

Towards an Enumerative Geometry of the Moduli Space of Curves

David Mumford

Dedicated to Igor Shafarevitch on his 60th birthday

Introduction

The goal of this paper is to formulate and to begin an exploration of the enumerative geometry of the set of all curves of arbitrary genus g . By this we mean setting up a Chow ring for the moduli space \mathcal{M}_g of curves of genus g and its compactification $\overline{\mathcal{M}}_g$, defining what seem to be the most important classes in this ring and calculating the class of some geometrically important loci in $\overline{\mathcal{M}}_g$ in terms of these classes. We take as a model for this the enumerative geometry of the Grassmannians. Here the basic classes are the Chern classes of the tautological or universal bundle that lives over the Grassmannian, and the most basic cycles are the loci of linear spaces satisfying various Schubert conditions: the so-called Schubert cycles. However, since Harris and I have shown that for g large, \mathcal{M}_g is not unirational [II-M] it is not possible to expect that \mathcal{M}_g has a decomposition into elementary cells or that the Chow ring of \mathcal{M}_g is as simple as that of the Grassmannian. But in the other direction, J. Harer [Ha] and E. Miller [Mi] have strong results indicating that at least the low dimensional homology groups of \mathcal{M}_g behave nicely. Moreover, it appears that many geometrically natural cycles are all expressible in terms of a small number of basic classes.

More specifically, the paper is divided into 3 parts. The goal of the first part is to define an intersection product in the Chow group of $\overline{\mathcal{M}}_g$. The problem is that due to curves with automorphisms, $\overline{\mathcal{M}}_g$ is singular, but in a mild way. In fact it is a “ \mathbb{Q} -variety”, locally the quotient of a smooth variety by a finite group. If it were *globally* the quotient of a smooth variety by a finite group, it would be easy to define a product in $A \cdot (\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$. Instead we have used the fact that $\overline{\mathcal{M}}_g$ is globally the quotient of a Cohen-Macaulay variety by a finite group, plus many of the ideas of Fulton and MacPherson, and especially a strong use of both the Grothendieck and Baum-Fulton-MacPherson forms of the Riemann-Roch theorem to achieve this goal. To

handle an arbitrary Q -variety, Gillet has proposed using higher K -theory ($H^n(K^n) \cong A^n$) and this may well be the right technique.

The goal of the second part is to introduce a sequence of "tautological" classes $\kappa_i \in A^i(\overline{M}_g) \otimes \mathbb{Q}$, derive some relations between them, and calculate the fundamental class of certain subvarieties, such as the hyperelliptic locus, in terms of them. Again the Grothendieck Riemann-Roch theorem is one of the main tools. Some of these results have been found independently by E. Miller [Mi], and it seems reasonable to guess, in view of the results of Harer and Miller (op. cit.), that in low codimensions $H^i(M_g) \otimes \mathbb{Q}$ is a polynomial ring in the κ_i .

Finally, to make the whole theory concrete, we work out $A^*(\overline{M}_2)$ completely in Part III. An interesting corollary is the proof, as a consequence of general results only, that M_2 is affine. It seems very worthwhile to work out $A^*(\overline{M}_g)$ or $H^*(M_g)$ for other small values of g , in order to get some feeling for the properties of these rings and their relation to the geometry of \overline{M}_g . The techniques of Atiyah-Bott [A-B] may be very useful in doing this.

Part I: Defining a Chow Ring of the Moduli Space

§1. Fulton's Operational Chow Ring

If X is any quasi-projective variety, Fulton and Fulton-MacPherson have defined in two papers ([F1], p. 157, [F-M], p. 92) two procedures to attach to X a kind of Chow cohomology theory: a ring-valued contravariant functor. We combine here the 2 definitions taking the simplest parts of them in a way that is adequate for our applications. The theory also becomes substantially simpler if we take the *coefficients for our cycles to be* \mathbb{Q} , and we assume that $\text{char } k = 0$. This is the case we are interested in, *so we will restrict ourselves to this case henceforth*. We call the resulting ring $\text{op}A^*(X)$. To form this ring, take:

generators : $\text{elts}(f, \alpha)$,
 $f: X \rightarrow Y$ a morphism
 Y smooth, quasi-projective
 α a cycle on Y

relations : $(f, \alpha) \sim 0$ if, for all $g: Z \rightarrow X$, we have:
 $(f \circ g)^* \alpha$ rationally equivalent to 0 on Z
 via the induced map

$$A^k(Y) \xrightarrow{(f \circ g)^*} A_{n-k}(Z)$$

$(k = \text{codim of } \alpha, n = \text{dim of } Z)$

Equivalently, we may define

$$opA^*(X) = \text{Image} \left\{ \lim_{(x \xrightarrow{f} Y)} A^*(Y) \longrightarrow \prod_{(Z \xrightarrow{g} X)} \text{End}(A_*(Z)) \right\}$$

where the map is given by cap product

$$A^k(Y) \times A_l(Z) \xrightarrow{-\cap} A_{l-k}(Z)$$

(cf. [F1], §2). This makes it clear that opA^* is a ring and a contravariant functor and that $opA^*(X)$ acts on $A_*(X)$ by cap product. If X is smooth, then $opA^*(X) \cong A^*(X)$.

Moreover, as in Fulton [F1], §3.2, for all coherent sheaves \mathcal{F} with finite projective resolutions, we can define the Chern classes $c_k(\mathcal{F}) \in opA^k(X)$, (by resolution of \mathcal{F} , twisting and pull-back of Schubert cycles from maps of X to Grassmannians).

Using a resolution of X , we can give a very simple description of the relations in $opA^*(X)$:

Proposition 1.1. *If $\pi: \tilde{X} \rightarrow X$ is a resolution of X , then*

$$(f, \alpha) \sim 0 \Leftrightarrow (f \circ \pi)^*(\alpha) = 0 \quad \text{in } A^k(\tilde{X}),$$

i.e.,

$$opA^*(X) \subset A^*(\tilde{X}).$$

Proof. We must show that if

$$g: Z \rightarrow X$$

is any test morphism, then $l(f \circ \pi)^* \alpha$ rationally equivalent to 0 on \tilde{X} implies $l'(f \circ g)^* \alpha$ rationally equivalent to 0 on Z for some l' . But by taking a

suitable subvariety of $Z \times_X \tilde{X}$ we get a diagram

$$\begin{array}{ccccc} \tilde{Z} & \xrightarrow{\tilde{g}} & \tilde{X} & & \\ p \downarrow & & \downarrow \pi & & \\ Z & \xrightarrow{g} & X & \xrightarrow{f} & Y \end{array}$$

where p is proper, surjective, generically finite of degree l'' . Therefore

$$\begin{aligned} l''(f \circ g)^* \alpha &= p_*((f \circ g \circ p)^* \alpha) \\ &= p_*(\tilde{g}^*((f \circ \pi)^* \alpha)) \end{aligned}$$

hence

$$l.l''(f \circ g)^* \alpha = p_*(\tilde{g}^*(l(f \circ \pi)^* \alpha)) \sim 0.$$

This uses the formula:

$$\begin{array}{l} (*) \text{ For all } \begin{array}{ccc} \tilde{Z} & \xrightarrow{\tilde{h}} & Y \\ p \downarrow & \parallel & \\ Z & \xrightarrow{h} & Y \end{array} \quad \begin{array}{l} Y \text{ smooth, } \alpha \text{ a cycle on } Y, \\ p \text{ proper, surjective, generically finite of degree } d \end{array} \\ p_* \tilde{h}^*(\alpha) \sim dh^*(\alpha). \end{array}$$

(See [F1], §2.2, part 2 of lemma).

Q.E.D.

In fact, we can say more:

Proposition 1.2. *In the situation of Prop. 1.1, the image of $opA^k(X)$ in $A^k(\tilde{X})$ is contained in the subgroup of $A^k(\tilde{X})$ generated by irreducible subvarieties W of \tilde{X} such that $W = \pi^{-1}(\pi(W))$.*

Proof. Let (f, α) be a generator of $opA^k(X)$, where $f: X \rightarrow Y$ is a morphism and Y is smooth, quasi-projective. Let $p = f \circ \pi$, and

$$Y_k = \{y \in Y \mid \dim p^{-1}(y) \geq k\}.$$

Then use the moving lemma on Y to represent α as a cycle $\sum n_i W_i$, whose components W_i meet properly all the Y_k . Then each component W_{i_j} of $p^{-1}(W_i)$ meets the open set of \tilde{X} where π is an isomorphism and

$p: \tilde{X} \rightarrow p(\tilde{X})$ is equidimensional. Therefore, $p^*\alpha$ is represented by a combination of the W_{ij} with suitable multiplicities and each W_{ij} satisfies $W_{ij} = \pi^{-1}(\pi(W_{ij}))$. Q.E.D.

There is also another natural way to give generators for $opA'(X)$:

Proposition 1.3. *Using rational coefficients in $K(X)$ too, the homomorphism*

$$ch : K(X) \rightarrow opA'(X)$$

is surjective, hence $opA'(X)$ can be defined to be

$$\begin{aligned} \text{Image} [K(X) \rightarrow A'(\tilde{X})] \\ \mathcal{E} \mapsto ch(\pi^* \mathcal{E}) \end{aligned}$$

$(\pi: \tilde{X} \rightarrow X \text{ a resolution of } X)$.

Proof. It is well known that for any smooth quasi-projective Z , $ch: gr K(Z) \rightarrow A'(Z)$ is a graded isomorphism, hence taking the total Chern characters, $ch: K(Z) \rightarrow A'(Z)$ is also an isomorphism¹. Therefore $ch: K(X) \rightarrow opA'(X)$ is surjective.

$opA'(X)$ has a much subtler *covariance* for certain morphisms f , whose existence is tied up with the version of the Grothendieck-Riemann-Roch theorems for opA' . The result is this:

Theorem 1.4 (Fulton). *Let $f: X \rightarrow Y$ be a projective local-complete-intersection morphism. Define $Td_f \in opA'(X)$ in the usual way. Then there is a homomorphism*

$$f_*: opA'(X) \rightarrow opA'(Y)$$

such that

¹This sounds a bit odd, but it is perhaps clarified by the observation that for any rank r and dimension n , there are universal polynomials P_k such that for all vector bundles \mathcal{E} of rank r on n -dimensional varieties,

$$c_k(\mathcal{E}) = P_k(ch \mathcal{E}, ch \Lambda^2 \mathcal{E}, \dots, ch \Lambda^r \mathcal{E})$$

where $ch \mathcal{E}$ is the total Chern character, and these elements lie in any cohomology ring with the usual Chern formalism and rational coefficients.

1) for all cartesian diagrams

$$\begin{array}{ccc}
 X' & \xrightarrow{h} & X \\
 f' \downarrow & & \downarrow f \\
 Y' & \xrightarrow{g} & Y
 \end{array}$$

and all $\alpha \in A.(Y')$, $\beta \in opA'(X)$

$$f_*\beta \cap \alpha = f'_*(\beta \cap [f]_{Y'}(\alpha)) \quad \text{in } A.(Y')$$

($[f]_{Y'}$ defined as in Fulton-MacPherson [F-M], p. 95).

2) for all locally free sheaves \mathcal{E} on X ,

$$ch(f_!\mathcal{E}) = f_*(ch \mathcal{E} \cdot Td_f).$$

This is proven in Fulton [F2], Ch.18: here opA' is a possibly larger Chow Ring in which (1) is the definition of f_* . In this ring (2) is proven, and (2) shows that $f_*\beta$ actually lies in the subring opA' considered here.

§2. Q -Varieties and \overline{M}_g

The moduli space \overline{M}_g of stable curves is an example of a variety which is locally in the étale topology a quotient of a smooth variety by a finite group. The approach we take to defining a Chow ring for \overline{M}_g is best studied in this more general context. Because these varieties are quite close to the objects introduced by Matsusaka [Ma], we shall call them quasi-projective Q -varieties. We define a quasi-projective Q -variety to be:

- 1) a quasi-projective variety X ,
- 2) a finite atlas of charts:

$$\begin{array}{c}
 X_\alpha \\
 \downarrow \\
 X_\alpha/G_\alpha \\
 \downarrow p'_\alpha \\
 X
 \end{array}
 \quad p_\alpha \left(\begin{array}{l} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} \right.$$

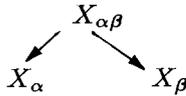
where p'_α is étale, G_α is a finite group acting, faithfully on a quasi-projective smooth X_α and

$$X = \bigcup_{\alpha} (\text{Im } p_\alpha),$$

3) The charts should be compatible in the sense that for all α, β , let

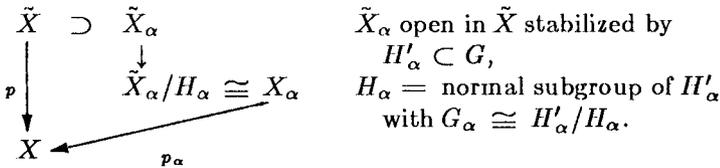
$$X_{\alpha\beta} = \text{normalization of } X_\alpha \times_X X_\beta.$$

Then the projections



should be étale.

Here, of course, a new chart can be added if it satisfies the compatibility conditions (3) with the old ones. For any such Q -variety, we can normalize X in a Galois extension of its function field $k(X)$ containing copies of the field extensions $k(X_\alpha)$ for all α . This leads to a covering $p: \tilde{X} \rightarrow X$ with a group G acting faithfully on \tilde{X} and $X = \tilde{X}/G$. The fact that $k(\tilde{X}) \supset k(X_\alpha)$ leads to a factorization of p locally:



Q -varieties come in various different grades. The best ones are those such that for some atlas, \tilde{X} can be chosen to be itself smooth. Not so nice, but still amenable to the techniques we shall use are those where \tilde{X} can be chosen to be Cohen-Macaulay. We call these Q -varieties *with a smooth global cover* and with a *Cohen-Macaulay global cover* respectively.

