Henri's Crystal Ball

Philip J. Davis and David Mumford

On April 10, 1908, at the general session of the Fourth International Congress of Mathematicians held in Rome, Gaston Darboux presented a talk by Henri Poincaré (1854–1912) entitled *The Future of Mathematics* (Poincaré was unfortunately unable to deliver the talk himself). The original can be found on the web at: http://gallica.bnf.fr/ark:/12148/bpt6k17083c/f934n10.capture.

It is now a full century since this date, and it is of some interest (and amusement) to see how its contents shape up in view of the historically turbulent years and the tremendously productive mathematical ones that followed. Poincaré's talk has long been available in an English translation. This translation can now be downloaded either in pdf or html form: http://www-history.mcs.st-andrews.ac.uk/Extras/Poincare_Future.html or http://portail.mathdoc.fr/BIBLIOS/PDF/Poincare.pdf. The remarks in this paper are based on the latter translation.

The talk is divided into two parts; the first presents generalities. In the second part, numerous specific problems are mentioned in ten different areas, in which the current status of the problems is described and suggestions are made that research would be welcome along certain follow-up lines. The language of the first part is vivid and clear. This is not the case with the second part, but what impresses there is the wide range of mathematical material that Poincaré had at his command.

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Poincaré's talk should be compared and contrasted with the earlier talk by David Hilbert (1862–1943) at the Second International Congress of Mathematics held in Paris in 1900. Hilbert specified twenty-three problems that he said were important and open for solution. History has given the accolade and notoriety to Hilbert's problems, whereas Poincaré, who did not list specific problems, has attracted not nearly as much publicity and éclat within the mathematical world.

Poincaré's Generalities

First: a brief summary of the first part of Poincaré's paper. He proposes that some people considered mathematics in 1908 to be rich in ideas, having developed in "every sense". But then he says, if this was absolutely true, "Our riches would become an encumbrance" and produce an incomprehensible increase in knowledge. One answer to the plethora of material is professional specialization. But this may be a "vexatious obstacle to the progress of our science". Instead, he affirms that we must fight specialization by seeking unifying ideas:

> If a new result has value it is when, by binding together long known elements, until now scattered and appearing unrelated to each other, it suddenly brings order where there reigned apparent disorder.

Poincaré is fond of the Viennese physicist/philosopher Ernst Mach (1838–1916). "The role of science," Poincaré quotes Mach as saying, is "the production of economy of thought, just as a machine produces economy of labor." Poincaré carries this over to mathematics, citing both formulas and unifying theories.

He further asserts that the aesthetic element is often bound up with an achieved economy of thought as well as labor. The aesthetic element in methods and results is thus of great importance. It is not pure "dilettantism" because it brings "a comprehension at the same time of the whole and the parts". He asserts that long calculations alone cannot reveal the general structure of the originating problem:

> When a somewhat long calculation has led us to a simple and striking result we are not fully satisfied until we have shown that we could have foreseen…its most characteristic details. … how vain would be any attempt to replace by any mechanical process the free initiative of the mathematician.

Poincaré appreciates the rigor that the preceding fifty years had brought to mathematics but is wary of making a fetish out of it:

> In mathematics rigor is not everything, but without it there would be nothing…But is it necessary to repeat every time this discussion?…I fear that in this lengthening of our demonstrations they will lose that appearance of harmony.

The linguistic element—i.e., the creation of new terms—is also of great importance. An older example is the word "convergence", but he gives as newer examples "group", "invariant", "isomorphic", and "transformation".

> One of those marks by which we recognize the pregnancy of a result is in that it permits a happy innovation in our language. The mere fact is oftentimes without interest; it has been noted many times, but it is of no service to science. It becomes of value only on that day when some happily advised thinker perceives a relationship which he indicates and symbolizes by a word.

Poincaré acknowledges "the study of postulates, of unusual geometries, of functions having unusual values" as showing us "the workings of the human mind… when freed from the tyranny of the external world." But he is not impressed: "It is to the opposite side—the side of nature—against which we must direct the main corps of our army."

