Oscar Zariski

1899–1986

Oscar Zariski was born on April 24, 1899, in the town of Kobryn, which lies on the border of Poland and the U.S.S.R. It was Russian at the time of Zariski's birth, was Polish between the two world wars, and is now Russian again. He was the son of Bezalel and Chana Zaristky, and was given the name of Asher Zaristky, which he changed to Oscar Zariski when he came to Italy. Kobryn was a small town where his mother ran a general store, his father having died when he was two. In 1918, he went to the University of Kiev in the midst of the revolutionary struggle. He was seriously wounded in one leg when caught in a crowd that was fired at by troops, but recovered after two months in the hospital. As a student, he was attracted to the fields of algebra and number theory as well as to the revolutionary political ideas of the day. He supported himself partly by writing for a local Communist paper. This is most surprising for those of us who only knew him much later, but calls to mind the quip—A man has no heart if he is not a radical in his youth and no mind if he is not a conservative in his mature years.

Because of the limitations of the education available in the U.S.S.R. at the time, in 1921 Zariski went first to the University of Pisa and six months later to the University of Rome where the famous Italian school of algebraic geometry, Castelnuovo, Enriques, and Severi, was flourishing. He had no money and the fact that universities in Italy were free to foreign students was an important consideration. Zariski was especially attracted to Castelnuovo, who immediately recognized his talent. Castelnuovo took him on a three hour walk around Rome after which Zariski realized that he had been given an oral exam in every area of mathematics! Castelnuovo saw in Zariski a man who would not only push their subject further and deeper, but would find radically new ways to overcome its present limitations. Zariski was fond of quoting Castelnuovo as saying "Oscar, you are here with us, but are not one of us," referring to Zariski's doubts even then of how rigorous their proofs were. Zariski met his wife, Yole Cagli, while a student in Rome and they were married on September 11, 1924, in Kobryn.

He received his doctorate in the same year. His thesis ([1],[2]) classified all rational functions $y = P(x)/Q(x)$ of x such that 1) x can be solved for in terms of radicals starting with y, and 2) given any two solutions $x1$ and $x2$, all other solutions x are rational functions of $x1$ and $x2$. Already in his first work he strongly showed his ability to combine algebraic ideas (the Galois group), topological ideas (the fundamental group), and the "synthetic" ideas of classical geometry. The interplay of these different tools was to characterize his life's work.

He pursued these ideas with the support of a Rockefeller fellowship in Rome during the years 1925–1927. His son Raphael was born there on July 18, 1925. In 1927, he accepted a position at Johns Hopkins and in 1928 his family moved to the U.S.A. to join him. Here his daughter Vera was born on September 14, 1932.

A crucial paper in this phase of his career is his analysis [3] of an incomplete proof by Severi that the Jacobian of a generic curve of genus g has no nontrivial endomorphisms. Severi's paper reads as though the proof were complete. Zariski discovered the problem and found a very ingenious argument to remedy it, but neither were well received by Severi who published his own correction independently.

The effect of this discovery seems to have been to turn Zariski's interests to the study of the topology of algebraic varieties, especially of the fundamental group, where the rigor of the techniques was beyond question and the tools were clean and new. He travelled frequently to Princeton to discuss his ideas with Lefschetz. In this phase of his career, roughly from 1927 to 1935, he studied the fundamental group of a variety through the fundamental group of projective n-space minus a divisor. This work is characterized by the spirit of exploration and discovery and, in spite of much recent interest, it remains a largely uncharted area. One result will give the flavor of the new things he turned up: according to another incomplete paper of Severi, it was widely believed that all plane curves of fixed degree with a fixed number of nodes (ordinary double points) belonged to a single algebraic family. What Zariski found was that curves with a fixed degree and a fixed number of *cusps* (the next most complicated type of double point) could belong to several families. He exhibited curves $C1$ and $C2$ of degree 6 with 6 cusps such that the fundamental groups of their complements were not isomorphic!

In 1935, however, Zariski completed his monumental review of the central results of the Italian

D. Mumford, *Selected Papers*, Vol. II,

school, his Ergebnisse monograph *Algebraic Surfaces* [4]. His goal had been to disseminate more widely the ideas and results of their research, but the result for him was "the loss of the geometric paradise in which I so happily had been living"[1]. He saw only too clearly that the lack of rigor he had touched on was not a few isolated sores but a widespread disease. His goal now became the problem of restoring the main body of algebraic geometry to proper health. Algebra had been his early love and algebra was blooming, full of beautiful new ideas in the hands of Noether and Krull, and various applications to algebraic geometry had already been proposed by van der Waerden. Zariski threw himself into this new discipline. He spent the year 1935–1936 at the Institute for Advanced Study in Princeton, and met regularly with Noether, then at Bryn Mawr, learning the new field through first hand contact with the master.