Another important concept is that of a Q -sheaf on a Q -variety X . By this, we do not mean a coherent sheaf on X , but rather a family of coherent sheaves \mathcal{F}_α on X_α , plus isomorphisms

$$\mathcal{F}_\alpha \otimes_{\mathcal{O}_{X_\alpha}} \mathcal{O}_{X_{\alpha\beta}} \cong \mathcal{F}_\beta \otimes_{\mathcal{O}_{X_\beta}} \mathcal{O}_{X_{\alpha\beta}}$$

compatible on the triple overlaps. Note that such a coherent sheaf pulls back by tensor product to a family of coherent sheaves on \tilde{X}_α , which

glue together to one coherent sheaf $\tilde{\mathcal{F}}$ on \tilde{X} on which G acts. Therefore, equivalently we can define a coherent sheaf on the Q -variety X to be a coherent G -sheaf $\tilde{\mathcal{F}}$ on \tilde{X} such that for all α , $\tilde{\mathcal{F}}|_{\tilde{X}_\alpha}$ with its H'_α -action is the pull-back of a coherent sheaf on X_α . The importance of \tilde{X} being Cohen-Macaulay is illustrated by the simple fact:

Proposition 2.1. *If \tilde{X} is Cohen-Macaulay, then for any coherent sheaf \mathcal{F} on the Q -variety X , $\tilde{\mathcal{F}}$ has a finite projective resolution.*

Proof. In fact, if \tilde{X} is Cohen-Macaulay and X_α is smooth, $\tilde{X}_\alpha \rightarrow X_\alpha$ is flat. But \mathcal{F}_α has a finite projective resolution, so this resolution pulls back to such a resolution for $\tilde{\mathcal{F}}|_{\tilde{X}_\alpha}$. Q.E.D.

Consider the case of the moduli space $\overline{\mathcal{M}}_g$. Choose an integer $n \geq 3$ prime to the characteristic. Fix a free $\mathbb{Z}/n\mathbb{Z}$ -module V of rank $2g$ with an alternating non-degenerate form

$$e: V \times V \rightarrow \mu_n.$$

Fix, moreover, a flag of isotropic free submodules:

$$(0) \subset V_1 \subset V_2 \subset \dots \subset V_g = V_g^\perp \subset \dots \subset V_2^\perp \subset V_1^\perp \subset V$$

where $rk(V_i) = i$. Then for all stable curves C of genus g , let h be the sum of the genera of the components of its normalization. Then there is an isomorphism

$$H^1(C, \mathbb{Z}/n\mathbb{Z}) \cong V_{g-h}^\perp$$

such that the form

$$H^1 \times H^1 \xrightarrow{\cup} H^2 \xrightarrow{\text{f'al class}} \mu_n$$

corresponds to e . We may consider the auxiliary moduli space

$$\begin{aligned} \left(\overline{\mathcal{M}}_g^{(n,h)}\right)' &= \text{set of pairs } (C, \phi), C \text{ stable curve,} \\ \phi: V_{g-h}^\perp &\xleftarrow{\phi \text{ injective}} H^1(C, \mathbb{Z}/n\mathbb{Z}) \text{ a sympl. map} \end{aligned}$$

which can be constructed by standard arguments. Inside this space, define an open subset by:

$$\begin{aligned} \overline{\mathcal{M}}_g^{(n,h)} = & \text{those pairs } (C, \phi) \text{ such that every automorphism} \\ & \alpha: C \rightarrow C \text{ fixes the submodule } \text{Im } \phi \subset H^1(C, \mathbb{Z}/n\mathbb{Z}) \\ & \text{and, if } \alpha \neq 1_C, \text{ then } \alpha \text{ acts non-trivially on } \text{Im } \phi. \end{aligned}$$

Since the pairs (C, ϕ) in this subset have no automorphisms, $\overline{\mathcal{M}}_g^{(n,h)}$ is smooth and represents the universal deformation space of any curve occurring in it. Note that every curve C occurs in the space $\overline{\mathcal{M}}_g^{(n,h)}$ such that $g + h = rk H^1(C)$ (see [D-M], Th. 1.13). Next consider the finite groups

$$G = Sp(V, \mathbb{Z}/n\mathbb{Z})$$

∪

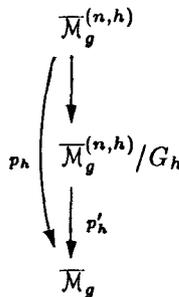
$$H'_h = (\text{stabilizer of } V_{g-h})$$

∪

$$H_h = \left(\begin{array}{l} \text{elements which} \\ \text{act identically} \\ \text{on } V_{g-h}^\perp \end{array} \right)$$

$$G_h = H'_h/H_h = (\text{induced group of automorphisms of } V_{g-h}^\perp).$$

Then G_h acts on $\overline{\mathcal{M}}_g^{(n,h)}$ and we have canonical morphisms



I claim that p'_h is étale. In fact, if $(C, \phi) \in \overline{M}_g^{(n,h)}$, then $\text{Aut}(C)$ can be identified with a subgroup of G_h and *formally* near $[C, \phi]$, the isomorphism

$$\overline{M}_g^{(n,h)} \xleftarrow{\text{formal isom.}} \text{Def}(C)$$

commutes with the action of $\text{Aut}(C)$.

Therefore we have a diagram

$$\begin{array}{ccc} \overline{M}_g^{(n,h)} & \xleftarrow{\quad} & \text{Def}(C) \\ \downarrow & & \downarrow \\ \overline{M}_g^{(n,h)} / \text{Aut}(C) & \xleftarrow{\quad} & \text{Def}(C) / \text{Aut}(C) \\ p \downarrow & & \downarrow \\ \overline{M}_g^{(n,h)} / G_h & & \\ q \downarrow & & \\ \overline{M}_g & \xleftarrow{\quad} & \text{Spec } \hat{\mathcal{O}}_{\overline{M}_g, [C]} \end{array}$$

where the horizontal arrows are formal isomorphisms. Now $\text{Def}(C) / \text{Aut}(C)$ i.e., Spec of the $\text{Aut}(C)$ -invariants in the complete local ring representing the deformations of C , is isomorphic to Spec of the complete local ring of \overline{M}_g at $[C]$. Thus the morphism indicated by $q \circ p$ in the diagram is étale at $[C, \phi]$. Therefore so is q .

This proves that the atlas $\{p_h: \overline{M}_g^{(n,h)} \rightarrow \overline{M}_g\}$ puts a structure of \mathbb{Q} -variety on \overline{M}_g . In this setting, what is the variety \tilde{X} dominating all the charts? This is

$$\overline{M}_g^{(n)} = (\text{normalization of } \overline{M}_g \text{ in this field extension } \overline{M}_g^{(n,g)}).$$

Note that the full group G acts on $\overline{M}_g^{(n)}$. Moreover, $\overline{M}_g^{(n,g)}$ is the open subset of $\overline{M}_g^{(n)}$ lying over the open set M_g of smooth curves.

$$\begin{array}{ccc} p^{-1}(M_g) = \overline{M}_g^{(n,g)} & \subset & \overline{M}_g^{(n)} \\ \downarrow & & \downarrow \\ \overline{M}_g^{(n,g)} / G & \subset & \overline{M}_g^{(n)} / G \\ \parallel & & \parallel \\ M_g & \subset & \overline{M}_g \end{array} \quad \begin{array}{c} \curvearrowright \\ p \end{array}$$

What we call $\overline{\mathcal{M}}_g^{(n,h)}$, for $h < g$, however, is recovered by dividing an open subset of $\overline{\mathcal{M}}_g^{(n)}$ by H_h .

Life would be particularly simple if $\overline{\mathcal{M}}_g^{(n)}$ were smooth. However, it is not, nor is it known whether the normalization of $\overline{\mathcal{M}}_g^{(n)}$ in any finite extension field is smooth. However, fortunately $\overline{\mathcal{M}}_g$ does have a Cohen-Macaulay global cover, because:

Proposition 2.2. $\overline{\mathcal{M}}_g^{(n,g)} \subset \overline{\mathcal{M}}_g^{(n)}$ is a toroidal embedding, i.e., $\overline{\mathcal{M}}_g^{(n)}$ is formally isomorphic to \mathbb{A}^{3g-3} modulo an abelian group acting diagonally, and with $\overline{\mathcal{M}}_g^{(n)} - \overline{\mathcal{M}}_g^{(n,g)}$ isomorphic to the image of a union of coordinate hyperplanes in \mathbb{A}^{3g-3} .

Proof. At every point of $\overline{\mathcal{M}}_g^{(n)}$, $\overline{\mathcal{M}}_g^{(n)}$ is a Galois covering of one of the smooth varieties $\overline{\mathcal{M}}_g^{(n,h)}$ with group H_h . Note that H_h is abelian of order prime to $\text{char}(k)$. The covering is ramified only on $\overline{\mathcal{M}}_g^{(n,h)} - p_h^{-1}(\mathcal{M}_g)$. Since $\overline{\mathcal{M}}_g^{(n,h)}$ is formally the universal deformation space of some curve C , this is formally a ramified cover of \mathbb{A}^{3g-3} , ramified only in coordinate hyperplanes. But if $\text{char}(k) \nmid n$, the n -cyclic extensions $k[[x_1, \dots, x_{3g-3}]]$ ramified only over the ideals (x_i) are all given by

$$\left(\prod_{i \in I} x_i^{a_i} \right)^{1/n}, \quad I \subset \{1, \dots, 3g-3\}.$$

Thus the covering in hand is sandwiched between $k[[x_1, \dots, x_{3g-3}]]$ and $k[[x_1^{1/n}, \dots, x_{3g-3}^{1/n}]]$, hence, by Galois theory, is as described.

Corollary 2.3. $\overline{\mathcal{M}}_g^{(n)}$ is Cohen-Macaulay.

§2b. Q -Stacks

Unfortunately, the concept of Q -variety, although adequate to deal with $\overline{\mathcal{M}}_g$, $g \geq 3$, or with any moduli variety whose general object has no automorphisms, breaks down for $\overline{\mathcal{M}}_2$ and $\overline{\mathcal{M}}_{1,1}$ where the general object has automorphism group $\mathbb{Z}/2\mathbb{Z}$. Consider, for instance, $\overline{\mathcal{M}}_2$. Let $\mathcal{M}_2^o \subset \overline{\mathcal{M}}_2$

be the open set of smooth curves C such that $\text{Aut}(C) \cong \mathbb{Z}/2\mathbb{Z}$. Then, although M_2^o gives a local deformation space for its curves, it does not carry a universal family of curves. And if M'_2 is an étale cover of M_2^o carrying some family

$$p: C' \rightarrow M'_2$$

the sheaf $\mathbb{E} = p_*\Omega_{C'/M'}$ on M'_2 will not be a Q -sheaf. In fact, to compare $p_1^*\mathbb{E}, p_2^*\mathbb{E}$ on $M'_2 \times_{M_2} M'_2$, we want an isomorphism of the 2 families

$$\begin{array}{ccc} C' \times_{M_2} M'_2 & \cong & M'_2 \times_{M_2} C' \\ & \searrow & \swarrow \\ & M'_2 \times_{M_2} M'_2 & \end{array}$$

and although these families are fibrewise isomorphic, the isomorphism is not unique and may not globalize.

To deal with this, one must use some variant of the ideal of stack (cf. [D-M], §4). The most natural thing is to replace the normalization of $X_\alpha \times_X X_\beta$ by a scheme $X_{\alpha\beta}$ which must be given as part of the data and cannot be derived from the rest. $X_{\alpha\beta}$ should map to $X_\alpha \times_X X_\beta$ and given $x \in X_{\alpha\beta}, y \in X_{\beta\gamma}$ with the same projection to X_β , a “composition” $x \circ y \in X_{\alpha\gamma}$ should be defined. A point $x \in X_{\alpha\beta}$ lying over $u \in X_\alpha, v \in X_\beta$ should be thought of as meaning an isomorphism from the object C_u corresponding to u to the object C_v corresponding to v .

Definition 2.4. A Q -stack is a collection of quasi-projective varieties and morphisms:

$$\coprod X_{\alpha\beta} \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} \coprod X_\alpha \xrightarrow{p} X$$

$X_\alpha, X_{\alpha\beta}$ smooth, X normal, p_1, p_2 étale

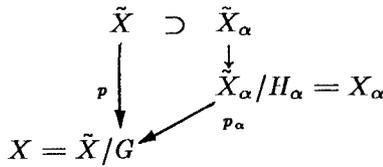
- $\coprod X_\alpha \rightarrow X$ surjective
- $X_{\alpha\beta} \rightarrow X_\alpha \times_X X_\beta$ surjective, finite
- $p_i \circ \epsilon = \text{identity}$

plus morphisms²:

$$\begin{array}{ccc} X_{\alpha\beta} \times_{X_\beta} X_{\beta\gamma} & \xrightarrow{\circ} & X_{\alpha\gamma} \\ & \searrow & \\ & X_{\alpha\beta} & \xrightarrow{-1} X_{\beta\alpha} \end{array}$$

making $\coprod X_{\alpha\beta}$ into a pseudo-group (i.e., \circ is associative where defined, -1 is its inverse and ϵ is an identity).

It is an interesting exercise in categorical style constructions to show that this collection of data can be derived from a finite group G acting on a normal variety \tilde{X} , plus open sets $\tilde{X}_\alpha \subset \tilde{X}$ stabilized by $H_\alpha \subset G$, very much as above:



and satisfying:

- (2.5) a) For all $x \in \tilde{X}_\alpha \cap g(\tilde{X}_\beta)$ and all $h \in H_\alpha$ such that $h(x) = x$ then $g^{-1}hg \in H_\beta$.
 b) H_α acts faithfully on X_α .