He imagines a physicist or engineer coming to a mathematician with a problem. Sometimes but not often the solution can be expressed explicitly in terms of known functions. But there is likely to be a power series solution: still, does it converge fast enough to be useful? His engineer has a time constraint and cares little for what "the engineer of the twenty-second century" can do (is he imagining a future of ultra-fast computers?). But for the mathematician, the conclusion is that "there are no longer some problems which are solved and others which are not" because one has qualitative approaches as well as quantitative, computationally useful as well as computationally useless power series in addition to traditional solutions. Poincaré promotes the qualitative as opposed to the quantitative when the latter is not immediately or easily forthcoming.

Henri Poincaré

Poincaré's overall conclusion is that the best that one can do in predicting the future of mathematics is to start with the present and give heed to these rubrics: take the various general lines along which progress has been accomplished, extrapolate these by generalization, abstraction, analogies, etc. But we should expect the greatest advances when two branches of mathematics find a "similarity of their forms despite the dissimilarity of material", where "each takes profit from the other."

A Few Comments

The comments that follow draw, of course, on our knowledge of post-Poincaré developments in mathematics.

Predicting the future

Is the possession of a crystal ball a specific skill possessed by some and not others, and do the people who are considered the most brilliant and prominent in a field have a better crystal ball? Perhaps their prominence and their lines of thought in part shape the future. Can we do no better than, as Poincaré suggests, to extrapolate from the present by calling for intensifications, generalizations, analogies, abstractions, etc., of what is already around? What about the genuinely new? Historians always find seeds of such developments in the past, but these are post-hoc judgments.

Freeman Dyson analyzed how well one can predict the future and came to the conclusion that, in science, unexpected technological breakthroughs were often the events that led to the discovery of wholly new and unexpected phenomena and thus to new theories. In mathematics it is harder to separate technology and theory, but we shall find much of Dyson's caution reflected in mathematical developments that Poincaré did not predict when we take up specific fields below.

Increase in the corpus of mathematics and specialization

Poincaré's remarks on the increase of the mathematical corpus, of the subsequent specialization and the limitations it causes, are certainly valid. Specialization and specialized vocabulary have exploded and create huge barriers to sharing ideas both within mathematics and to neighboring fields. When Alexander Ostrowski (1893–1986) came up for his doctoral examination in 1920 (under the supervision of Hilbert and Landau) he once confided to one of us—perhaps in jest—that he was the last student in mathematics who would ever be expected to answer questions in any part whatever of mathematics. Poincaré himself has been called "the last of the universalists".

At least the number of Ph.D.'s in mathematics who have been able to find gainful employment has increased. Specialization has led to a vast increase in the number of journals, societies, meetings, papers (the last can be accurately tracked in the exponential growth of *Math Reviews* and *Zentralblatt*).

Rigor

As Poincaré feared, there has occurred a widespread tendency towards more and more lengthy rigorous expositions and a resulting increase in the difficulty of finding the essential ideas often buried in a mathematician's papers. Sometimes an increase in abstraction as well as the explosion of new technical terms has compounded the bad effects of meticulous rigor (although they are not the same). All of these make papers in any but your own narrow field harder to read.

But the question may be asked, insofar as absolute rigor is an unattainable ideal, how is it to be attained and how much rigor suffices? An extreme, not envisioned in Poincaré's day, is computer verification of proofs carried out within a precisely defined set of predicate calculus axioms, as in the work of R. S. Moore and J. S. Boyer. The history of mathematical proof shows that great mathematicians of the past were not hung up on rigor, that the standards of rigor have waned and waxed. "Sufficient unto the day is the rigor thereof." However, the twentieth century did bring some debacles: many papers in algebraic geometry in the period 1920–1950 did contain "proofs" which were wrong or uncorrectable.

Computation

Poincaré seems rather hard on what might be called "naked" computation, denying it a role in the discovery process. Of course, the electronic digital computer was not around in Poincaré's day, and for him computation meant hand calculation, algebraic as well as numerical. We believe computation does have an important role. It has had differing impacts on different areas of mathematics, but few areas have not felt its impact. We will discuss several examples below.