The fifteen years or so that followed, 1938–1951, if you take the years between his paper [5] recasting the theory of plane curve singularities in terms of valuation theory and his monumental treatise [6] on his so-called "holomorphic functions" (sections of sheaves formed from completions of rings in I-adic topologies), saw the most incredible outpouring of original and creative ideas in which tool after tool was taken from the kit of algebra and applied to elucidate basic geometric ideas. Many mathematicians in their forties reap the benefits of their earlier more original work; but Zariski undoubtedly was at his most daring exactly in this decade. He corresponded extensively in this period with Andre Weil, who was also interested in rebuilding algebraic geometry and extending it to characteristic p with a view to its number-theoretic applications. Although they only rarely agreed, they found each other very stimulating, Weil saying later that Zariski was the only algebraic geometer whose work he trusted. They managed to get together in 1945 while both were visiting the University of Sao Paulo in Brazil.

At the same time, these were years of terrible personal tragedy. During the war, all his relatives in Poland were killed by the Nazis. Only his immediate family and those of two of his siblings who had moved to Israel escaped the holocaust. He told the story of how he and Yole were halfway across the U.S., driving back to the East Coast, the day Poland was invaded. They listened each hour to the news broadcasts on their car radio, their only link to the nightmare half a world away. There was nothing they could do.

In this period of his work, Zariski solved many problems with his algebraic ideas. Three themes in his work are particularly beautiful and deep and I want to describe them in some detail.

[1] Preface by Zariski to his Collected Works, MIT Press.

The first theme is the study of birational maps which lead him to the famous result universally known as "Zariski's Main Theorem". This was the final result in a foundational analysis of birational maps between varieties, "maps" which are one to one and onto outside of a finite set of subvarieties of the range and domain, but which "blow up" or "blow down" special points. Zariski showed that if there are points P and Q in the range and domain which are isolated corresponding points, i.e. the set of points corresponding to P contains Q but no curve through Q, and the set of points corresponding to Q contains P but no curve through P, and if, further, P and Q satisfy an algebraic restriction—they are **normal** points—then in fact Q is the only point corresponding to P and vice versa (slightly stronger: the map is **biregular** between P and Q). Zariski's proof of this was astonishingly subtle, yet short.

The second theme from this period is the resolution of singularities of algebraic varieties, which culminated in his proof that all algebraic varieties of dimension at most 3 (in characteristic zero) have "nonsingular models," i.e., are birational to nonsingular projective varieties. In dimension 3, this was a problem that had totally eluded the easy-going Italian approach. Even in dimension 2, although some classical proofs were essentially correct, many of the published treatments definitely were not. Zariski attacked this problem with a whole battery of techniques, pursuing it relentlessly over 6 papers and 200 pages. Perhaps the most striking new tool was the application of the theory of general valuations in function fields to give a birationally invariant way to describe the set of all places which must be desingularized. The result proved to the mathematical world the power of the new ideas. For many years, this work was also considered by everyone in the field to be technically the most difficult proof in all algebraic geometry. Only when the result was proven for surfaces in characteristic p by Abhyankar and later for varieties of arbitrary dimension in characteristic 0 by Hironaka[2] was this benchmark surpassed!

The third theme is his theory of abstract "holomorphic functions." The idea was to use the notion of formal completion of rings with respect to powers of an ideal as a substitute for the idea of convergent power series, and to put elements of the resulting complete rings to some of the same uses as classical holomorphic functions. The most striking application was to a stronger version of the "Main Theorem," known as the connectedness theorem. The connectedness theorem states that if a birational map from X to

[2] S.S. Abhyankar, *Local uniformization on algebraic surfaces of characteristic* $p \neq 0$, Annals of Math., **63** (1956); H. Hironaka, *Resolution of singularities of an algebraic variety of characteristic* 0, Annals of Math., **79** (1964).

Y is single-valued and if a point Q of Y is normal, then the inverse image of Q on X is connected (we are assuming X and Y are complete, e.g. projective). This result was later one of the inspirations for Grothendieck's immense work in rebuilding with yet newer tools the foundations of algebraic geometry[3].