Then define

$$X_{\alpha\beta} = \coprod_{\substack{g = \text{repres. of double} \\ \text{cosets } H_\alpha \backslash G / H_\beta}} \tilde{X}_\alpha \cap g(\tilde{X}_\beta) / H_\alpha \cap gH_\beta g^{-1}$$

$$p_1 = \text{natural map } \tilde{X}_\alpha \cap g(\tilde{X}_\beta) / H_\alpha \cap gH_\beta g^{-1} \rightarrow \tilde{X}_\alpha / H_\alpha$$

$$p_2 = \text{the map induced by } g^{-1} \\ \tilde{X}_\alpha \cap g(\tilde{X}_\beta) / H_\alpha \cap gH_\beta g^{-1} \rightarrow \tilde{X}_\beta / H_\beta$$

and if

$$\begin{array}{l} x \in \tilde{X}_\alpha \cap g(\tilde{X}_\beta) \text{ maps to } \bar{x} \in X_{\alpha\beta} \\ y \in \tilde{X}_\beta \cap g'(\tilde{X}_\gamma) \text{ maps to } \bar{y} \in X_{\beta\gamma} \end{array}$$

²The maps \circ can also be introduced by giving as extra data more of a semi-simplicial variety:

$$\coprod X_{\alpha\beta\gamma} \rightrightarrows \coprod X_{\alpha\beta}$$

in the usual way.

so that

$$g^{-1}x = hy, \quad h \in H_\beta$$

then let

$$\bar{x} \circ \bar{y} = (\text{image of } x \in \tilde{X}_\alpha \cap ghg'(\tilde{X}_\gamma) \text{ in } X_{\alpha\gamma}).$$

The object so constructed is a Q -variety if G acts faithfully on \tilde{X} , but in general only a Q -stack.

Note that \bar{M}_2 and $\bar{M}_{1,1}$ are in a natural way Q -stacks. We let $\tilde{M}_2, \tilde{M}_{1,1}$ be the normalization of $\bar{M}_2, \bar{M}_{1,1}$ in the level n covering, some $n \geq 3$ and let $G = Sp(4, \mathbb{Z}/n\mathbb{Z}), SL(2, \mathbb{Z}/n\mathbb{Z})$ resp. The open sets X_α and subgroups H_α are defined exactly as in the case $g \geq 3$ treated above. The most general chart is any $X_\alpha \rightarrow \bar{M}_2$ (resp. $\bar{M}_{1,1}$) such that X_α comes with a family of the corresponding curves over it which represents locally everywhere the universal deformation space. Given 2 charts X_α, X_β , with families C_α, C_β , then $X_{\alpha\beta}$ is by definition:

$$\text{Isom}(C_\alpha, C_\beta) = \{(x, y, \phi) \mid x \in X_\alpha, y \in X_\beta, \phi \text{ an isom. of } C_{\alpha,x} \text{ with } C_{\beta,y}\}$$

Finally morphisms between Q -stacks X, Y are given by sets of morphisms and commuting diagrams:

$$\begin{array}{ccc} \coprod X_{\alpha\beta} & \xrightarrow{\cong} & \coprod X_\alpha \rightarrow X \\ \downarrow f_{\alpha\beta} & & \downarrow f_\alpha \quad \downarrow f \\ \coprod Y_{\alpha\beta} & \xrightarrow{\cong} & \coprod Y_\alpha \rightarrow Y \end{array}$$

provided the atlas for X is suitably refined. For suitable \tilde{X} and \tilde{Y} the morphism will be induced by a morphism

$$\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$$

which is equivariant with respect to a homomorphism $G_X \rightarrow G_Y$ of the finite groups acting on \tilde{X}, \tilde{Y} . However, if the Q -stack X is already presented as \tilde{X}/G_X for one \tilde{X} , one may have to pass to a bigger covering before \tilde{f} will be defined. This gives a diagram

$$\begin{array}{ccc} \tilde{X}' & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \downarrow & & \downarrow \\ \tilde{X} & & Y \\ \downarrow & & \downarrow \\ \tilde{X}'/G_X = \tilde{X}/G_X = X & \xrightarrow{f} & Y = \tilde{Y}/G_Y \end{array}$$

Among morphisms of Q -stacks, the simplest class consists of those that satisfy:

$$(2.6) \quad \left\{ \begin{array}{l} \forall \alpha, \text{ let } X'_{\alpha\alpha} = \{x \in X_{\alpha\alpha} \mid f_{\alpha\alpha}(x) = c_Y(f_\alpha(p_1(x)))\}. \\ \text{Then } X'_{\alpha\alpha} \text{ acts freely on } X_\alpha \end{array} \right.$$

These are the morphisms whose fibres are bona fide varieties, not just Q -varieties or Q -stacks. For such morphisms, it is possible to choose \tilde{X}, \tilde{Y} with the same finite group G acting and \tilde{f} G -equivariant:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \downarrow & & \downarrow \\ \tilde{X}/G = X & \rightarrow & Y = \tilde{Y}/G \end{array}$$

such that, moreover, locally, $\tilde{X}_\alpha \cong (\tilde{X}_\alpha/H_\alpha) \times_{(\tilde{Y}_\alpha/H_\alpha)} \tilde{Y}_\alpha$. The fibres of \tilde{f} are then the "true fibres" of f . The typical example of this is

$$\begin{array}{ccc} \bar{\mathcal{C}}_g^{(n)} & \rightarrow & \bar{\mathcal{M}}_g^{(n)} \\ \downarrow & & \downarrow \\ \bar{\mathcal{C}}_g & \rightarrow & \bar{\mathcal{M}}_g \end{array}$$

where

- $\bar{\mathcal{C}}_g =$ moduli space of pairs (C, x) , C a stable curve, $x \in C$
- \cong moduli space of pairs (C, x) , a 1-pointed stable curve³
- $\bar{\mathcal{C}}_g^{(n)} =$ normalization of $\bar{\mathcal{C}}_g$ in the covering defined by the moduli space of triples (C, x, ϕ) , C a smooth curve, $x \in C$ and ϕ a level n structure on C .

Such morphisms may be called *representable* morphisms of Q -stacks. We do not need to develop the theory of Q -stacks for our applications, so we stop at merely these definitions.

³An n -pointed stable curve (C, x_1, \dots, x_n) is a reduced, connected curve C with at most ordinary double points plus n distinct smooth points $x_i \in C$ such that every smooth rational component E of C contains at least 3 points which are either x_i 's or double points of C .

§3. The Chow Group for \mathcal{Q} -Varieties with Cohen-Macaulay Global Covers

We want to study a quasi-projective \mathcal{Q} -variety $\{p_\alpha: X_\alpha \rightarrow X\}$ of dimension n such that the big global cover \tilde{X} is Cohen-Macaulay. We use the notation of §2, esp. $X = \tilde{X}/G$. We also choose a resolution of singularities

$$\pi: \tilde{X}^* \rightarrow \tilde{X}.$$

Then we assert:

Theorem 3.1. *With the above hypotheses, there is a canonical isomorphism γ between the Chow group of X and G -invariants in the operational Chow ring of \tilde{X} , (as usual after extending the coefficients to \mathbb{Q}):*

$$\gamma: A_{n-k}(X) \cong opA^k(\tilde{X})^G, \quad 0 \leq k \leq n.$$

This key result does two important things for us:

- a) it defines a ring structure on $A.(X)$,
- b) for all \mathcal{Q} -sheaves \mathcal{F} on the \mathcal{Q} -variety X , we can define Chern classes

$$c_k(\mathcal{F}) \in A.(X).$$

I don't know if these things can be done if we drop the hypothesis that \tilde{X} is Cohen-Macaulay. My guess is that this hypothesis can be dropped, but more powerful tools seem to be needed to treat this case.

Proof. The first step is to define, for all subvarieties $Z \subset X$, an element $\gamma(Z) \in opA^*(\tilde{X})^G \otimes \mathbb{Q}$. We use the local covers $p_\alpha: X_\alpha \rightarrow X$ and let $p_\alpha^{-1}(Z)$ be the *reduced* subscheme of X_α with support $p_\alpha^{-1}(Z)$. Then $\{\mathcal{O}_{p_\alpha^{-1}(Z)}\}$ is a \mathcal{Q} -sheaf on the \mathcal{Q} -variety X . As above, let p factor locally

$$\tilde{X}_\alpha \xrightarrow{q_\alpha} \tilde{X}_\alpha/H_\alpha \cong X_\alpha \xrightarrow{p_\alpha} X$$

and lift $p_\alpha^{-1}(Z)$ to its *scheme-theoretic* inverse image $q_\alpha^*(p_\alpha^{-1}(Z))$. Define

$$\tilde{Z} = \left\{ \begin{array}{l} G\text{-invariant subscheme of } \tilde{X} \text{ supported} \\ \text{on } p^{-1}(Z) \text{ such that } \tilde{Z}|_{\tilde{X}_\alpha} = q_\alpha^*(p_\alpha^{-1}(Z)) \end{array} \right\}.$$

Note that because \tilde{X} is Cohen-Macaulay, and X_α is smooth, q_α is flat. Since $\mathcal{O}_{p_\alpha^{-1}(Z)}$ has a finite projective resolution on X_α , this implies that \mathcal{O}_Z has a finite projective resolution on \tilde{X} . Therefore, by Fulton's theory [F1], the Chern classes $c_k(\mathcal{O}_{\tilde{Z}})$ are defined in $opA^k(\tilde{X})$. Next define the "ramification index":

$$e(Z) = \text{order of the stabilizer in } G_\alpha \text{ of almost all points of } p_\alpha^{-1}(Z).$$

Here α is any index such that $p_\alpha^{-1}(Z) \neq \emptyset$: the definition does not depend on α . Then let

$$\begin{aligned} \gamma(Z) &= \frac{(-1)^{k-1} e(Z)}{(k-1)!} \cdot c_k(\mathcal{O}_{\tilde{Z}}), \quad \text{if } k = \text{codim } Z, \\ (\text{or } &= e(Z)ch_k(\mathcal{O}_{\tilde{Z}}), \text{ using lemma 3.3 below).} \end{aligned}$$

An important point in the study of \tilde{Z} is that the family of subschemes $\{p_\alpha^{-1}(Z)\}$ can be simultaneously resolved:

Theorem 3.2 (Hironaka). *For all subvarieties $Z \subset X$, there is a birational map $\pi: Z^* \rightarrow Z$, Z^* normal, such that for all α*

$$(p_\alpha^{-1}(Z) \times_Z Z^*)_{nor}$$

is smooth.

This is a Corollary of Hironaka's strong resolution theorem, giving a resolution compatible with the pseudo-group of all local analytic isomorphisms between open sets in the original variety: see [H], Theorem 7.1, p. 164. One may proceed as follows: first resolve $p_{\alpha_1}^{-1}(Z)$ as in [H]. Since $((\tilde{X}_{\alpha_1}/H_{\alpha_1}) \times_X (\tilde{X}_{\alpha_1}/H_{\alpha_1}))_{nor}$ is étale over $\tilde{X}_{\alpha_1}/H_{\alpha_1}$ this equivalence relation extends to one on the resolution $p_{\alpha_1}^{-1}(Z)^*$, hence there is a blow-up $\pi_1: Z_1 \rightarrow Z$ such that $p_{\alpha_1}^{-1}(Z)^* \cong (p_{\alpha_1}^{-1}(Z) \times_Z Z_1)_{nor}$. Secondly, resolve $(p_{\alpha_2}^{-1}(Z) \times_Z Z_1)_{nor}$ by a blow-up over $X - p(X_{\alpha_1})$ so as not to affect the first step. Again, descend this to a further blow-up ($\pi_2: Z_2 \rightarrow Z_1 \rightarrow Z$) of Z . Eventually, we get the needed resolution.

Lemma 3.3. *For all \mathcal{Q} -sheaves \mathcal{F} on the \mathcal{Q} -variety X , let $S \subset X$ be the support of \mathcal{F} . Then in $opA^*(\tilde{X})$,*

- a) $c_k(\tilde{\mathcal{F}}) = 0$ if $k < \text{codim } S$
- b) if $k = \text{codim } S$, and S_1, \dots, S_n are the codimension k components of S , then

$$c_k(\tilde{\mathcal{F}}) = \sum_{i=1}^n l_i c_k(\mathcal{O}_{\tilde{S}_i})$$

where l_i is the length of the stalk of $\tilde{\mathcal{F}}_\alpha$ at the generic point of $p_\alpha^{-1}(S_i)$ when S_i meets $p(\tilde{X}_\alpha)$.