In every generation, some mathematicians have used calculation extensively and some not. Gauss is a clear example that some of the most brilliant mathematicians have loved to calculate. With contemporary computers, those who do like to calculate have the power of a race car at their disposal compared with Gauss's horse and buggy.

One can cite the existence of journals devoted to "experimental mathematics". Virtually all the papers published in them draw sustenance from computer results. In this connection, it is appropriate and revealing to quote from the announced philosophy of the *Journal of Experimental Mathematics*:

> Experiment has always been, and increasingly is, an important method of mathematical discovery. (Gauss declared that his way of arriving at mathematical truths was "through systematic experimentation".) Yet this tends to be concealed by the tradition of presenting only elegant, well-rounded, and rigorous results. While we value the theorem-proof method of exposition, and do not depart from the established view that a result can only become part of mathematical knowledge once it is supported by a logical proof, we consider it anomalous that an important component of the process of mathematical creation is hidden from public discussion. It is to our loss that most of us in the mathematical community are almost always unaware of how new results have been discovered. It is especially deplorable that this knowledge is not made part of the training of graduate students, who are left to find their own way through the wilderness.

While we agree largely with the sentiments expressed here, we disagree with the above stated "established view" that all mathematical knowledge rests on logical proof. We would assert that in various ways "mathematical knowledge" goes beyond that which is supported by logical proof.

The quantitative vs. the qualitative

Poincaré, of course, was one of the discoverers of the whole field of topology, and this is the prime area where qualitative approaches to geometry superseded quantitative ones. But in many other fields of pure and applied mathematics these two approaches still vie for dominance.

In analysis, Poincaré uses the term "quantitative" to indicate not just isolated numbers but the whole theory of particular special functions from which specific numbers relevant to a theory or experiment have been derived and which then allow automatic application to other parallel theories.

From Poincaré's own work to the present day, the replacement of the "quantitative" by the "qualitative" has played a great role in the theory of differential equations. It was challenged by an old comment of the Nobelist in physics, Ernest Rutherford, that "the qualitative is naught but poor quantitative." What might have been in Rutherford's mind was set out for us recently by a physicist friend:

> Suppose, for example, someone has a theory of capillary attraction. Try it: water rises in the tube. Is the theory correct? Unless measurement agrees with a quantitative prediction you can't possibly know; a qualitative experiment would be a waste of time and money. Of course, it can happen that a false theory gives a correct answer but it happens rarely; also no experiment is foolproof, but everybody knows there is room for coincidence and error. Science does not deal with facts but with probable facts. These would be a logician's nightmare but they are part of a scientist's everyday thinking.

The so-called "catastrophe theory" of Thom, Zeeman, Arnold, and others is a prime example of a qualitative theory whose validity many have questioned. Its mathematical elegance is obvious but since it avoids ever committing to specific models and differential equations, its applicability is uncertain. Self-similar "fractal" models are another example of a theory in this gray area. There is extensive numerical evidence in extremely diverse fields for self-similarity over several orders of magnitude but there are relatively few physical models that demonstrably exhibit this.

In many areas of applied mathematics, models are proposed for some aspect of highly complex systems that cannot be modeled in their entirety. These models are qualitatively reasonable but, in order to argue for their validity, they are fleshed out in quantitative guise in order to make "predictions" about experimental results and/or computational simulations. This is sometimes a dubious procedure that can be summarized by the skeptical sentiment "every model is doomed to succeed." However, there is much that can be said pro qualitative analysis, and there is an ongoing and red-hot debate between the qualitative and the quantitative that will most certainly be argued beyond our lifetimes.

Aesthetics as an element of discovery and presentation

Poincaré's sentiments, when he asserts that for mathematicians elegance means a quality of a proof that makes the whole comprehensible, have been echoed by Gian-Carlo Rota. "A proof is beautiful," Rota wrote, "when it gives away the secret of the theorem, when it leads us to perceive the actual and not the logical inevitability of the statement that is proved." Aesthetics are certainly an important part of mathematics, one that has attracted much comment and speculation. But it should not be overstressed:

> I once heard [Paul] Dirac (1902–1984, British physicist) say in a lecture, which largely consisted of students, that students of physics shouldn't worry too much about what the equations of physics mean, but only about the beauty of the equations. The faculty members present groaned at the prospect of all our students setting out to imitate Dirac.
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> —Steven Weinberg
> *Towards the Final Laws of Physics*

The aesthetic and the useful should not be confused. Thus, there are very many computer programs useful in promoting mathematical discovery and scientific computation that are hardly aesthetic by the criterion of simplicity or any other criterion such as that proposed by Gian-Carlo Rota. The proofs of the four-color theorem and of Kepler's conjecture, which rely heavily on computation, attest to their usefulness.