This phenomenal string of papers caught the attention of the mathematical world. Zariski received the Cole Prize from the American Mathematical Society in 1944. In 1945, he moved to a Research Professorship at the University of Illinois. Early in the forties, his work had caught the attention of G.D. Birkhoff who decided he must come to Harvard[4] and, indeed, in 1947 he received and accepted an offer to come to Harvard University, where he remained for the rest of his life. He was a very strong influence on the mathematical environment at Harvard and he enjoyed the opportunity of luring the best people he could to Harvard and bringing out the best in each of his students. While he was chairman, the Dean, McGeorge Bundy, used to refer to him as that "Italian pirate," so shrewd was he in getting his way, inside or outside the usual channels. Whenever Harvard's baroque appointment rules, known as the Graustein Plan (after the earlier mathematician who invented it), jibed with his plans, he used them; but whenever they did not, he feigned ignorance of all that nonsense and insisted the case be considered on its own merits. Over the next thirty years, he made Harvard into the world center of algebraic geometry. His seminar welcomed Weil, Hodge, Nagata, Kodaira, Serre, Grothendieck, and many others. The stimulating evenings at his home and the warm welcome extended by Oscar and Yole were not easily forgotten.

His work of reconstruction of algebraic geometry had started with the writing of the monograph *Algebraic Surfaces*, and now that Zariski felt he had reliable and powerful general tools, it was natural for him to see if he could put all the main results of the theory of surfaces in order. He initiated the modern work on the duality theorems for cohomology (called by him the "lemma of Enriques-Severi" [7], before the topic was taken up by Serre and Grothendieck), the questions of the existence of minimal nonsingular models in each birational equivalence class of varieties [8], and on the classification of varieties following Enriques [9] (now known as the classification by Kodaira dimension). In each of these areas he spread before his students the vision of many possible areas to explore, many exciting prospects.

Although he himself had developed a fully worked out theory of the foundations of algebraic geometry, he welcomed the prospect of yet newer definitions and techniques being introduced because they would make the subject itself stronger. He embraced the new language of sheaf theory and cohomology, working through the basic ideas methodically as was his custom in the Summer Institute in Colorado in 1953 [10], although he never adopted this language as his own. When Grothendieck appeared in the field, he immediately invited him to Harvard. Grothendieck, for his part, welcomed the prospect of working with Zariski. Because Grothendieck's political beliefs did not allow him to swear the oaths of loyalty required in those unfortunate days, he even asked Zariski to investigate the feasibility of continuing his mathematical research from a Cambridge jail cell, i.e., how many books and visitors would be allowed!

The final phase of Zariski's mathematical career was a return to the problems of singularities. Zariski had absolutely no use for the concept of retirement and he dedicated his sixties and seventies and as much of his eighties as he could to a broad-based attack on the problem of "equisingularity". The goal was to find a natural decomposition of an arbitrary variety X into pieces Y_i, each one made up of a subvariety of X from which a finite set of lower dimensional subvarieties have been removed, such that *along* each subvariety Y_i, the big variety X had essentially the same type of singularity at each point. Zariski made major strides towards the achievement of this goal, but the problem has turned out to be quite difficult and is still unsolved.

Zariski's last years were disturbed by his fight with his hearing problem. Zariski was always very lively both in mathematical and in social interactions with his friends and colleagues, picking up every nuance. He was struck with tinnitus, which produced a steady ringing in his ears, a greater sensitivity to noise, and a gradual loss of hearing. This forced him into himself, into his research and kept him close to home. Only the boundless love of his family sustained him in his last years. He died at home on July 4, 1986.

Many honors flowed to Zariski in well-deserved appreciation of the truly extraordinary contribution he had made to the field of algebraic geometry. He received honorary degrees from Holy Cross in 1959, Brandeis in 1965, Purdue in 1974, and from Harvard in 1981. He received the National Medal of Science in 1965, and the Wolf Prize, awarded by the government of Israel, in 1982. His friends, his students, and his colleagues

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[3] Grothendieck's style was the opposite of Zariski's. Whereas Zariski's proofs always had a punch-line, a subtle twist in the middle, Grothendieck would not rest until every step looked trivial. In the case of holomorphic functions, Grothendieck liked to claim that the result was so deep for Zariski because he was just proving it for the 0th cohomology group. The easy way, he said, was to prove it first for the top cohomology group, then use descending induction!

[4] The story, which I have from reliable sources, is that Birkhoff approached Zariski and said in his magisterial way: "Oscar, you will probably be at Harvard within the next five years."

will remember not only the beautiful theorems he found, but the forcefulness and the warmth of the man they knew and loved.

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