Proof. This results from an application of Fulton's Grothendieck-Riemann-Roch Theorem 1.4 and Hironaka's resolution 3.2. By the usual dévissage, reduce the lemma to the case where S is irreducible of codimension k , and $\tilde{\mathcal{F}}$ is an $\mathcal{O}_{\tilde{S}}$ -module. Let $S^* \rightarrow S$ be a "resolution" as in 3.2. This gives a family of resolutions

$$S_\alpha^* = (p_\alpha^{-1}(S) \times_S S^*)_{\text{nor}} \xrightarrow{\pi_\alpha} p_\alpha^{-1}(S) \subset \tilde{X}_\alpha/H_\alpha$$

which are local complete intersection (or l.c.i.) morphisms. Therefore, by fibre product with the flat morphism $\tilde{X}_\alpha \rightarrow \tilde{X}_\alpha/H_\alpha$,

$$S_\alpha^* \times_{(\tilde{X}_\alpha/H_\alpha)} \tilde{X}_\alpha \rightarrow \tilde{X}_\alpha$$

are l.c.i. morphisms. These glue together to an l.c.i. morphism

$$\tilde{S}^* \xrightarrow{\pi} \tilde{X}$$

such that $\tilde{\mathcal{F}}$ is a $\pi_* \mathcal{O}_{\tilde{S}^*}$ -module. Let $\tilde{\mathcal{F}}^* = \pi^*(\tilde{\mathcal{F}})$. Then by 1.4:

$$ch \pi_!(\tilde{\mathcal{F}}^*) = \pi_*(ch \tilde{\mathcal{F}}^* \cdot Td_\pi).$$

Now if $i > 0$, $R^i \pi_*(\tilde{\mathcal{F}}^*)$ are \mathcal{Q} -sheaves on the \mathcal{Q} -variety X with supports properly contained in S , so by induction we can assume

$$c_l(R^i \pi_* \tilde{\mathcal{F}}^*) = 0 \quad \text{if } l \leq k, i > 0.$$

Therefore

$$ch_l(\pi_! \tilde{\mathcal{F}}^*) = ch_l(\pi_* \tilde{\mathcal{F}}^*) = ch_l(\tilde{\mathcal{F}}) \quad \text{if } l \leq k.$$

But $\pi_*: opA^1(\tilde{S}^*) \rightarrow opA^1(\tilde{X})$ raises the codimension of a cycle by k . So

$$ch_l(\tilde{\mathcal{F}}) = 0 \quad \text{if } l < k$$

and

$$\begin{aligned} ch_k(\tilde{\mathcal{F}}) &= (\text{generic rank of } \tilde{\mathcal{F}}^* \text{ as free } \mathcal{O}_{\tilde{S}}\text{-module}) \cdot \pi_*(1) \\ &= (\text{length of } \mathcal{F}_\alpha \text{ at generic point of } p_\alpha^{-1}(S)) \cdot ch_k(\mathcal{O}_{\tilde{S}}). \end{aligned}$$

Q.E.D.

Lemma 3.4. *If two cycles $\sum n_i Z_i, \sum m_i W_i$ on X are rationally equivalent, then*

$$\sum n_i \gamma(Z_i) = \sum m_i \gamma(W_i) \quad \text{in } opA^*(\tilde{X}).$$

Proof. Let L be an ample line bundle on X . Rational equivalence on X may be defined by requiring that for all subvarieties $Y \subset X$ and all $s_1, s_2 \in \Gamma(Y, L^n \otimes \mathcal{O}_Y)$, if D_i is the divisor of zeroes of s_i on X , then

$$D_1 \underset{\text{rat}}{\sim} D_2.$$

So to prove the lemma, it will suffice to prove that for all $s \in \Gamma(Y, L^n \otimes \mathcal{O}_Y)$, if $D = \sum n_i Z_i$ is the divisor of zeroes of s , then

$$\sum n_i \gamma(Z_i) = e(Y) \cdot [ch_k(\mathcal{O}_{\tilde{Y}}) - ch_k(L^{-n} \otimes \mathcal{O}_Y)].$$

To see this, use the exact sequence of \mathcal{Q} -sheaves

$$0 \rightarrow L^{-n} \otimes \mathcal{O}_{\tilde{Y}} \xrightarrow{\otimes^s} \mathcal{O}_{\tilde{Y}} \rightarrow \mathcal{O}_{\tilde{D}} \rightarrow 0$$

where $\tilde{D} \subset \tilde{X}$ is the scheme of zeroes of s in \tilde{Y} and the local calculation⁴ that on $p^{-1}(Y)$:

$$(*) \quad (\text{Divisor of zeroes of } p_\alpha^*(s) \text{ on } p_\alpha^{-1}(Y)) = \sum n_i \frac{e(Z_i)}{e(Y)} p_\alpha^{-1}(W_i)$$

⁴We use the lemma that if a finite group G acts faithfully on a variety Y and ϕ is a G -invariant function on Y , zero on a subvariety $W \subset Y$ of codimension 1, and $\bar{\phi}$ is the induced function on Y/G , then:

$$\text{ord}_W(\phi) = \#\{g \in G \mid g = id. \text{ on } W\} \cdot \text{ord}_{W/G}(\bar{\phi}).$$

(n.b., $p_\alpha^{-1}(Y)$, $p_\alpha^{-1}(W_i)$ are the *reduced* inverse images). Therefore

$$\begin{aligned}
 ch_k \mathcal{O}_{\tilde{Y}} - ch_k L^{-n} \otimes \mathcal{O}_{\tilde{Y}} &= ch_k \mathcal{O}_{\tilde{D}} \\
 &= \frac{(-1)^{k-1}}{(k-1)!} c_k \mathcal{O}_{\tilde{D}} \\
 &= \sum_i \left(\text{length of } \mathcal{O}_{D_\alpha} \text{ at} \right. \\
 &\quad \left. \text{gen.pt. of } p_\alpha^{-1}(W_i) \right) \cdot \frac{(-1)^{k-1}}{(k-1)!} c_k \mathcal{O}_{\tilde{W}_i} \quad \text{by (3.3)} \\
 &= \sum_i \text{ord}_{p_\alpha^{-1}(W_i)}(p_\alpha^* s) \cdot \frac{\gamma(W_i)}{e(W_i)} \quad \text{by def}^n \text{ of } \gamma \\
 &= \frac{1}{e(Y)} \sum n_i \gamma(W_i) \quad \text{by (*)}.
 \end{aligned}$$

This proves (3.4), which shows that γ factors:

$$\gamma: A.(X) \rightarrow opA'(\tilde{X})^G.$$

Lemma 3.5. *The composition of maps*

$$A.(X) \xrightarrow{\gamma} opA'(\tilde{X})^G \xrightarrow{n[\tilde{X}]} A.(\tilde{X})^G \xrightarrow{p_*} A.(X)$$

is multiplication by n , the degree of p .

Proof. To prove this we use another Riemann-Roch theorem: the version of Baum-Fulton-MacPherson [BFM]. This says that there is a natural transformation $\tau: K_o(Z) \rightarrow A.(Z)$ for all varieties Z such that

$$\begin{array}{ccc}
 K_o(Z) \otimes K_o(Z) & \xrightarrow{\otimes} & K_o(Z) \\
 \downarrow ch \otimes \tau & & \downarrow \tau \\
 opA'(Z) \otimes A.(Z) & \xrightarrow{\cap} & A.(Z)
 \end{array}$$

commutes. By the lemma, p. 129 of [BFM] and dévissage, τ satisfies:

for all \mathcal{F} with support $\cup Z_i$ of codimension k , $\tau(\mathcal{F})$ has codimension k and

$$\tau(\mathcal{F})_k = \text{class of } \sum \left(\begin{array}{c} \text{length of } \mathcal{F} \text{ at} \\ \text{gen. pt. of } Z_i \end{array} \right) [Z_i].$$

We apply this to $Z = \tilde{X}$ and $\mathcal{F} = \mathcal{O}_{\tilde{Z}}$ where $Z \subset X$ is a subvariety. It follows that

$$ch(\mathcal{O}_{\tilde{Z}}) \cap \tau(\mathcal{O}_{\tilde{X}}) = \tau(\mathcal{O}_{\tilde{Z}}).$$

Therefore if $k = \text{codimension } Z$,

$$\begin{aligned} p_*(\gamma(Z) \cap [\tilde{X}]) &= e(Z) \cdot p_*(ch_k(\mathcal{O}_{\tilde{Z}}) \cap [\tilde{X}]) \\ &= e(Z) \cdot p_*([\tilde{Z}]) \\ &= e(Z) \cdot [\tilde{Z} : Z] \cdot \text{class of } Z \\ &= n \cdot \text{class of } Z. \end{aligned}$$

Q.E.D.

Lemma 3.6. *If $\pi: \tilde{X}^* \rightarrow \tilde{X}$ is a resolution, and $Z \subset X$ is a subvariety of codimension k such that for all components Z_i of $p^{-1}(Z)$, $\pi^{-1}(Z_i)$ is irreducible of codimension k , then $\pi^*(\gamma(Z))$ is represented by a cycle*

$$c \cdot \sum \pi^{-1}(Z_i), \quad \text{some } c \in \mathbb{Q}, c > 0.$$

Proof. Let $U \subset \tilde{X}$ be the open set over which π is an isomorphism. Then $Z_i \cap U \neq \emptyset$, all i . Now

$$\begin{aligned} \pi^*(\gamma(Z)) &= e(Z) \cdot \pi^*(ch_k(\mathcal{O}_{\tilde{Z}})) \\ &= e(Z) \cdot \left(\sum (-1)^l ch_k(\text{tor}_l(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{X}^*})) \right). \end{aligned}$$

But these tor_l are supported on proper subsets of $\pi^{-1}(Z_i)$, hence have no k^{th} Chern character. Therefore:

$$\begin{aligned} \pi^*(\gamma(Z)) &= e(Z) \cdot ch_k(\mathcal{O}_{\tilde{Z}} \otimes \mathcal{O}_{\tilde{X}^*}) \\ &= e(Z) \cdot \text{class of } \pi^{-1}(\tilde{Z}) \end{aligned}$$

by the Riemann-Roch theorem on \tilde{X}^* .

Q.E.D.

Corollary 3.7. γ is bijective.

Proof. 3.5, 3.6 and 1.2.

This proves the theorem. A few comments can be made on the ring structure that this introduces in $A(X)$. First of all, suppose W_1, W_2 are two cycles on X that intersect properly. Then the product $[W_1], [W_2]$ in the above ring structure can also be defined directly by assigning suitable multiplicities to the components of $\text{Supp } W_1 \cap \text{Supp } W_2$. In fact, define:

$$W_1 \cdot W_2 = \sum_{\substack{\text{comp. } U \text{ of} \\ \text{Supp } W_1 \cap \text{Supp } W_2}} i(W_1 \cap W_2; U) \cdot U$$

where if $p_\alpha^{-1}(U) \neq \emptyset$, then

$$i(W_1 \cap W_2; U) = \frac{e(W_1) \cdot e(W_2)}{e(U)} \cdot i(p_\alpha^{-1}(W_1) \cap p_\alpha^{-1}(W_2); p_\alpha^{-1}U).$$

Note that the intersection multiplicity on the right is taken on the smooth ambient variety $\tilde{X}_\alpha/H_\alpha$, hence is defined, e.g., by

$$\sum_l (-1)^l \binom{\text{length at gen. pt.}}{\text{of } p_\alpha^{-1}U} (\text{tor}_l(\mathcal{O}_{p_\alpha^{-1}W_1}, \mathcal{O}_{p_\alpha^{-1}W_2})).$$

The proof that this is the same as the product in $opA'(\tilde{X})$ is straightforward, i.e.,

$$\begin{aligned} \gamma(Z_1) \cdot \gamma(Z_2) &= e(Z_1)e(Z_2)ch_{k_1}(\mathcal{O}_{\tilde{Z}_1}) \cdot ch_{k_2}(\mathcal{O}_{\tilde{Z}_2}) \\ &= e(Z_1)e(Z_2)ch_{k_1+k_2}(\mathcal{O}_{\tilde{Z}_1} \overset{L}{\otimes} \mathcal{O}_{\tilde{Z}_2}) \\ &\quad (\overset{L}{\otimes} \text{ means take tensor product of projective resol.}) \\ &= \sum_U e(Z_1)e(Z_2)i(p_\alpha^{-1}(W_1) \cap p_\alpha^{-1}(W_2); p_\alpha^{-1}U) \cdot ch_{k_1+k_2}(\mathcal{O}_U) \\ &= \sum \frac{e(Z_1)e(Z_2)}{e(U)} i(p_\alpha^{-1}(W_1) \cap p_\alpha^{-1}(W_2); p_\alpha^{-1}U) \cdot \gamma(U). \end{aligned}$$

This product could be introduced directly without relating it to the product in $opA'(\tilde{X})$. This has been done by Matsusaka in his book "Theory of Q -varieties", [Ma], where associativity and other standard formulae are proven. The missing ingredient, however, is the moving lemma. This follows as a Corollary of the isomorphism of $A(X)$ with $opA'(\tilde{X})$, i.e., by representing a cycle on X as the projection from \tilde{X} of the Chern class of

a sheaf with finite resolution. In particular, I don't know any way to get a moving lemma unless some \tilde{X} is Cohen-Macaulay.

Henceforth, in the study of the Chow rings of \mathbb{Q} -varieties we shall identify $A_{n-k}(X)$ and $opA^k(\tilde{X})^G$ via the map γ , and write this as $A^k(X)$ just like the k -codimension piece of the Chow ring of an ordinary non-singular variety. This does not usually lead to any confusion, except with regard to the concept of the *fundamental class* of a subvariety $Y \subset X$. The important thing to realize here is that there are really two different notions of fundamental class, differing by a rational number, and both are important. Thus for all Y of codimension k , we will write

$$[Y] = \text{class of the cycle } Y \text{ in the Chow group}$$

$$A_{n-k}(X) = A^k(X)$$

and

$$[Y]_{\mathbb{Q}} = \text{the class } ch_k(\mathcal{O}_{\tilde{Y}}) \text{ in } opA^k(\tilde{X})^G = A^k(X).$$

Since we are using the identification γ , we have:

$$[Y]_{\mathbb{Q}} = \frac{1}{e(Y)} \cdot [Y].$$

When one makes calculations of intersections in local charts $\tilde{X}_{\alpha}/H_{\alpha}$, then one is verifying an identity between classes $[Y]_{\mathbb{Q}}$. But when one has a rational equivalence between cycles on X , one has an identity between $[Y]'$'s: e.g., if X is unirational, then for all points $P_1, P_2 \in X$,

$$[P_1] = [P_2],$$

but the point classes $[P]_{\mathbb{Q}}$ are fractions $1/e(P)$ of the basic point class $[P] \in A^1(X)$.