The linguistic element

Poincaré's comment on the role of the linguistic element in mathematics is both sharp and prophetic. It has been explored only in recent decades and deserves more elaboration and attention. The famous linguist, Benjamin Whorf, proposed that the structure of your language affects, in fact constrains, your understanding of a situation, the way you think about it. In a recent letter to us, semioticist and mathematician Kay O'Halloran put Poincaré's perception in current semiotic terminology (or jargon):

> The "word" gives rise to existence![1] The relationship symbolized by the word undergoes processes of co-contextualisation and re-contextualisation to enter

[1] *This sounds almost theological. Cf. John: 1:1 "In the beginning was the word." But see: Kay L. O'Halloran,* Mathematical Discourse: Language, Symbolisms and Visual Images, *Continuum, London & New York, 2005.*

> into other relationships: a never-ending ongoing phenomenon.

On the other hand, the development of twentieth century mathematics has seen the explosion of specialized vocabularies in each sub-sub-area of mathematics. Who is conversant with all the concepts of Woodin cardinals, Lie superalgebras, algebraic stacks, perverse sheaves, Weyl tensor, Thom spectra, Besov spaces, semi-martingales, chromatic index, and trapdoor functions, each basic in its own field? It is not that these concepts are minor—they are each part of the standard vocabulary in their area. But sadly, they are a huge impediment to Poincaré's dream that "interlockings" between diverse fields will drive the deepest future discoveries.

Poincaré's specific predictions

In the second half of his talk, Poincaré takes up each of the areas of mathematics and makes specific comments. Poincaré, in contrast to Hilbert, does not pinpoint the problems to be worked on; he merely points in a general way to certain "subareas" and issues in each field, sometimes with frustratingly vague phrases. Ideally, the second part of the talk should be responded to by experts in the various fields or sub-fields—thus confirming Poincaré's concern about the specialization and the fragmentation of mathematics. Nonetheless, we will do our best to say something about what he had right and what he missed! In some instances, what followed after 1908 cannot be adequately described except at the monograph level.

In each section, the heading is Poincaré's, and, in parentheses, we put in some cases the more standard contemporary name. A word of caution: much of what he says is pretty vague, and one needs (or at least we need) to interpret his text and guess what he is suggesting.

Arithmetic (number theory)

In this field, Poincaré is quite successful in predicting the twentieth century developments. His first point seems to us to foreshadow clearly the work of André Weil creating characteristic p algebraic geometry alongside traditional algebraic geometry over the complex numbers:

> The first example which comes to mind is the theory of congruences where we find a perfect parallelism with that of algebraic equations. And we will certainly complete this parallelism which must exist between the theory of algebraic curves and that of congruences of two variables, for instance. And when the problems relative to congruences of several variables are solved we shall have taken the first step toward the solution of many of the questions of indeterminate analysis.

By "congruences",[2] he clearly means polynomial equations mod p. Although "indeterminate analysis" is pretty vague, a sympathetic interpretation would be that he is asking for connections between solutions of polynomials in many variables mod p and their solutions over the integers, Diophantine equations. And this is what Weil's conjectures made precise and Dwork, Grothendieck, and Deligne proved. He pursues the analogy between number theory and algebraic geometry in the next paragraph:

> Another example where the analogy has not always been seen at first sight is given to us by the theory of corpora and ideals. For a counterpart let us consider the curves traced upon a surface; to the existing numbers correspond the complete intersections, to the ideals the incomplete intersections, and to the prime ideals the indecomposable curves; the various classes of ideals thus have their analogs.

