THE WORK OF C. P. RAMANUJAM IN ALGEBRAIC GEOMETRY

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IT WAS a stimulating experience to know and collaborate with C. P. Ramanujam. He loved mathematics and he was always ready to take up a new thread or to pursue an old one with infectious enthusiasm. He was equally ready to discuss a problem with a first year student or a colleague, to work through an elementary point or to puzzle over a deep problem. On the other hand, he had very high standards. He felt the spirit of mathematics demanded of him not merely routine developments but the *right* theorem on any given topic. He wanted mathematics to be beautiful and to be clear and simple. He was sometimes tormented by the difficulty of these high standards, but, in retrospect, it is clear to us how often he succeeded in adding to our knowledge, results both new, beautiful and with a genuinely original stamp.

Our lives and researches intertwined considerably. I first met him in Bombay in 1967-68, when he took notes on my course in Abelian Varieties and we worked jointly on refining and understanding better many points related to this theory. Later, in 1970-71, we were together in Warwick where he ran seminars on étale cohomology and on classification of surfaces. His excitement and enthusiasm was one of the main factors that made that "Algebraic Geometry year" a success. We discussed many topics involving topology and algebraic geometry at that time, and especially Kodaira's Vanishing Theorem. My wife and I spent many evenings together with him, talking about life, religion and customs both in India and the West and we looked forward to a warm and continuing friendship. His premature death was a great shock to all who knew him. I will always miss his companionship and collaboration in the enterprise of mathematics.

I will give a short survey of his contributions to algebraic geometry. Perhaps his most perfect piece of work is his proof that a smooth affine complex surface X, which is contractable *and* simply connected at ∞, is isomorphic to the plane $\mathbf{C}^2$. The proof of this is not simple and uses many techniques; in particular, it shows how well he knew his way about in the classical geometry of surfaces! What is equally astonishing is his very striking counter-example showing that the hypothesis "simply connected at ∞" cannot be dropped. The position of this striking example in a general theory of 4-manifolds and particularly in a general theory of the topology of algebraic surfaces is yet to be understood. As mentioned above, the Kodaira Vanishing Theorem was an enduring interest of his. Both of us were particularly fascinated by this "deus ex machina", an intrusion of analytic tools (i.e., harmonic forms) to prove a purely algebraic theorem. His two notes on this subject went a long way to clarifying this theorem: (a) he proves it by merely topological, not analytic, techniques and (b) he finds a really satisfactory definitive extension of the theorem to a large class of non-ample divisors on surfaces. This second point is absolutely essential for many applications and was used immediately and effectively by Bombieri in his work on the pluricanonical system $|nK|$ for surfaces of general type. His result is that if D is a divisor on X, such that $(D^2) > 0$ and $(D.\ C) \geqslant 0$ for all effective curves C, then $H^1(X, \mathcal{O}(-D)) = (0)$.

His earliest paper, on automorphisms group of varieties, is a definitive analysis of the way this group inherits an algebraic structure from the variety itself. This work employs the techniques of functors, e.g., families of automorphisms developed by Gröthendieck at about the same time. His paper "On a certain purity theorem" addresses itself to a question of Lang that puzzled almost all algebraic geometers at that time: *given a proper surjective morphism* $f: X \to Y$ *between smooth varieties, is the set*

$$\{y \in Y \mid f^{-1}(Y) \text{ singular}\}$$

of codimension 1 *in* Y? Here he provides a topologico-algebraic analysis of one good case where it is true, and describes a counter-example to the general case worked out jointly with me. We again see his fascination with the interactions between purely topological techniques and algebro-geometric ones.

This interest comes out again in his joint paper with Le Dung-Trang, whose Main Result is described in the title: "The invariance of Milnor's number implies the invariance of the topological type." Here they are concerned with a family of hypersurfaces in $\mathbf{C}^{n+1}$: $F_t(z_0, \dots, z_n) = 0$, with isolated singularities at the origin, whose coefficients are C^∞ functions of $t \in [0, 1]$. They show that when Milnor's number μ_t, giving the number of vanishing cycles at the origin, is independent of t, then if $n \neq 2$, the germs of the maps F_t: $\mathbf{C}^{n+1} \to \mathbf{C}$ near 0 are independent of t, up to homeomorphism (if $n = 2$, they get a slightly weaker result). A beautiful and intriguing corollary is that the Artin local ring

$$\mathbf{C}\,[[z_0, \dots, z_n]] \Big/ \left(F, \frac{\partial F}{\partial z_0}, \dots, \frac{\partial F}{\partial z_n} \right)$$

already determines the topology of the map F near 0.

Finally, his paper "On a geometric interpretation of multiplicity" proves essentially the following elegant theorem: *If $Y \subset X$ is a closed subscheme defined by $I_Y \subset \mathcal{O}_X$, which blows up to a divisor $E \subset X'$, then*

$$\frac{(-1)^{n-1}(E^n)}{n!} = \begin{bmatrix} \textit{leading coefficient of the polynomial} \\ P(k) = \chi(\mathcal{O}_X/I_Y^k),\ k \gg 0 \end{bmatrix}.$$

In addition to these published papers, Ramanujam made many contributions to my book "Abelian Varieties", while writing up notes from my lectures. Reprinted here is the Appendix by him on Tate's Theorem on abelian varieties over finite fields; and the following extraordinary theorem: It had been proven by Weil that if X is a projective variety and m: $X \times X \to X$ is a morphism, then if m makes X into a group, m must satisfy the commutative law too. Ramanujam proved that if m merely possessed a 2-sided identity ($m(x, e) = m(e, x) = x$), then m must also have an inverse and satisfy the associative law, hence make X into a group!