If X is a \mathbb{Q} -stack, exactly the same theorem holds and we have an isomorphism

$$\gamma: A_*(X) \cong opA^*(\tilde{X})^G.$$

The only difference is that a subgroup $Z \subset G$ acts identically on \tilde{X} . If $\#Z = z$, then the effect of this is merely to modify the ring structure on $A^*(X)$ as follows. Let W_1, W_2 be cycles on X and consider:

- i) the \mathbb{Q} -variety structure on X given by the action of G/Z on \tilde{X} , and the multiplication $W_1 \cdot_{var} W_2$,

ii) the Q -stack structure on X given by the action of G on \tilde{X} , and the multiplication $W_{1 \cdot st} W_2$.

Then by the moving lemma plus the formula above for proper intersections, it follows:

$$W_{1 \cdot st} W_2 = z \cdot W_{1 \cdot var} W_2.$$

In particular, the identity in the Chow ring of a Q -stack X is $[X]_Q$, not $[X]$.

The Chow ring for Q -varieties, or more generally Q -stacks, has good contravariant functorial properties. We consider morphisms of Q -stacks with global Cohen-Macaulay covers:

$$X \xrightarrow{f} Y$$

as defined in §2. Then I claim:

Proposition 3.8. *There is a canonical ring homomorphism*

$$f^* : A(Y) \rightarrow A(X)$$

satisfying:

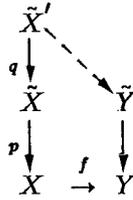
- i) $f_*(a \cdot f^* b) = f_* a \cdot b$ (f_* being defined from $A(X)$ to $A(Y)$ as usual),
- ii) $f^*(c_k \mathcal{E}) = c_k(f^* \mathcal{E})$ for all Q -vector bundles \mathcal{E} on Y
- iii) if W is a subvariety of Y such that $\text{codim } f^{-1}(W) = \text{codim } W$ then

$$f^*([W]) = \text{class of } \sum_{\substack{\text{comp. } V_k \\ \text{of } f^{-1}(W)}} i_k \cdot [V_k]$$

where i_k is calculated on suitable charts $f_\alpha: X_\alpha \rightarrow Y_\alpha$ by pull-backs in the smooth case adjusted by $e(W)/e(V_k)$.

Proof. Although the moving lemma plus (iii) provides us with the simplest formula for f^* , to see that f^* is well-defined, we use opA . There is one complication. X and Y have global Cohen-Macaulay covers \tilde{X}, \tilde{Y} but f may

not lift to $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$. Instead, we may have to ‘refine’ \tilde{X} :



and then \tilde{X}' may not longer be Cohen-Macaulay. Still, once one has one Cohen-Macaulay \tilde{X} with which to set up the theory, one proves that

$$(3.9) \quad opA \cdot (X)^{G_X} \xrightarrow{q^*} opA \cdot (\tilde{X}')^{G_X}$$

is an isomorphism, hence f^* may be defined by:

$$A \cdot (Y) \cong_{\gamma_Y} opA \cdot (\tilde{Y})^{G_Y} \xrightarrow{\tilde{f}_*} opA \cdot (\tilde{X}')^{G_X} \cong opA \cdot (\tilde{X})^{G_X} \cong A \cdot (X).$$

To check (3.9), use

$$\begin{array}{ccc}
 A \cdot (X) \cong_{\gamma_X} opA \cdot (\tilde{X})^{G_X} & \xrightarrow{q^*} & opA \cdot (\tilde{X}')^{G_X} \hookrightarrow A \cdot (\tilde{X}') \\
 & & \downarrow \cap [\tilde{X}'] \\
 & \searrow^{(p \circ q)_*} & A \cdot (\tilde{X}')
 \end{array}$$

(\tilde{X}'' = resolution of \tilde{X}') and argue as in lemmas 3.5 and 3.6. There is one hitch: namely, in 3.5, we get

$$\begin{aligned}
 (p \circ q)_*(q^*(\gamma(Z)) \cap [\tilde{X}']) &= e(Z)(p \circ q)_*(q^*(ch_k(\mathcal{O}_{\tilde{Z}})) \cap [\tilde{X}']) \\
 &= e(Z)(p \circ q)_*(ch_k(\mathcal{O}_{\tilde{Z}} \overset{L}{\otimes} \mathcal{O}_{\tilde{X}'})) \cap [X']
 \end{aligned}$$

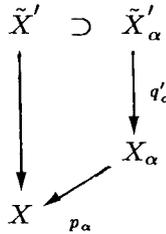
where $\overset{L}{\otimes}$ means take a resolution of $\mathcal{O}_{\tilde{Z}}$ and tensor it with $\mathcal{O}_{\tilde{X}'}$. But now if \tilde{X}' is not Cohen-Macaulay, $\mathcal{O}_{\tilde{Z}} \overset{L}{\otimes} \mathcal{O}_{\tilde{X}'}$ will not be a resolution of some $\mathcal{O}_{\tilde{Z}'}$ and we get instead:

$$\begin{aligned}
 &= e(Z) \sum (-1)^l (p \circ q)_*(ch_k \operatorname{tor}_l(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{X}'}) \cap [\tilde{X}']) \\
 &= e(Z) \cdot (p \circ q)_* \left(\sum_n i_n \cdot [\tilde{Z}_n] \right)
 \end{aligned}$$

where \tilde{Z}_l are the components of $(p \circ q)^{-1}Z$ and

$$\begin{aligned} i_n &= \sum (-1)^l (\text{length at gen. pt. of } \tilde{Z}_n) \left(\text{tor}_l^{\mathcal{O}_{\tilde{X}}} (\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{X}'}) \right) \\ &= \sum (-1)^l (\text{length at gen. pt. of } \tilde{Z}_n) \left(\text{tor}_l^{\mathcal{O}_{X_\alpha}} (\mathcal{O}_{p_\alpha^{-1}(Z)}, \mathcal{O}_{\tilde{X}'}) \right) \\ &= \text{mult. of } \tilde{Z}_n \text{ in the cycle } q'_\alpha{}^*(p_\alpha^{-1}(Z)) \end{aligned}$$

where we factor $p \circ q$:



Thus

$$\begin{aligned} e(Z)(p \circ q)_* \left(\sum_n i_n [\tilde{Z}_n] \right) &= e(Z) p_{\alpha,*} (\text{class of } q'_{\alpha,*} (q'_\alpha{}^*(p_\alpha^{-1}Z))) \\ &= \text{deg } q'_\alpha \cdot e(Z) \cdot p_{\alpha,*} (p_\alpha^{-1}Z) \\ &= n \cdot \text{class of } Z. \end{aligned}$$

f^* being defined, the rest of the proof is straightforward.

For representable morphisms $f: X \rightarrow Y$ of \mathcal{Q} -stacks, there is a further important compatibility. For such f , let

$$\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$$

be a G -equivariant morphism such that $X = \tilde{X}/G, Y = \tilde{Y}/G$ and

$$(3.9) \quad \tilde{X}_\alpha \cong X_\alpha \times_{Y_\alpha} \tilde{Y}_\alpha$$

as in §2b. Then we have:

Proposition 3.10. *If the morphism f on local charts*

$$f_\alpha: X_\alpha \rightarrow Y_\alpha$$

is a local complete intersection and \tilde{Y} is Cohen-Macaulay, then \tilde{f} is l.c.i. and the diagram

$$\begin{array}{ccc}
 \text{op}A^*(\tilde{X})^G & \xrightarrow{\tilde{f}_*} & \text{op}A^*(\tilde{Y})^G \\
 \uparrow \gamma & & \uparrow \gamma \\
 A(X) & \xrightarrow{f_*} & A(Y)
 \end{array}$$

commutes.

Proof. Let W be a codimension k subvariety of X such that W is generically finite over $f(W)$. Then we must check

$$\tilde{f}_*(\gamma W) = [W : fW] \cdot \gamma(fW),$$

i.e.,

$$e(X) \cdot \tilde{f}_*(ch_k(\mathcal{O}_{\tilde{W}})) = [W : fW] \cdot e(fW) \cdot ch_k(\mathcal{O}_{f\tilde{W}}).$$

But by Riemann-Roch for \tilde{f} ,

$$\begin{aligned}
 \tilde{f}_*(ch_k(\mathcal{O}_{\tilde{W}})) &= ch_k(\tilde{f}_*\mathcal{O}_{\tilde{W}}) \\
 &= [W_\alpha : f_\alpha(W_\alpha)] \cdot ch_k(\mathcal{O}_{f\tilde{W}})
 \end{aligned}$$

because $\tilde{f}_*(\mathcal{O}_{\tilde{W}})$ is generically a locally free $\mathcal{O}_{f\tilde{W}}$ -algebra of length $[W_\alpha : f_\alpha W_\alpha]$ by (3.9). But now use:

$$[W_\alpha : W] \cdot [W : fW] = [W_\alpha : f_\alpha W_\alpha] \cdot [f_\alpha W_\alpha : fW]$$

and

$$\begin{aligned}
 [W_\alpha : W] \cdot e(W) &= [X_\alpha : X] \cdot e(X) \\
 &= [G : H_\alpha] \\
 &= [Y_\alpha : Y] \cdot e(Y) \\
 &= [f(W)_\alpha : fW] \cdot e(fW)
 \end{aligned}$$

and the equality of the coefficients follows.

Part II: Basic Classes in the Chow Ring of the Moduli Space

§4. Tautological Classes

Whenever a variety or topological space is defined by some universal property, one expects that by virtue of its defining property, it possesses certain cohomology classes called tautological classes. The standard example is a Grassmannian, e.g., the Grassmannian Grass of k -planes in \mathbb{C}^n . By its very definition, there is a universal bundle E on Grass of rank k , and this induces Chern classes $c_l(E)$, $1 \leq l \leq k$, in both the cohomology ring of Grass and the Chow ring of Grass . These two rings are, in fact, isomorphic and generated as rings by $\{c_l(E)\}$. Moreover, one gets tautological relations from the fact that E is a sub-bundle of the trivial bundle $\mathbb{C}^n \times \text{Grass}$. This gives an exact sequence:

$$0 \rightarrow E \rightarrow \mathcal{O}^n \rightarrow F \rightarrow 0, \quad F \text{ a bundle of rank } n - k,$$

hence

$$(1 + c_1(E) + \cdots + c_k(E))_l^{-1} = 0, \quad l > n - k.$$

As is well known, these are a complete set of relations for the cohomology and Chow rings of Grass .

We shall begin a program of the same sort for the Chow ring (or cohomology ring) of $\overline{\mathcal{M}}_g$. Our purpose is merely to identify a natural set of tautological classes and some tautological relations. To what extent these lead to a presentation of either ring is totally unclear at the moment.

The natural place to start is with the universal curve over $\overline{\mathcal{M}}_g$. This is the same as the coarse moduli space of 1-pointed stable curves (C, P) (see Knudsen [K], Harris-Mumford [H-M]), which we call $\overline{\mathcal{M}}_{g,1}$ or $\overline{\mathcal{C}}_g$ alternatively. $\overline{\mathcal{C}}_g$ is a Q -variety, too, and everything we have said about $\overline{\mathcal{M}}_g$ applies to $\overline{\mathcal{C}}_g$ too. The morphism $\overline{\mathcal{C}}_g \rightarrow \overline{\mathcal{M}}_g$ is a representable morphism, and via level n structures, we have a covering $\tilde{\mathcal{C}}_g$ of $\overline{\mathcal{C}}_g$ and a morphism:

$$\pi: \tilde{\mathcal{C}}_g \rightarrow \tilde{\mathcal{M}}_g$$

which is a flat, proper family of stable curves, with a finite group G acting on both, and $\overline{\mathcal{C}}_g = \tilde{\mathcal{C}}_g/G$, $\overline{\mathcal{M}}_g = \tilde{\mathcal{M}}_g/G$. If $g \geq 3$, then

$$\tilde{\mathcal{C}}_g = (\text{normalization of } \overline{\mathcal{C}}_g \times_{\overline{\mathcal{M}}_g} \tilde{\mathcal{M}}_g),$$

but if $g = 2$, the generic curve has automorphisms, $Sp(4, \mathbb{Z}/n\mathbb{Z})$ does not act faithfully on \tilde{M}_2 , and \tilde{C}_2 is a double cover of this normalization. In any case, \tilde{C}_g has a \mathbb{Q} -sheaf $\omega_{\tilde{C}_g/\tilde{M}_g}$ represented by the invertible sheaf $\omega_{\tilde{C}_g/\tilde{M}_g}$ on \tilde{C}_g . Henceforth, whenever we talk of sheaves on \tilde{C}_g or \tilde{M}_g we shall mean \mathbb{Q} -sheaves and they are always represented by usual coherent sheaves on \tilde{C}_g and \tilde{M}_g with G -action. Furthermore, we shall make calculations in $A^*(\tilde{M}_g)$ and $A^*(\tilde{C}_g)$ by implicitly identifying these with $opA^*(\tilde{M}_g)^G$ and $opA^*(\tilde{C}_g)^G$.