Here he seems to be talking about the theory of divisors on varieties, the "Picard" group or ideal class group and the analogy again between the number-theoretic situation and the algebro-geometric situation. In the twentieth century, class field theory in the number-theoretic case, and the theory of generalized Jacobians and Picard varieties in the algebro-geometric case, have developed this analogy. But note that Hilbert's famous *Zahlbericht* had appeared when this lecture was given and contained clearer leads to later developments.

His next topic is the theory of quadratic forms for which he says "(It) was one of the first to take shape … when the arithmeticians introduced unity through the consideration of groups of linear transformations." He suggests that further groups may yield more fruit, and he brings up discontinuous groups and Minkowski's *Geometrie der Zahlen*. Although this is a bit of a leap, one might say that his ideas are leading to the theory of semi-simple algebraic groups and their discrete subgroups. This has been one of the major themes of work in the twentieth century.

Finally, there is a paragraph about prime numbers, where, he says,

> I believe I have a glimpse of the wished for unity…All leads back without doubt to the study of a family of transcendental functions which, through the study of their singular points and the application of the method of M. Darboux, will permit the calculation asymptotically of certain functions of very great numbers.

[2] *The translator mistakenly wrote "congruents" for Poincaré's word "congruences".*

In this rather mysterious passage, it is possible to guess that he is foreshadowing the tremendously successful use of L-functions in number theory. If so, he has touched on all the major themes of twentieth century number theory.

Algebra

In Section II (Algebra), Poincaré focuses narrowly on polynomial equations. He starts by saying "the most important [subject here] is that of groups…," obviously meaning Galois groups, but he will treat groups in a separate section. He discusses instead "the question of the calculation of the numerical value of roots and the discussion of the number of real roots."

Concerning the numerical calculation of roots of polynomials, it is hard to decipher Poincaré's specific remarks but there has certainly been much work and much success both experimental and theoretical—all stimulated by the appearance of increasingly powerful digital computers. Indeed, the whole of numerical analysis, relatively stagnant, burst forth in the digital age like the desert cactus that blooms when it suddenly rains. Now, there is hardly a package for scientific computation that does not have a reasonably high precision polynomial root finder. Many diverse attacks on the problem have been made, each with its pluses and minuses.

An allied problem, perhaps of more applied significance than "mere" root finding, is that of the numerical calculation of the eigenvalues of a square matrix. The roots of a polynomial are the eigenvalues of its companion matrix. The QR algorithm gives a reliable method for eigenvalue calculation. So a method of choice, valid for polynomials of degree, say, less than several hundred, first inaugurated by Cleve Moler of (Matlab fame) and later provided a substantial theoretical underpinning by Edelman and Murakami, also Trefethen, is to go that route. The companion matrix must first be "balanced" by a standard similarity transformation to reduce the condition of the matrix. For polynomials of enormously high degree, arising in special problems, effective special algorithms have been devised. Future work on root finding will very likely be stimulated by improvements in digital computers combined with demands from scientific/technological applications.

Poincaré goes on to talk about invariants of homogeneous polynomials, i.e., functions of their coefficients invariant by linear substitutions, and mentions Gordon and Hilbert's work here. He then writes "If we have an infinity of whole polynomials, depending algebraically on a finite number among them, can we always deduce them from a finite number among them by addition and multiplication?" This would seem to be Hilbert's 14th problem. It was disproved by Nagata in 1959 for the ring of invariants of a representation of a power of the additive group. Both Hilbert and Poincaré seem to have been overly optimistic about finiteness results for rings of polynomials. He ends this section by proposing that questions about algebra should be done over rings of polynomials with integer or other coefficients but not pursuing this.

Differential equations (dynamical systems)

Poincaré starts off with a very astute proposal: we need a group of transformations that will group dynamical systems into classes that are easier to describe. He proposes the analogy with using birational transformations to classify algebraic curves. One can read this as foreshadowing Smale's idea of using the full group of homeomorphisms to classify dynamical systems, more precisely defining two systems to be topologically equivalent if there is a homeomorphism taking the orbits of one system to the orbits of the other. As an example, Poincaré raises the question of counting the number of limit cycles of two-dimensional dynamical systems.

