEs)—an eighteenth century calculus invention. Or take biology and evolution; the folding of proteins, the process by which a neuron transmits information, and the evolution of new species have all led to major mathematical advances.

The reader now sees how easy it has been for pure and applied mathematics to drift apart. One group of practitioners are immersed in abstract worlds where all the rules have to be painstakingly guessed and proven, with no help from real life experience. The other is in constant touch with scientists and engineers and has to keep up with new data and new experiments. Their goal is always to make the right mathematical model, which captures the essential features of some practical situation, usually by simplifying the messiness of reality and often replacing rigorous derivations by numerical simulations. The example I mentioned above, of von Neumann's work on the atomic and hydrogen bombs, illustrates this approach. Von Neumann and his colleagues started by trying to model the explosion of the bomb using conventional PDEs. At a certain point, working with Nicholas Metropolis and Stan Ulam, his colleagues at Los Alamos, they had an inspiration: Let's imagine that a hundred neutrons in the bomb are gambling. Here gambling means that each of them collides with a uranium atom when the roll of the dice comes out right. You set up your dice so that its odds mimic those of the real neutron moving at the same speed. This game is a lot easier than following the

approximately 1,000,000,000,000,000,000,000,000 neutrons that are really whizzing about—and it turned out to work well, unless you regret the legacy von Neumann's inspiration left the world.

But the drifting apart of pure and applied mathematics is not the whole story. The two worlds are tied more closely than you might imagine. Each contributes many ideas to the other, often in unexpected ways. Perhaps the most famous example is Einstein's need of new mathematical tools to push to deeper levels the ideas of special relativity. He found that Italian mathematicians, dealing with abstract *n*-dimensional space, had discovered tools for describing higher dimensional versions of curvature and the equations for shortest paths, called geodesics. Adapting these ideas, Einstein turned them into the foundations of general relativity (without which your global positioning system [GPS] wouldn't work). In the other direction, almost a century after Einstein discovered general relativity, working out the implications of Einstein's model is a hot area in pure mathematics, driving the invention of new techniques to deal with the highly nonlinear PDEs underlying his theory. In other words, pure mathematics made Einstein's physics possible, which in turn opened up new fields for pure mathematics.

A spectacular recent example of the interconnections between pure and applied mathematics involves prime numbers. No one (especially G. H. Hardy, as I mentioned) suspected that prime numbers could ever be useful in the real world, yet they are now the foundation of the encryption techniques that allow online financial transactions. This application is a small part of an industry of theoretical work on new algorithms for discrete problems—in particular, their classification by the order of magnitude of their speed—which is the bread and butter of computer science.

I want to describe another example of the intertwining of pure and applied mathematics in which I was personally involved. Computer vision research concerns writing computer code that will interpret camera and video input as effectively as humans can with their brains, by identifying all the objects and actions present. When this problem was first raised in the 1960s, many people believed that it was a straightforward engineering problem and would be solved in a few years. But fifty years later, computers still cannot recognize specific individuals from their faces very well or name the objects and read all the signs in a street scene. We are getting closer: Computers are pretty good at least at finding all the faces in a scene, if not identifying who they are. A computer can even drive a car (at least when the traffic isn't too bad).

Several things have been crucial for this progress. The first thing was the recognition that visual analysis is not a problem of deduction—i.e., combining the rules of logic with a set of learned rules about the nature of the objects that fill our world. It turns out that our knowledge is always too incomplete and our visual data is too noisy and cluttered to be interpreted by deduction. In this situation, the method of reasoning needed to parse a real-world scene must be statistical, not deductive. To implement this form of reasoning, our knowledge of the world must be encoded in a probabilistic form, known as an *a priori* probability distribution. This distribution tells you things like this: The likelihood of seeing a tiger walk around the corner is smaller than that of seeing a dog walk around the corner. It is called *a priori* because we know this data before we start to analyze the scene present now to our eyes. Analyzing noisy, incomplete data using *a priori* knowledge is called Bayesian inference, after the Reverend Thomas Bayes, who proposed this form of statistical analysis in the eighteenth century.

But what kinds of probability distributions are going to be used? Here, computer vision drew on a wide variety of mathematical tools and, in turn, stimulated the development of new variants and new algorithms in many fields of mathematics and physics. In particular, statistical mechanics contributed a tool known as Gibbs probability models and a variety of techniques for analyzing them. The conversion of an image into a cartoon, in which the main objects are outlined, turns out to have much in common with a set of pure math problems called "free boundary value problems." These are problems that call for solving for an unknown and changing boundary between two distinct areas or volumes, such as a melting ice cube in water. For instance, analyzing an MRI to see if an organ of the body is diseased or normal has stimulated work in the mathematics of infinite dimensional spaces. This is because from a mathematical viewpoint, the set of possible shapes of the organ is best studied as the set of points in a space; there are infinitely many ways in which shape can vary, so this space must have infinitely many dimensions.

I hope I have convinced you that one of the striking features of the spectrum of related fields—pure mathematics, applied mathematics, and physics—is how unexpected connections are always being discovered. I talked about a variety of such connections *among* these fields. But even within pure mathematics, amazing connections between remote areas are uncovered all the time. In the last decade, for example, ideas from number theory have led to progress in the understanding of the topology of high-dimensional spheres.

It will be difficult to fully repair the professional split between pure and applied mathematics and between mathematics and physics. One reason for this difficulty is that each academic field has grown so much, so that professionals have limited time to read work outside their specialties. It is not easy to master more than a fraction of the work in any single field, let alone in more than one. What we need, therefore, is to work harder at explaining our work to each other. This book, though it is addressed mostly to lay people, is a step in the right direction.

As I see it, the major obstacle is that there are two strongly conflicting traditions of writing and lecturing about mathematics. In pure mathematics (but not exclusively), the twentieth century saw the development of an ideal exposition as one that started at the most abstract level and then gradually narrowed the focus. This style was especially promoted by the French writing collaborative "Bourbaki." In the long tradition of French encyclopedists, the mathematicians forming the Bourbaki group sought to present the entire abstract structure of all mathematical concepts in one set of volumes, the *Éléments de Mathématique*. In that treatise the real numbers, which most of us regard as a starting point, only appeared midway into the series as a special "locally compact topological field." In somewhat less relentless forms, their orientation has affected a large proportion of all writing and lecturing in mathematics.

An opposing idea, promoted especially in the Russian school, is that a few well-chosen examples can illuminate an entire field. For example, one can learn stochastic processes by starting with a simple random walk, moving on to Brownian motion, its continuous version, and then to more abstract and general processes. I remember a wonderful talk on hyperbolic geometry by the mathematician Bill Thurston, where he began by scrawling with yellow chalk on the board: He explained that it was a simple drawing of a fire. His point was that in hyperbolic space, you have to get much, much closer to the fire to warm up than you do in Euclidean space. Along with such homey illustrations, there is also the precept "lie a little." If we insist on detailing all the technical qualifications of a theorem, we lose our readers or our audience very fast. If we learn to say things simply and build up slowly from the concrete to the abstract, we may be able to build many bridges among our various specialties. For me, this style will always be *The Best Writing on Mathematics,* and this book is full of excellent examples of it.