Now define the *tautological classes*:

$$\begin{aligned} K_{\tilde{C}_g/\tilde{M}_g} &= c_1(\omega_{\tilde{C}_g/\tilde{M}_g}) \in A^1(\tilde{C}_g) \\ \kappa_l &= (\pi_* K_{\tilde{C}_g/\tilde{M}_g}^{l+1}) \in A^l(\tilde{M}_g) \\ E &= \pi_*(\omega_{\tilde{M}_g/\tilde{M}_g}) : \text{a locally free } \mathbb{Q}\text{-sheaf of rank } g \text{ on } \tilde{M}_g \\ \lambda_l &= c_l(E), \quad 1 \leq l \leq g. \end{aligned}$$

I believe that the κ_l are the natural tautological classes to consider on \tilde{M}_g . On the other hand, the λ_l are the natural classes for abelian varieties. Let me sketch this link, which will not be used subsequently. In fact, if

$$\mathcal{A}_g^* = \left(\begin{array}{l} \text{Satake's compactification of the moduli} \\ \text{space of principally polarized abelian varieties} \end{array} \right)$$

then there is a natural morphism

$$t: \tilde{M}_g \rightarrow \mathcal{A}_g^*$$

carrying the point $[C]$ to the point of \mathcal{A}_g^* defined by the Jacobian of C . This morphism lifts to a G -equivariant morphism

$$\tilde{t}: \tilde{M}_g \rightarrow \tilde{\mathcal{A}}_g$$

where $\tilde{\mathcal{A}}_g$ is a suitable toroidal compactification of the level n covering of \mathcal{A}_g : see Namikawa [N]. Moreover, $\tilde{\mathcal{A}}_g$ carries a universal family

$$\pi: \tilde{\mathcal{G}}_g \rightarrow \tilde{\mathcal{A}}_g$$

of semi-abelian group schemes, i.e., $\tilde{\mathcal{G}}/\tilde{\mathcal{A}}$ is a group scheme whose fibres are extensions of abelian varieties by algebraic tori $(\mathbb{C}^*)^h$. The family $\tilde{\mathcal{G}}_g$ pulls

back on \tilde{M}_g to the family of Jacobians and generalized Jacobians of \tilde{C}_g . Over \tilde{A}_g , define

$$\begin{aligned} E' &= \Omega_{\tilde{C}_g/\tilde{A}_g}^1|_{0\text{-section}}, \text{ a locally free sheaf of rank } g \\ \lambda'_l &= c_k(E'), \quad 1 \leq l \leq g. \end{aligned}$$

Then it follows that

$$\tilde{t}^* E' \cong E$$

and

$$\tilde{t}^* \lambda'_l = \lambda_l.$$

The class $K_{\tilde{C}_g/\tilde{M}_g}$ played a central role in the basic paper [A] of Arakelov, who proved the essential case of:

Theorem 4.1 (Arakelov). *The divisor $K_{\tilde{C}_g/\tilde{M}_g}$ is numerically effective on \tilde{C}_g , i.e., for all curves $C \subset \tilde{C}_g$,*

$$\deg_C K_{\tilde{C}_g/\tilde{M}_g} \geq 0.$$

Proof. In fact, Arakelov proved that for all normal surfaces F fibred in stable curves over a smooth curve C , $\omega_{F/C}$ is ample on F . This implies that for all curves $C \subset \tilde{C}_g$ such that $\pi(C) \cap M_g \neq \emptyset$,

$$\deg_C K_{\tilde{C}_g/\tilde{M}_g} > 0.$$

Now suppose $C \subset \tilde{C}_g$ and $\pi(C) \subset \overline{M}_g - M_g$.

Case 1: $\pi(C) = \text{one pt.}$ Then $\deg_C K > 0$ because ω is ample on all fibres of $\tilde{C}_g \rightarrow \tilde{M}_g$.

Case 2: $d\pi|_C \equiv 0$, i.e., C is in the locus $\text{Sing } C$ of double points of the fibres. But $\text{Sing } C$ has an étale double cover $\text{Sing}' C$ parametrizing pairs consisting of a double point of a fibre of π and a branch through this point. By residue

$$\omega_{\tilde{C}_g/\tilde{M}_g} \otimes \mathcal{O}_{\text{Sing}' C} \cong \mathcal{O}_{\text{Sing}' C}$$

so $\deg_C K = 0$.

Case 3: Other. After a suitable case change

$$C' \rightarrow \pi(C) \subset \tilde{M}_g$$

we can assume that the pull-back family $\tilde{C}_g \times_{\tilde{M}_g} C'$ is obtained by gluing several generically smooth stable families $Y_\alpha \rightarrow C'$ along a set of sections $t_{\alpha\beta}: C' \rightarrow Y_\alpha$. Lying over C there will be a curve C'' contained in one of the Y_α 's, say Y_{α_0} , mapping onto C' and not equal to $t_{\alpha_0\beta}(C')$, any β . The pull-back of $\omega_{\tilde{C}_g/\tilde{M}_g}$ to Y_α will be equal to $\omega_{Y_\alpha/C'}(\sum_\beta t_{\alpha_0,\beta}(C''))$ and, by Arakelov, this will have non-negative degree on C' if genus $C' \geq 2$. If genus $C' = 0$ or 1, it is easy to check that this is still the case. Q.E.D.

Corollary 4.2. *The classes κ_l are numerically effective, i.e., for all subvarieties $W \subset \tilde{M}_g$ of dimension l ,*

$$(W.\kappa_l) \geq 0.$$

Proof. $K_{\tilde{C}/\tilde{M}}$ numerically effective implies $K_{\tilde{C}/\tilde{M}}^{l+1}$ numerically effective (see [K1]), hence $\pi_*(K_{\tilde{C}/\tilde{M}}^{l+1})$ is numerically effective.

In fact, κ_1 is ample, see [M], §5.

§5. Tautological Relations via Grothendieck-Riemann-Roch

Grothendieck's Riemann-Roch theorem (G-R-R) is, in many cases, tailor-made to find relations among tautological classes. For example, see Atiyah-Bott [A-B], §9. We can compute the classes λ_k in terms of the classes κ_k . To do this, we apply the G-R-R to the morphism

$$\pi: \tilde{C}_g \rightarrow \tilde{M}_g.$$

This gives us

$$ch \pi_! \omega_{\tilde{C}/\tilde{M}} = \pi_*(ch \omega_{\tilde{C}/\tilde{M}} Td^V(\Omega_{\tilde{C}/\tilde{M}}^1)).$$

Here we use the notation $Td^V(\mathcal{E})$ to write the universal multiplicative polynomial in the Chern classes of \mathcal{E} such that for line bundles L ,

$$\begin{aligned} Td^V(L) &= \frac{\lambda}{e^\lambda - 1}, \quad \lambda = c_1(L) \\ &= 1 - \frac{1}{2}\lambda - \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} \lambda^{2k}, \end{aligned}$$

(i.e., the usual $Td(L)$ is $\lambda/1 - e^{-\lambda}$ or $1 + \frac{1}{2}\lambda + \dots$). Since $R^1\pi_*\omega_{\bar{C}/\bar{M}} \cong \mathcal{O}_{\bar{M}}$, this means:

$$ch E = 1 + \pi_*(e^K Td^\vee(\Omega_{\bar{C}/\bar{M}}^1)).$$

Now use the exact sequence:

$$0 \rightarrow \Omega_{\bar{C}/\bar{M}}^1 \rightarrow \omega_{\bar{C}/\bar{M}} \rightarrow \omega_{\bar{C}/\bar{M}} \otimes \mathcal{O}_{\text{Sing } \bar{C}} \rightarrow 0$$

(compare [M], pf. of 5.10). Let $\text{Sing}' \bar{C}$ be the double cover of $\text{Sing } \bar{C}$ consisting of singular points plus branches: as a \mathbb{Q} -variety, it is an étale double cover, i.e., the map between the charts

$$(\text{Sing}' \bar{C})_\alpha \rightarrow (\text{Sing } \bar{C})_\alpha$$

which are local universal deformation spaces, is étale. Then via residue

$$\omega_{\bar{C}/\bar{M}} \otimes \mathcal{O}_{\text{Sing}' \bar{C}} \cong \mathcal{O}_{\text{Sing}' \bar{C}}.$$

Therefore:

$$\begin{aligned} ch E &= 1 + \pi_*(e^K \cdot Td^\vee(\omega_{\bar{C}/\bar{M}}) \cdot Td^\vee(\mathcal{O}_{\text{Sing } \bar{C}})^{-1}) \\ &= 1 + \pi_*(e^K \cdot \frac{K}{e^K - 1} + [Td^\vee(\mathcal{O}_{\text{Sing } \bar{C}})^{-1} - 1]) \end{aligned}$$

since K intersects any cycle on $\text{Sing } \bar{C}$ in zero. Now use the lemma:

Lemma 5.1. *There is a universal power series P such that for all $i: Z \rightarrow X$, an inclusion of a smooth codimension two subvariety in a smooth variety,*

$$(Td^\vee \mathcal{O}_Z)^{-1} - 1 = i_*[P(c_1N, c_2N)]$$

where N is the normal bundle I_Z/I_Z^2 .

Proof. In fact

$$(Td^\vee \mathcal{O}_Z)^{-1} = 1 + (\text{polyn. in } ch_k(\mathcal{O}_Z), \quad k \geq 1)$$

and by G-R-R for i , $ch_k(\mathcal{O}_Z)$ is i_* of a polynomial in c_1N, c_2N .

To compute this polynomial P , say $Z = D_1 \cdot D_2$. Then use

$$0 \rightarrow \mathcal{O}_X(-D_1 - D_2) \rightarrow \mathcal{O}_X(-D_1) \oplus \mathcal{O}_X(-D_2) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0.$$

This gives us

$$\begin{aligned} Td^V \mathcal{O}_Z &= (Td^V \mathcal{O}_X(-D_1))^{-1} \cdot (Td^V \mathcal{O}_X(-D_2))^{-1} \cdot Td^V \mathcal{O}_X(-D_1 - D_2) \\ &= \left(\frac{-D_1}{e^{-D_1} - 1} \right)^{-1} \cdot \left(\frac{-D_2}{e^{-D_2} - 1} \right)^{-1} \cdot \left(\frac{-D_1 - D_2}{e^{-D_1 - D_2} - 1} \right). \end{aligned}$$

Thus

$$\begin{aligned} D_1 D_2 \cdot P(D_1 + D_2, D_1 \cdot D_2) &= Td^V(\mathcal{O}_Z)^{-1} - 1 \\ &= \frac{D_1}{1 - e^{-D_1}} \cdot \frac{D_2}{1 - e^{-D_2}} \cdot \frac{1 - e^{-D_1 - D_2}}{D_1 + D_2} - 1 \\ &= \frac{1}{D_1 + D_2} \cdot \left[D_1 \cdot \left(\frac{D_2}{1 - e^{-D_2}} - 1 \right) + \right. \\ &\quad \left. D_2 \cdot \left(\frac{D_1}{1 - e^{-D_1}} - 1 \right) - D_1 \cdot D_2 \right] \\ &= \frac{D_1 D_2}{D_1 + D_2} \cdot \sum_{k=1}^{\infty} \frac{(-1)^{k-1} B_k}{(2k)!} (D_1^{2k-1} + D_2^{2k-1}) \end{aligned}$$

So

$$\begin{aligned} P(D_1 + D_2, D_1 \cdot D_2) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} B_k}{(2k)!} \left(\frac{D_1^{2k-1} + D_2^{2k-1}}{D_1 + D_2} \right) \\ &= \frac{1}{12} - \frac{1}{720} ((D_1 + D_2)^2 - 3D_1 D_2) + \\ &\quad \frac{1}{30,240} ((D_1 + D_2)^4 - 5D_1 D_2 (D_1 + D_2)^2 + 5D_1^2 D_2^2) + \dots \end{aligned}$$

Therefore

$$ch \mathbb{E} = 1 + \pi_* \left(\frac{K}{1 - e^{-K}} \right) + (\pi \circ i)_* P(c_1 N, c_2 N).$$

Now $Sing \bar{C}$ breaks up into pieces depending on whether the double point disconnects the fibre in which it lies or not, and if it does, what the genera are of the two pieces. Thus:

$$Sing \bar{C} = \coprod_{0 \leq h \leq [g/2]} \Delta_h^*$$

where Δ_0^* are the non-disconnecting double points and if $h \geq 1$, Δ_h^* are the points for which one piece has genus h . Moreover, looking at the two pieces, one sees that

$$\Delta_h^* \cong \bar{C}_h \times \bar{C}_{g-h} \quad \text{if } 1 < h < g/2$$

while

$$\begin{aligned} \Delta_{g/2}^* &\cong \bar{C}_{g/2} \times \bar{C}_{g/2}/(\mathbb{Z}/2\mathbb{Z}) \quad \text{if } g \text{ is even} \\ \Delta_0^* &\cong \bar{M}_{g-1,2}/(\mathbb{Z}/2\mathbb{Z}) \end{aligned}$$

where $\bar{M}_{g-1,2}$ is the space of stable curves with two ordered points P_1, P_2 and $\mathbb{Z}/2\mathbb{Z}$ permutes either the two factors or the two points. In fact, specifying a branch too, we get:

$$\begin{aligned} \text{Sing}'\bar{C} &= \coprod_{0 \leq h \leq [g/2]} \Delta'_h \\ \Delta'_h &\cong 2 \text{ copies of } \bar{C}_h \times \bar{C}_{g-h} \quad 1 \leq h \leq g/2 \\ &\cong \bar{C}_{g/2} \times \bar{C}_{g/2} \quad \text{if } h = g/2 \\ &\cong \bar{M}_{g-1,2} \quad \text{if } h = 0. \end{aligned}$$

Let K_1, K_2 be the divisor classes defined

- a) on $\bar{C}_h \times \bar{C}_{g-h}$ by $K_1 = p_1^*K_{\bar{C}_h/\bar{M}_h}$, $K_2 = p_2^*K_{\bar{C}_{g-h}/\bar{M}_{g-h}}$
- b) on $\bar{M}_{g-1,2}$ by $K_i =$ conormal bundle at the i^{th} point.

