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SCIENCE

Fields Medals (IV): An Instinct for the Key Idea

Pierre Deligne was born in Brussels, Belgium, in 1944. When he was 14 an enthusiastic high school teacher, M. J. Nijs, lent him several volumes of the *Elements of Mathematics* by N. Bourbaki. This work develops a solid foundation for all of modern mathematics, in a most logically efficient manner, proceeding from the general to the particular; for example, the real number system is discussed only in the fourth chapter of the third long book, after general topology and abstract algebra have been extensively treated. In the whole treatment there is (except perhaps for the excellent historical notes) no motivation given at all, other than the internal logic of the development itself. That Deligne not only survived but even thrived on his exposure to such a work at such a tender age was perhaps already an indication of his genius, as well as of Nijs' good judgment.

Thus when Deligne went to the University of Brussels he already knew the fundamentals of most of modern mathematics. There he learned much from group theorist Jaques Tits now at the College de France, and Tits gave him excellent advice on his general mathematical development. In 1965, at Tits' suggestion, Deligne went to Paris to pursue further his interests in algebraic geometry and number theory. It would be hard to imagine a better place for this at the time. Among other activities there were

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the seminars in algebraic geometry of Alexander Grothendieck (Fields Medal, 1966) and the lectures of Jean-Pierre Serre (Fields Medal, 1954), which had a more number-theoretical flavor. Deligne was strongly influenced by both these men.

Deligne's association with Grothendieck during the late 1960's at the IHES (European Institute for Advanced Study, in Bures-sur-Yvette just south of Paris) was especially close. We personally first heard of Deligne in 1966 from Grothendieck, who was more impressed than we had ever seen him be by a young mathematician. At that time Deligne was 21 and Grothendieck immediately recognized him as his equal. The significance of this and of their collaboration will be clearer if we explain the situation in algebraic geometry at this time. In the 1930's algebraic geometry had an antiquated air, with many appealing charming results but an embarrassingly handmade and dusty look. During the period 1940 to 1960 several of the greatest mathematicians of this century contributed to building suitable foundations for algebraic geometry and fitting it into the abstract conceptual framework that had by then been built for most of the rest of mathematics. After the great contributions of Oscar Zariski now at Harvard University, André Weil, and Serre of the Institute for Advanced Study in Princeton, it was Grothendieck who pushed this program through to its ultimate logical conclusion. Grothendieck was an untiring, implacably logical, almost fanatical force. He was guided in his thinking perhaps more than any other mathematician has ever been by the desire to view each concept in the greatest possible degree of generality with no artificial restrictions—that is, no restrictions not absolutely forced by the logic of the situation. The result, as Grothendieck wrote his monumental works on the foundations of algebraic geometry, was an utter transformation of the subject. As he pursued the ultimate in generality the volume of the work increased exponentially, and algebraic geometry became a vast structure, gleaming, hard to grasp, overpowering. The key ideas seemed hidden, let alone the appealing artifacts of the previous century.

Deligne mastered this structure of Grothendieck's seemingly without effort, but his style was not to add a whole new layer of systematic development to the theory unless it was absolutely necessary. He preferred to find an elegant fundamental new idea suddenly clarifying a whole area or an old problem. Deligne was able to use the extensive de-

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velopments of Grothendieck as well as any one, but his own ideas were often more concise, more particular. To contrast their styles metaphorically, one could say that Grothendieck liked to cross a valley by filling it in, Deligne by building a suspension bridge.

During the next few years Deligne touched on virtually all areas of algebraic geometry, making extraordinary contributions. In 1970, at the age of 26, he was promoted to a permanent professorship at the IHES, the position he now holds. We will not try to describe his early work but will focus instead on his most exciting and deepest result, his proof in 1973 of the last and hardest of Weil's conjectures. Fortunately this result is relatively easy to state in simple language, and it may convey an idea of the almost mystical flavor of the direction in which this frontier of mathematics is growing.