Curiously, Poincaré does not talk about the complexities of dynamical systems that he had encountered in his work on the three-body problem theory. The modern theory of dynamical systems has been dominated by the struggle to find a satisfactory theory for such chaotic systems, the split between the relatively simple hyperbolic systems and those with strange attractors. Simple chaotic systems, such as the famous Lorenz system modeling convection cells in the atmosphere, were found to be ubiquitous in three or more dimensions.

Instead Poincaré mentions holomorphic vector fields in the plane and asks when they have integrals and what you can say about the functions that uniformize their orbits. One can imagine links with the discovery and exploration in the century to follow of the many unexpected completely integrable dynamical systems such as KdV, the Toda lattice, etc.

Equations with partial derivatives (linear PDEs)

Poincaré reviews what was then recent work of Fredholm on integral equations and clearly envisions the idea that linear PDEs are going to require an understanding of infinite-dimensional space and the extension of linear algebra to these spaces. He describes the analogy he sees between Hill's work on infinite determinants and Fredholm's theory, the analogy between an infinite-dimensional space of sequences and an infinite-dimensional space of functions. At the end, he acknowledges that "Thanks to M. Hilbert, who has been doubly an initiator, we are already on that path." That path is unifying these "two methods" and applying it to problems such as the Dirichlet problem.

As it turned out, linear PDEs were essentially mastered using function space techniques, distributions, and Fourier analysis a little after the middle of the twentieth century. Then the cutting edge turned to nonlinear PDEs which remain an area full of mysteries. Poincaré says nothing about, for example, the Euler and Navier-Stokes fluid flow equations.

The Abelian functions

This very short section is remarkably specific. The question Poincaré raises is

> What is the relationship of the Abelian functions begot by the integrals relative to an algebraic curve to the general Abelian functions and how shall we classify the latter?

The question in the last part of this quote has proven to be by far the more important one. It leads directly to the construction of Siegel's modular variety, the moduli space that indeed classifies what are now called principally polarized Abelian varieties. These spaces are the simplest arithmetic quotients of Hermitian symmetric spaces and all such modular varieties and the more general arithmetic quotients are a key component of the theories linking number theory, algebraic geometry and representations of Lie groups (especially the "Langlands conjecture"). Poincaré was certainly on the right track in raising this classification question.

The first part of his question is a much more special one, although it has been studied by quite a few mathematicians. In modern algebro-geometric language, it asks what is special about the Jacobian varieties of curves in the bigger set of all Abelian varieties? Finding ways to characterize Jacobians is now called the "Schottky problem". There are remarkably very many quite different ideas for solving this problem whose interrelations are still not completely clear: a review up to 1996 is in an appendix to one of the second author's books.[3]

The theory of functions (complex variables)

In another short section, Poincaré's main concern is the theory of analytic functions of several variables as opposed to one:

> The analogy with the functions of a single variable gives a valuable but insufficient guide; there is an essential difference between the two classes of functions (one and more than one variable) and every time a generalization is attempted by passing from one to the other, an unexpected obstacle has been encountered…

Thus:

> Why is a conformal representation more often impossible in the domain of four dimensions and what shall we substitute for it? Does not the true generalization of functions of one variable come in the harmonic functions of four variables …In what sense may we say that the transcendental functions of two variables are to transcendental functions of one variable as (algebraic or) rational functions of two variables are to (algebraic or) rational functions of one variable?

Poincaré certainly hit on a ripe area here. The work of William Fogg Osgood on functions of several complex variables that date shortly after 1908 can be found in his influential *Lehrbuch der Funktionentheorie*. The field opened up wide in the first half of the twentieth century, and we have the later theories and books of e.g., Behnke and Thullen, Bochner and Martin, Bergman, Kodaira and Spencer, Hörmander, Remmert, Krantz, Scheidemann. It gave birth to topics such as pseudo-convexity, Stein manifolds, and sheaf theory. The link with the theory of algebraic varieties of dimension two or more has been extremely fruitful, and the algebraic and transcendental theory have intertwined continuously. The ghost of Poincaré is very pleased.