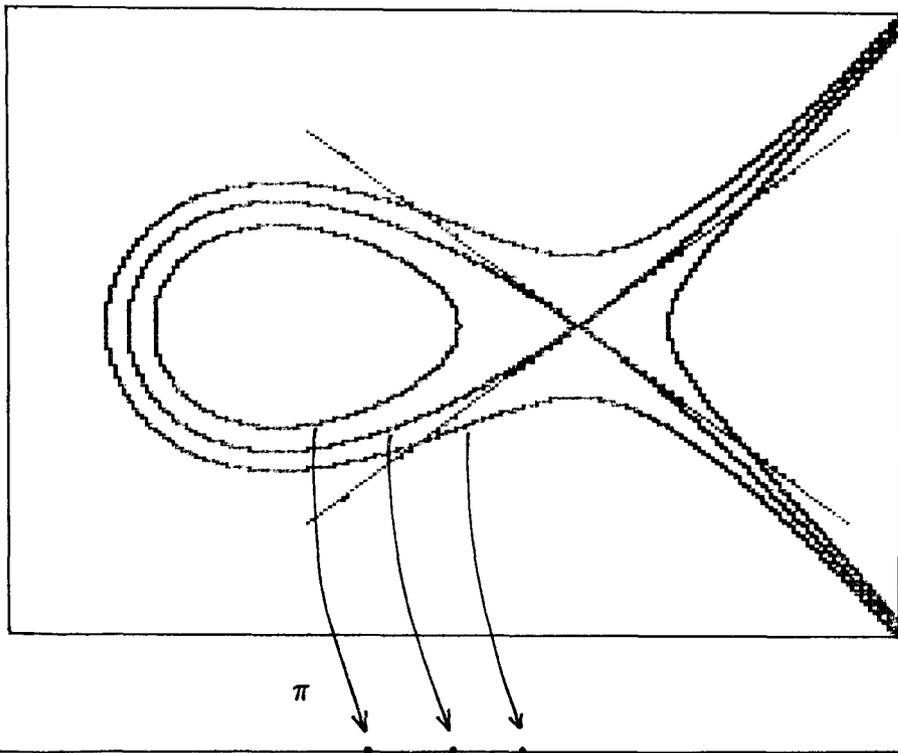
Writing out chE finally we get

$$(5.2) \quad chE = g + \sum_{l=1}^{\infty} \frac{(-1)^{l+1} \cdot B_l}{(2l)!} \cdot \left[\kappa_{2l+1} + \frac{1}{2} \sum_{h=0}^{g-1} i_{h,*} (K_1^{2l-2} - K_1^{2l-3} \cdot K_2 + \dots + K_2^{2l-2}) \right].$$

Here we have expanded $K/1 - e^{-K}$ and used the fact that π_*K is $(2g-2)$ times the fundamental class of \bar{M}_g . The morphism i_h is

$$\begin{aligned} i_0: \bar{M}_{g-1,2} &\rightarrow \text{Sing } \bar{C} \rightarrow \bar{M}_g \\ i_h: \bar{C}_h \times \bar{C}_{g-h} &\rightarrow \text{Sing } \bar{C} \rightarrow \bar{M}_g, \quad 1 \leq h \leq g-1. \end{aligned}$$

Note that i_0 and $i_{g/2}$ have degree 2 and the other i_h 's are repeated twice in the sum: hence the factor $1/2$. Moreover, we have evaluated the normal bundle to $\text{Sing } \bar{\mathcal{C}}$ in $\bar{\mathcal{C}}_g$ as the direct sum of the tangent bundle to the two branches of the curve at the singular point:



(In a transversal to $\text{Sing } \bar{\mathcal{C}}$, $\bar{\mathcal{C}}_g/\bar{\mathcal{M}}_g$ looks like $xy = t$, and the tangent bundle to the x, y -surface at $(0, 0)$ is the sum of the tangent line to the branch $x = 0$ and to the branch $y = 0$.)

The formula (5.2) specializes in codimension 1 to the formula of [M], p. 102:

$$\lambda_1 = c_1(\mathbb{E}) = \frac{1}{12}(\kappa_1 + \delta)$$

where

$$\begin{aligned} \delta &= \frac{1}{2} \sum_{h=0}^{g-1} i_{h,*}(1) \\ &= \text{fundamental class of } \overline{\mathcal{M}}_g - \mathcal{M}_g. \end{aligned}$$

Moreover, it proves

Corollary (5.3). *For all even integers $2k$,*

$$(ch \mathbb{E})_{2k} = 0.$$

This formula can be proven in *cohomology* from the Gauss-Manin connection. We sketch this proof. First look at the smooth curves $\mathcal{C}_g/\mathcal{M}_g$. For these we have the DeRham complex

$$\Omega_{\mathcal{C}_g/\mathcal{M}_g}: \mathbf{0} \rightarrow \mathcal{O}_{\mathcal{C}_g} \xrightarrow{d} \Omega^1_{\mathcal{C}_g/\mathcal{M}_g} \rightarrow \mathbf{0}$$

along the fibres of π . This gives:

$$\mathbf{0} \rightarrow \pi_* \Omega^1_{\mathcal{C}_g/\mathcal{M}_g} \rightarrow \mathbb{R}^1 \pi_* \Omega^1_{\mathcal{C}_g/\mathcal{M}_g} \rightarrow R^1 \pi_* \mathcal{O}_{\mathcal{C}_g} \rightarrow \mathbf{0}.$$

By Serre duality, this gives:

$$\mathbf{0} \rightarrow \mathbb{E} \rightarrow \mathbb{R}^1 \pi_* \Omega^1_{\mathcal{C}_g/\mathcal{M}_g} \rightarrow \mathbb{E}^\vee \rightarrow \mathbf{0}.$$

The vector bundle in the middle has rank $2g$, is isomorphic to $R^1 \pi_* \mathbb{C}$ and possesses the Gauss-Manin connection. Therefore its Chern classes are zero and over \mathcal{M}_g :

$$(5.4) \quad c(\mathbb{E}) \cdot c(\mathbb{E}^\vee) = 1.$$

This identity can be extended to $\overline{\mathcal{M}}_g$ if we use the complex

$$\omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}: \mathbf{0} \rightarrow \mathcal{O}_{\overline{\mathcal{C}}_g} \xrightarrow{d} \omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g} \rightarrow \mathbf{0}$$

from which we get the sequence:

$$(5.5) \quad 0 \rightarrow E \rightarrow \mathbb{R}^1 \pi_* \omega_{\tilde{C}_g/\overline{M}_g} \rightarrow E^\vee \rightarrow 0.$$

Although the Gauss-Manin connection does not extend regularly to $\mathbb{R}^1 \pi_* \omega_{\tilde{C}_g/\overline{M}_g}$, it has regular singularities with a polar part which is nilpotent. This is enough to conclude that its Chern classes zero, extending (5.5) to \overline{M}_g . This means equivalently that

$$ch(E) + ch(E^\vee) = 0$$

or

$$ch(E)_{2k} = 0, \quad k \geq 1.$$

This identity in fact holds on \tilde{A}_g , the toroidal compactification of A_g . It can be deduced, for instance, from the extension of Hirzebruch's proportionality theorem to \tilde{A}_g (see [M2]).

The conclusion to be drawn from (5.2) and (5.3) is that the even λ_k 's are polynomials in the odd ones, and that all the λ_k 's are polynomials in the κ_k 's and in boundary cycles. Moreover, applying (5.2) in odd degree above g , we can express κ_k for k odd, $k > g$, in terms of lower κ_l 's and boundary cycles. We shall strengthen this in the next section, where we find a simpler way to get identities on the κ_k 's.

The exact sequence (5.4) is remarkable in another way that reveals something of the nature of \overline{M}_g . Note that E tends to be a positive bundle: at least $c_1(E)$ is the pull-back of an ample line bundle by a birational map. But it is also a sub-bundle of a bundle with connection, i.e., the DeRham bundle $\mathbb{R}^1 \pi_* \omega^\cdot$ is unstable yet has a connection.

§6. Tautological Relations via the Canonical Linear System

There is another very different way to get relations on the λ_i and κ_i . For this, we will not try to get the full relations in $A^*(\overline{M}_g)$ as the boundary terms seem to be a bit involved, but instead get the relations in $A^*(M_g)$. Because of the exact sequence:

$$A^*(\overline{M}_g - M_g) \rightarrow A^*(\overline{M}_g) \rightarrow A^*(M_g) \rightarrow 0$$

this is the same as a relation in $A^*(\overline{M}_g)$ with an undetermined boundary term.

The method is based on the fact that for all smooth curves C , the sheaf ω_C is generated by its global sections.⁵

Now if we let \tilde{C}_g/\tilde{M}_g temporarily stand for the family of *smooth* stable curves, i.e., replace \tilde{C}_g by $\pi^{-1}(M_g)$, then we have an exact sequence:

$$0 \rightarrow \mathcal{F} \rightarrow \pi^* \pi_* \omega_{\tilde{C}_g/\tilde{M}_g} \rightarrow \omega_{\tilde{C}_g/\tilde{M}_g} \rightarrow 0$$

where all these sheaves are Q -sheaves and \mathcal{F} is locally free of rank $g - 1$. Taking Chern classes, we get:

$$c(\mathcal{F}) = \pi^*(1 + \lambda_1 + \dots + \lambda_g) \cdot (1 + K_{\tilde{C}_g/\tilde{M}_g})^{-1}.$$

Using the fact that $c_n(\mathcal{F}) = 0$ if $n \geq g$, this says:

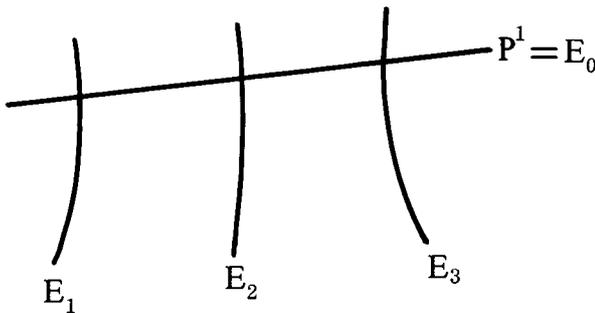
$$(K_{\tilde{C}/\tilde{M}}^n) - \pi^*(\lambda_1) \cdot (K_{\tilde{C}/\tilde{M}}^{n-1}) + \dots + (-1)^g \pi^*(\lambda_g) \cdot (K_{\tilde{C}/\tilde{M}}^{n-g}) = 0$$

for all $n \geq g$. Taking π_* , this means

$$\begin{aligned} \kappa_{n-1} - \lambda_1 \cdot \kappa_{n-2} + \dots + (-1)^g \lambda_g \cdot \kappa_{n-g-1} &= 0 && \text{if } n \geq g + 2 \\ \kappa_g - \lambda_1 \cdot \kappa_{g-1} + \dots + (-1)^g \lambda_g \cdot (2g - 2) &= 0 && \text{if } n = g + 1 \\ \kappa_{g-1} - \lambda_1 \cdot \kappa_{g-2} + \dots + (-1)^{g-1} \lambda_{g-1} \cdot (2g - 2) &= 0 && \text{if } n = g. \end{aligned}$$

⁵If C is a singular stable curve, then one can show that $\Gamma(\omega_C)$ generates the subsheaf of ω_C of sections which are zero

- i) at all double points P for which $C - P$ is disconnected,
- ii) on all components E_0 of C which are isomorphic to P^1 and such that all double



points P on E_0 are disconnecting double points.

Corollary 6.2. *For all g , all the classes λ_i, κ_i restricted to $A(M_g)$ are polynomials in $\kappa_1, \kappa_2, \dots, \kappa_{g-2}$.*

Proof. This is clear except for κ_g, κ_{g-1} . Here we must combine the above relations with (5.2). There are two cases depending on whether g is even or odd. Recalling that

$$ch_n E = \frac{(-1)^{n-1} \cdot c_n(E)}{(n-1)!} + \text{polyn. in lower Chern class}$$

it follows that if $g = 2k$ or $2k - 1$, then

$$\frac{(-1)^{k+1} \cdot B_k}{(2k)!} \kappa_{2k-1} = ch_{2k-1} E = \frac{\lambda_{2k-1}}{(2k-2)!} + (\text{polyn. in lower } \lambda\text{'s}).$$

If $g = 2k$, we want to show that the 2 equations

$$\begin{cases} (-1)^{k+1} B_k \cdot \kappa_{2k-1} = 2k \cdot (2k-1) \lambda_{2k-1} + \text{lower} \\ \kappa_{2k-1} = 2 \cdot (2k-1) \lambda_{2k-1} + \text{lower} \end{cases}$$

have independent leading terms, and if $g = 2k - 1$, then we want to do the same with

$$\begin{cases} (-1)^{k+1} B_k \cdot \kappa_{2k-1} = 2k \cdot (2k-1) \lambda_{2k-1} + \text{lower} \\ \kappa_{2k-1} = 4(k-1) \cdot \lambda_{2k-1} + \text{lower.} \end{cases}$$

This follows, however, by inspection if $k \leq 10$ and for larger k by the estimate:

$$B_k = \frac{2 \cdot (2k)!}{(2\pi)^{2k}} \zeta(2k) > \frac{2 \cdot (2k/e)^{2k}}{(2\pi)^{2k}} = 2 \cdot \left(\frac{k}{e \cdot \pi}\right)^{2k} > 2 \cdot k \quad \text{if } k \geq 11.$$

Q.E.D.

With this approach, the first relation between $\kappa_1, \dots, \kappa_{g-2}$ that we get occurs in codimension $g + 1$ or $g + 2$. One should, however, get the relation $\kappa_1^2 = 0$ in $A(M_3)$ so we clearly do not have all the relations on the κ_i 's and λ_i 's yet. It does seem reasonable to conjecture, however, that $\kappa_1, \dots, \kappa_{g-2}$ have no relations up to something like codimension g , e.g., $g - (\text{small constant})$.