One starts with a set of one or more simultaneous polynomial equations in several unknowns. This could be something as simple as one equation in two unknowns, such as $y^2 - x^3 + 1 = 0$, but in general would be $f_1(x, y, z, \ldots) = 0$, $f_2(x, y, z, \ldots) = 0, \ldots$. The f_i's, as stated, are to be polynomials, and we assume that their coefficients are whole numbers. The oldest question in arithmetic is to find, or give procedures for finding, all solutions in which the unknowns $x, y, z, \ldots$ are whole numbers. But this has turned out to be intractable in all but some elementary cases. Another question is to consider the set of solutions in which $x, y, z, \ldots$ are complex numbers. These solutions form a continuum, or manifold, X, of a certain dimensionality, called an algebraic variety because it is described by algebraic equations (sometimes one adds points at infinity to X to "complete" it). Such manifolds have been extensively studied, and in particular certain properties of X are described by its so-called Betti numbers $B_0, B_1, B_2, \ldots$. Thus B_0 is the number of connected pieces of X, and B_1 describes how many essentially different loops X contains. For example, in the case of the single equation $y^2 - x^3 + 1 = 0$, X turns out to be two-dimensional (remember that we are allowing complex values for x and y, not only real values) and to be like the surface of a doughnut (a space called a torus). In this case $B_0 = 1$, because X is connected, and $B_1 = 2$, because there are really two different ways around a torus (Fig. 1).

There is a third type of solution to our equations $f_1 = f_2 = \ldots = 0$ that is very important: one tries to put the unknowns $x, y, z, \ldots$ equal to whole numbers, but requires only that the values $f_i(x, y, z, \ldots)$ of the polynomials be divisible by a fixed prime number p (that is, be congruent to zero modulo p) instead of being 0. If $(x, y, z, \ldots)$ is one such set of values for the unknowns, then adding multiples of p to them, for example $(x + 2p, y - 3p, z + p, \ldots)$, gives another such set of values. So one can restrict $x, y, z, \ldots$ to be one of the p whole numbers $0, 1, 2, \ldots, p - 1$ and not miss anything. We then have in all only a finite set of values for the $x, y, z, \ldots$ to try, and there will be a finite number Np of solutions in the sense just described. For example, try the possible values 0, 1, and 2 for x and for y, and out of the nine possibilities you will find three of them such that 3 divides $y^2 - x^3 + 1$. Thus in this case $N_3 = 3$. With a bit more patience you can check $N_5 = 5$, $N_2 = 2$, and $N_7 = 3$ for the same equation.

We can now state a famous result of Weil, which is the leitmotiv of this whole development. Take the case of one irreducible polynomial equation in two variables. Also modify the number Np slightly to take into account infinite solutions and singularities; we omit describing this. Then

$$|Np - (p + 1)| \leq B_1 \sqrt{p} \qquad (1)$$

where B_1 is the first Betti number of the complex variety associated to the same equation. This variety will be like the surface of a doughnut with a certain number of holes, and B_1 is twice the number of holes. The point that is so startling here is that this sets up a connection between the solutions modulo p with whole numbers and the geometry of the continuum of complex solutions. What other cases can one find of such a miraculous connection between arithmetic and geometry? This question tantalizes many mathematicians today.

What should one expect for a general set of equations of the type we are considering? Weil guessed the answer in 1949, and Deligne proved that his guess was correct 24 years later. To explain this guess we must view the number Np described above in a more sophisticated way, as the number of solutions to our equations in the finite field with p elements. For each positive integer r there is an essentially unique finite field with p^r elements, and if Np^r denotes the number of solutions with $x, y, z, \ldots$ in that field, Weil conjectured that for each prime p there should exist complex number α_{ij} such that for each r

$$Np^r = \sum_{j=1}^{n} (-1)^j \sum_{i=1}^{B_j} \alpha_{ij}^r \qquad (2)$$

where n is the dimension of the space X of complex solutions and the B_j are the Betti numbers of X. Moreover the absolute values of the numbers x_{ij} should be given by

$$|\alpha_{ij}| = p^{j/2} \qquad (3)$$

(In this brief statement of Weil's conjectures we have exaggerated a bit: one must desingularize X and add some points at infinity, and make the corresponding modifications in counting the solutions in finite fields; also one must exclude a finite set of primes p, those for which X does not have "good reduction modulo p.") In the case of one equation in two unknowns, $n = 2$, $B_0 = B_2 = 1$, $x_{10} = 1$, and $x_{12} = p$, so that Eq. 1 is a consequence of Eqs. 2 and 3. A formula of the same type as Eq. 2 was proved by Bernard Dwork of Princeton University in 1959, and Eq. 2 was proved by Grothendieck in 1965. However, Eq. 3 is much harder, and it is this result for which Deligne is justly famous. Clearly, Eqs. 2 and 3 strengthen and confirm the link between the arithmetical problem of solving polynomial equations modulo p and the geometry of their complex solutions.