Though Poincaré failed to mention analytic functions of one complex variable, this field also flourished in the years following 1908. It also has a rich history. The Riemann mapping theorem for simply connected regions was worked on by Osgood, Carathéodory, and Perron. The Bieberbach conjecture and the extensive theory of conformal mappings, Nevanlinna theory and the work of Ahlfors on meromorphic curves, Teichmüller theory and its connection to three-manifolds via Thurston theory all drove the field far.

The theory of groups

In a third short section, Poincaré states he will talk only about Lie groups and Galois groups, thus ignoring both the growing general theory of finite groups and the discontinuous Kleinian groups on which he had worked himself. He recalls how Lie groups have been tamed by the use of Lie algebras (which he describes as a "special symbolism upon which you will excuse me for not dwelling"). He says justly that "The study of the groups of Galois is much less advanced" and hopes that, as in the links between number theory and algebraic geometry, links can be made between Lie theory and Galois theory. The search for a better understanding of Galois groups has proven to be very difficult and continues to this day.

Geometry

Poincaré first asks if geometry is nothing more than "the facts of algebra and analytical geometry expressed in another language?" No, he says, "Common geometry has a great advantage in that the senses may come to the help of our reason and aid it in finding what path to follow." But "our senses fail us when we try to escape from the

[3] *D. Mumford,* The Red Book of Varieties and Schemes, *2nd edition, Springer Lecture Notes 1358.*

classical three dimensions." One should not forget that it was during Poincaré's lifetime that mathematicians had come to accept higher-dimensional spaces as a matter of course:

> We have nowadays become so familiar with this notion of more than three dimensions that we may speak of it even in the university without arousing astonishment.

He states most eloquently that geometric intuition is more robust than one might expect and can be useful in higher dimensions:

> It guides us into that space which is too vast for us and which we may not see; it does this by ever bringing to mind the relationship of the latter space to our ordinary, visible space, which without doubt is only a very imperfect image, but which nevertheless is an image.

He then introduces Analysis Situs (topology) as a creation of Riemann and states that its importance is very great, that it is leading the way into higher dimensions and, indeed, must be studied in all dimensions. Of course, Poincaré is now usually considered as its creator. He is certainly on the money in foreseeing the central role topology will play in the twentieth century, creating key elements of our vocabulary (such as homology and homotopy groups) and giving us some intuition about higher-dimensional space.

It is interesting that he makes no speculations about how geometry in higher dimensions will differ from what we know in three dimensions. There had been one hint at his time: Schläfli's classification of regular polytopes showed that in dimension 5 or more, life got simpler. It is interesting that this is exactly what happened with the higher-dimensional versions of Poincaré's conjectured characterization of spheres: Stallings and Smale showed that this was true in dimension 5 or more because, in some sense, life was simpler due to there being more "elbow room". This phenomenon of things stabilizing as dimensions get higher has occurred over and over in many fields.

Poincaré goes on to doff his hat towards both algebraic geometry and differential geometry, "a vast field from which to reap a harvest". This is certainly right but he has nothing specific to say about them—a bit sad considering their great flowering in the twentieth century.

Cantorism (set theory and foundations)

One senses in this section considerable ambivalence of Poincaré towards Cantor's ideas. While acknowledging that "(His) services to science we all know," he ends the paragraph by saying "(with this theory) we can promise ourselves the joy of the physician called in to follow a beautiful pathological case!" It seems that uppermost in his mind in this short paragraph are the paradoxes that arise in this field, the apparent contradictions which "would have overwhelmed Zeno…with joy."

As we know, from our vantage point, it was Gödel's ingenious use of exactly these paradoxes that led to the deepest result in the foundations of mathematics, to Gödel's magnificent incompleteness theorem, whose philosophical significance continues to reverberate. Starting from Russell and Whitehead (1910–1913), the foundations of mathematics, the search for universal axioms for integers, real numbers, and set theory, developed into a field of its own. But Gödel showed that any finite set of axioms could not be a complete foundation for mathematics, and attempts to found mathematics on sets are not now universally accepted. Some of these skeptics (admittedly a minority of mathematicians) assert that mathematics can't have "ultimate" foundation stones, and, in any case, it doesn't need them.