To see how Deligne proved Eq. 3, we must go back again to Grothendieck. It was in order to prove a formula like Eq. 2, and with the hope of using it to prove Eq. 3, that Grothendieck began doing algebraic geometry. Weil had pointed out that Eq. 3 could be obtained as a "Lefschetz fixed point formula," if one had a "cohomology theory of varieties in characteristic p" (indeed Np^r is just the number of fixed points of the transformation F^r, where F is the Frobenius map of the set of solutions in characteristic p into itself). At the start of his work Grothendieck had guessed that such a

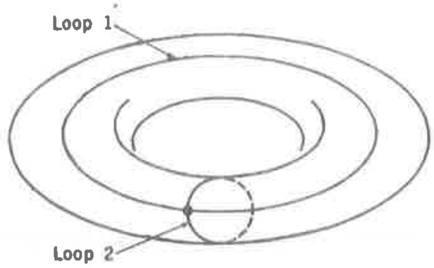

Fig. 1. The two ways to go around a torus.

cohomology theory could be obtained by systematically confusing the two mathematical senses in which the word covering is used (Fig. 2). This was the kind of abstract idea at which Grothendieck excelled, and in this case he was absolutely right. With the aid of Michael Artin of the Massachusetts Institute of Technology and Jean Louis Verdier of the University of Paris he constructed a new cohomology theory, known as "étale cohomology," yielding the numbers x_{ij} in a natural way. This theory was one of the building blocks of Deligne's proof.

The other main ingredient came from a little-known prewar (1939) paper of Robert Rankin in the *Proceedings of the Cambridge Philosophical Society*, in which Rankin made some progress on an analogous conjecture of the Indian mathematician Srinivara Ramanujan, by a squaring trick. It is hard to imagine two mathematical schools more different in spirit and outlook than were those of the British analytic number theorists in the 1930's and of the French algebraic geom-

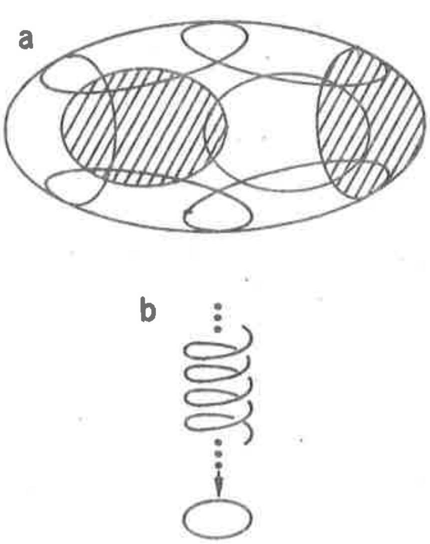

Fig. 2. (a) Covering of type I. A set of pieces that fill the whole. In this case, an oval region covered by nine smaller oval regions, two of which are shaded. (b) Covering of type II. One space lying smoothly over another. In this case, an infinite spring covering a closed loop.

eters in the 1960's. That Deligne's proof is a blend of ideas from both is an indication of the universality of his mathematical taste and understanding. He had a clue to the connection because already in 1968 he had shown that Weil's conjectures implied Ramanujan's. The ideas behind this were due to the Japanese mathematicians Kuga, Sato, Shimura, and Ihara, but it was Deligne who had the technical power to carry them out, and it was Serre who realized this and urged him to do it. At any rate, Deligne saw that Rankin's method could be understood geometrically and could be greatly extended. Combining this with a very delicate analysis of the cohomology via so-called Lefschetz pencils, using also a theorem of David Kazhdan now at Harvard University and Margoulis (one of this year's Fields Medalists), Deligne put together his sensational proof of Weil's conjecture. Besides its own intrinsic interest, this result has also already yielded several important consequences in number theory and algebraic geometry.

Since 1973 Deligne's center of interest has shifted slightly from geometry toward number theory. He has made several key contributions to problems connected with the vast program of Robert P. Langlands of the Institute for Advanced Study to relate the way in which the numbers x_{ij} mentioned above vary with p to the theory of automorphic forms.

Deligne's economy and clarity of thought are amazing. His writings contain few unnecessary words, little or no redundancy. The ideas are there, simply and clearly stated, but so densely that almost every phrase is relevant.

Deligne's nonmathematical interests and activities exhibit the same simplicity. For years he has cultivated a large vegetable garden in the rich soil of the housing project of the IHES. He enjoys organizing Easter egg hunts for the children living there. For transportation he prefers a bicycle to a car, and his vacations are usually spent hiking. There is nothing artificial about him. He is self-assured but modest and able and willing to discuss almost any mathematical subject with anyone. There are few subjects that his questions and comments do not clarify, for he combines powerful technique, broad knowledge, daring imagination, and unfailing instinct for the key idea.

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