In the century following 1908, brilliant mathematicians have created a large corpus of material that goes under the rubric of logic, sets, and foundations. To name but a few: Zermelo, Fraenkel, Ramsey, Łukasiewicz, Post, von Neumann, Bernays, Gödel, Turing, Cohen, Martin Davis, Henkin, Feferman, Chaitin. Scanning a recent text on mathematical logic yields postulate systems such as PA (Peano arithmetic), ZF (Zermelo (1908), Fraenkel: (1891–1965), ZFC (ZF + the axiom of choice), ZFL (ZF + constructibility). It yields such topics as decidability, consistency, forcing, generalized continuum hypotheses, non-standard analysis, hyper-hyper inaccessible cardinals, alternate logics.

But, on the other hand, there is also a widespread feeling among working mathematicians that measurable cardinals and the like, that is to say, present day set theory, are indeed some kind of "pathological case" as Poincaré put it, ideas that can give the uninitiated existential angst. So Poincaré perhaps caught the future mainstream reaction to this area as well as pinpointing its arguably most significant idea.

The research of postulates (axiomatic analysis)

This short section of Poincaré's article may be differentiated from the previous one by saying that under "Cantorism", he was thinking of the theoretical side of the logical analysis of the foundations of mathematics, while in this section, he was thinking of the applied side. If he was skeptical of Cantor, he is even more so of the usefulness of axiomatic analysis:

> We are trying to enumerate the axioms and postulates, <u>more or less deceiving</u>, which serve as the foundation stones of our various mathematical theories. M. Hilbert has obtained the most brilliant results. It seems now that this domain must be very limited and there will not be any more to be done when this

inventory is finished, and that will be very soon.

Consideration of the underlined phrase might very well suggest that Poincaré believed such an inventory or enumeration of postulates was unnecessary or misleading. Poincaré was really wrong in this instance: Hilbert's initiative not only at listing all necessary postulates to complete Euclid, but at constructing alternate geometric universes where all but one axiom held, has had a major influence on twentieth century mathematics.

In the 1920s, the German school of "modern algebra", with its completely general rings, its abstract ideal theory, and Noether's spectacular generalizations of Hilbert's results, seems to us the spiritual descendent of Hilbert's inventory. Rings were now divorced from specific examples such as rings of algebraic integers, polynomials, or matrices and instead were considered as having a vast array of possible incarnations, cases where some standard axioms held and others did not. Likewise, the abstract theory of topological spaces and of Banach spaces developed by the Polish school followed the same path: look at all combinations of postulates and see what spaces they deliver.

This point of view was thoroughly absorbed in the culture of twentieth century mathematics and was clearly enunciated in Bourbaki's monumental treatise. It is now taken for granted as the "obvious" way to do things in the pure math community, the way to find the best abstract setting for every argument, the most general form for every theorem. But it seems that Poincaré missed it, that it was definitely not his cup of tea.

We summarize our thoughts in the presumptuous table below. It is interesting that in many areas he saw the possibility of links between fields and, perhaps because it was less exciting, deemphasized the deepening of existing fields. His negative feelings about the "Research of Postulates" seems to lie behind his missing the explosion of work in the first half of the twentieth century setting almost every area of mathematics, but especially algebra, in its most general abstract form and investigating all mathematical objects that this led to (e.g., all finite simple groups).

SCORECARD

FORESEEN	MISSED
Importance of linking number theory and algebraic geometry	Theory of general commutative, noncommutative rings
Importance of analytic methods in number theory, L-functions	
Topological equivalence of dynamical systems	Deeper theory of chaotic dynamical systems
Importance of function spaces and their linear algebra	Small successes, challenges of nonlinear PDEs
Differences of several complex variables from one, links of complex analytic geometry with algebraic geometry	Deeper theory of one complex variable (e.g., Teichmüller, Bieberbach)
Importance of Lie and Galois groups	Theory of general finite groups
Topology as the key to higher dimensions	Rich diversity of dimensions 3 and 4 and 7 (exotic spheres)
	Gödel and deep significance of the paradoxes
	Axiomatic treatment of every field (eventually: categories)
	Explosion of computational methods, computational experiments, numerical analysis
	Development of probability theory, stochastic differential equations, information theory