§7. The Tautological Classes via Arbarello's Flag of Subvarieties of M_g

We want to consider the following subsets of C_g and M_g :

$$\begin{aligned}
 W_l^* &= \{C, x \in C_g \mid h^0(C_C(l \cdot x)) \geq 2\} \\
 &= \left\{ C, x \in C_g \mid \exists \text{ a morphism } \pi: c \rightarrow \mathbb{P}^1 \text{ of degree } \right. \\
 &\quad \left. d \leq l \text{ with } \pi^{-1}(\infty) = d \cdot x \right\} \\
 W_l &= \pi(W_l^*) \subset M_g
 \end{aligned}$$

where $2 \leq l \leq g$. Thus W_g^* = locus of Weierstrass points in C_g , W_{g-1} = curves with an exceptional Weierstrass of one of the two simplest types, and W_2 = hyperelliptic curves. Note that:

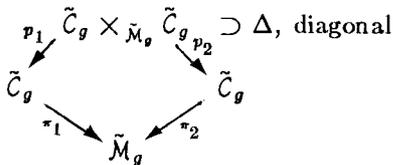
$$\begin{aligned}
 C_g \supset W_g^* \supset W_{g-1}^* \supset \dots \supset W_2^* \\
 M_g = W_g \supset W_{g-1} \supset \dots \supset W_2.
 \end{aligned}$$

I first heard of this flag from E. Arbarello who proposed (see [Arb]⁶) that they might be used as a ladder to climb from the reasonably well-known space W_2 to the still mysterious M_g .

Let me first recall and sketch the proof of the following well-known facts:

Proposition 7.1. *W_l^* is irreducible of codimension $g - l + 1$ and $W_l^* - W_{l-1}^*$ is an open dense subset smooth in the local charts for C_g , i.e., the local deformation space for the pairs (C, x) .*

Sketch of proof. Firstly, W_l^* is a determinantal subvariety of \tilde{C}_g . In fact, consider



Let

$$\mathcal{F}_l = R^1 p_{2,*}(\mathcal{O}_{\tilde{C} \times \tilde{C}}(l\Delta)).$$

Then over a point $[C, x] \in \tilde{C}_g$,

$$\mathcal{F}_l \otimes k([C, x]) \cong H^1(C, \mathcal{O}_C(l \cdot x)).$$

⁶Unfortunately, the proof of Theorem 3.27 in [Arb] is incomplete as it stands.

Now if $[C, x] \notin W_l^*$, $h^0(\mathcal{O}_C(lx)) = 1$ and $h^1(\mathcal{O}_C(lx)) = g - l$, while if $[C, x] \in W_l^*$, both numbers are bigger. Thus \mathcal{F}_l is locally free of rank $g - l$ on $\tilde{C}_g - W_l^*$ and not locally free anywhere on W_l^* . But look at the sequence:

$$0 \rightarrow \mathcal{O}_{\tilde{C} \times \tilde{C}} \rightarrow \mathcal{O}_{\tilde{C} \times \tilde{C}}(l\Delta) \rightarrow \mathcal{O}_{\tilde{C} \times \tilde{C}}(l\Delta) / \mathcal{O}_{\tilde{C} \times \tilde{C}} \rightarrow 0$$

which gives us:

$$(7.2) \quad 0 \rightarrow p_{2,*}(\mathcal{O}_{\tilde{C} \times \tilde{C}}(l\Delta) / \mathcal{O}_{\tilde{C} \times \tilde{C}}) \xrightarrow{\alpha} R^1 p_{2,*}(\mathcal{O}_{\tilde{C} \times \tilde{C}}) \rightarrow \mathcal{F}_l \rightarrow 0.$$

The first sheaf is locally free of rank l , the second locally free of rank g , hence

$$W_l^* = \{[C, x] \in \tilde{C}_g \mid rk_{[C,x]}(\alpha) < l\}.$$

Thus the codimension of W_l^* is at most $g - l + 1$. But describing $W_l^* - W_{l_1}^*$ as the set of l -fold covers of \mathbb{P}^1 , totally ramified at ∞ , one gets the *upper bound* $2g + l_1 - 3$ on $\dim W_{l_1}^* - W_{l_1-1}^*$ for all l_1 , hence the same upper bound on $\dim W_l^*$. Comparing the two, it follows that $\text{codim } W_l^*$ is *exactly* $g - l + 1$ and W_l^* is determinantal as well as that $W_l^* - W_{l_1}^*$ is dense in W_l^* . The irreducibility of $W_l^* - W_{l_1}^*$ is a classical result of Lüroth, describing all l -fold covers of \mathbb{P}^1 as branched covers with a standard set of transpositions.

The smoothness of $W_l^* - W_{l_1}^*$ in the universal deformation space may be checked by the following calculation: let f have an l -fold pole at $x \in C$ and make an infinitesimal deformation \tilde{C} of C over $\mathbb{C}[\epsilon]$ by glueing open sets $U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]$ via a 1-cocycle $D_{\alpha\beta}$ of derivations zero at x . Then f lifts to a rational function on \tilde{C} with l -fold pole at x if there are functions g_α with l -fold poles at x and:

$$(1 + \epsilon D_{\alpha\beta})(f + \epsilon g_\alpha) = f + \epsilon g_\beta.$$

This means that $\{D_{\alpha\beta}f\} \in H^1(C, \mathcal{O}(lx))$ is zero. But $D_{\alpha\beta}f = \langle D_{\alpha\beta}, df \rangle$ is the image:

$$(7.3) \quad \{D_{\alpha\beta}\} \in H^1(C, T_C(-x)) \xrightarrow{(\cdot, df)} H^1(C, \mathcal{O}_C(lx)).$$

Note that $H^1(C, T_C(-x))$ is the tangent space to the universal deformation space of (C, x) . Moreover (7.3) is dual to the injective map

$$H^0(C, \mathcal{O}(2K_C + x)) \xleftarrow{\otimes df} H^0(C, \mathcal{O}(K_C - lx))$$

hence (7.3) is *surjective*, i.e., the subscheme of the universal deformation space where f lifts is smooth of codimension $h^1(\mathcal{O}_C(lx)) = g - l$.

In order to work out the fundamental class of W_l^* , it is convenient to split up (7.2) into pieces as follows. Starting with

$$0 \rightarrow \mathcal{O}_{\tilde{C} \times \tilde{C}}((l-1)\Delta) \rightarrow \mathcal{O}_{\tilde{C} \times \tilde{C}}(l\Delta) \rightarrow \mathcal{O}_{\tilde{C} \times \tilde{C}}(l\Delta) / \mathcal{O}_{\tilde{C} \times \tilde{C}}((l-1)\Delta) \rightarrow 0$$

$$\parallel$$

$$\mathcal{O}_\Delta \otimes p_2^* \mathcal{O}_{\tilde{C}}(-lK_{\tilde{C}/\tilde{M}})$$

we get via $R^*p_{2,*}$:

$$(7.4) \quad 0 \rightarrow \mathcal{O}_{\tilde{C}}(-lK_{\tilde{C}/\tilde{M}}) \xrightarrow{\beta} \mathcal{F}_{l-1} \rightarrow \mathcal{F}_l \rightarrow 0.$$

It follows that on $\tilde{C}_g - W_{l-1}^*$ where \mathcal{F}_{l-1} is locally free:

$$W_l^* = \{[C, x] \in \tilde{C}_g \mid \beta_{[C, x]} = 0\}$$

$$= \text{zeroes of the section } \beta' \in \Gamma(\tilde{C}_g, \mathcal{F}_{l-1}(lK_{\tilde{C}/\tilde{M}})).$$

Moreover, on the universal deformation space of $[C, x]$, this section β' vanishes to 1^{st} order along W_l^* : in fact, the differential of β' at a point of W_l^* is a map

$$T_{[C, x], C_\alpha} \longrightarrow \mathcal{F}_{l-1}(lk) \otimes \mathbf{k}([C, x])$$

$$\parallel \qquad \qquad \qquad \parallel$$

$$H^1(C, T_C(-x)) \quad H^1(C, \mathcal{O}_C(lx)) \otimes (\mathfrak{m}_x / \mathfrak{m}_x^2)^{\otimes l}$$

which is readily seen to be the surjective map (7.3) (the factor $(\mathcal{M}_x / \mathcal{M}_x^2)^l$ is hidden in (7.3) in the choice of f). Thus

$$(7.5) \quad [W_l^*]_{\mathcal{Q}} = c_{g-l+1}(\mathcal{F}_{l-1}(lK_{\tilde{C}/\tilde{M}})), \quad \text{on } \tilde{C}_g - W_{l-1}^*.$$

But W_{l-1}^* has codimension $g - l + 2$ so

$$A^{g-l+1}(C_g) \cong A^{g-l+1}(C_g - W_{l-1}^*).$$

Thus (7.5) holds as an equation in $A^{g-l+1}(C_g)$, hence in $opA^{g-l+1}(\tilde{C}_g)$. Now let's calculate the fundamental class of W_l^* :

$$[W_l^*]_{\mathcal{Q}} = c_{g-l+1}(\mathcal{F}_{l-1}(lK))$$

$$= c_{g-l+1}(\mathcal{F}_l(lK))$$

$$= c_{g-l+1}(\mathcal{F}_l).$$

Here we have abbreviated $K_{\hat{C}/\hat{M}}$ to K , and the last equality follows from the general fact

$$c_n(\mathcal{G}(D)) = c_n(\mathcal{G}) + (r - n + 1)D \cdot c_{n-1}(\mathcal{G}) + \binom{r - n + 2}{2} D^2 c_{n-2}(\mathcal{G}) + \dots + \binom{r}{n} \cdot D^n$$

($r =$ generic rank \mathcal{G}), whence

$$c_{r+1}(\mathcal{G}(D)) = c_{r+1}(\mathcal{G}), \text{ all divisors } D.$$

But now

$$\begin{aligned} c(\mathcal{F}_l) &= c(\mathcal{F}_{l-1}) \cdot (1 - lK)^{-1} \\ &= c(\mathcal{F}_{l-2}) \cdot (1 - (l-1)K)^{-1} (1 - lK)^{-1} \\ &\dots \\ &= c(\mathcal{F}_0) \cdot (1 - K)^{-1} \cdot (1 - 2K)^{-1} \cdot \dots \cdot (l - lK)^{-1} \\ &= \pi_2^*(c(R^1\pi_1, *O_{\hat{C}/\hat{M}})) \cdot (1 - K)^{-1} \cdot \dots \cdot (1 - lK)^{-1} \\ &= \pi_2^*(1 - \lambda_1 + \lambda_2 - \dots + (-1)^g \lambda_g) \cdot (1 - K)^{-1} \cdot \dots \cdot (1 - lK)^{-1}. \end{aligned}$$

Thus

$$(7.6) \quad [W_l^*]_{\mathbb{Q}} = (g - l + 1)^{st} \text{ component of } \pi^*(1 - \lambda_1 + \lambda_2 - \dots + (-1)^g \lambda_g) \cdot (1 - K)^{-1} \cdot \dots \cdot (1 - lK)^{-1}.$$

If we define W_l as a cycle as $\pi_*(W_l^*)$, we get also

$$(7.7) \quad [W_l]_{\mathbb{Q}} = (g - l)^{th} \text{-component of } (1 - \lambda_1 + \lambda_2 - \dots + (-1)^g \lambda_g) \cdot \pi_*[(1 - K)^{-1} \cdot \dots \cdot (1 - lK)^{-1}].$$

This shows that $[W_l]_{\mathbb{Q}}$ is a polynomial in the tautological classes κ_j . Presumably the coefficient of κ_l is always non-zero and hence we can solve for the κ_i 's in terms of the classes $[W_j]$, but this looks like a messy calculation.

Let's work out the hyperelliptic locus \mathcal{X} as an example. Note that

$$W_2^* \rightarrow W_2$$

is a covering of degree $2g + 2$ because for all hyperelliptic curves C , there are exactly $2g + 2$ points x such that $h^0(O_C(2x)) \geq 2$ — namely the Weierstrass

points. Thus

$$\begin{aligned}
 [\mathcal{X}]_{\mathcal{Q}} &= \frac{1}{2g+2} [W_2]_{\mathcal{Q}} \\
 &= \frac{1}{2g+2} \left\{ (g-2)^{nd} \text{ component of} \right. \\
 &\quad \left. \left((1-\lambda_1 + \dots + (-1)^g \lambda_g) \cdot \pi_* \left((1-K)^{-1} \cdot (1-2K)^{-1} \right) \right) \right\} \\
 &= \frac{1}{2g+2} \left\{ (2^g - 1) \kappa_{g-2} - (2^{g-1} - 1) \lambda_1 \cdot \kappa_{g-3} + \dots + \right. \\
 &\quad \left. (-1)^{g-3} \cdot 7 \cdot \lambda_{g-3} \cdot \kappa_3 + (-1)^{g-2} \cdot (6g-6) \lambda_{g-2} \right\}.
 \end{aligned}$$

Finally every hyperelliptic curve has an automorphism of order 2, so

$$\begin{aligned}
 [\mathcal{X}] &= 2 \cdot [\mathcal{X}]_{\mathcal{Q}} \\
 &= \frac{1}{g+1} \left\{ (2^g - 1) \kappa_{g-2} - \dots + (-1)^{g-2} (6g-2) \lambda_{g-2} \right\}.
 \end{aligned}$$

Part III: The Case $g = 2$

§8. Tautological Relations in Genus 2

First of all, let's specialize the calculations of Part II to the case $g = 2$ and see what we have. From \mathbb{E} , we get 2 elements

$$\begin{aligned}
 \lambda_1 &\in A^1(\overline{\mathcal{M}}_2) \\
 \lambda_2 &\in A^2(\overline{\mathcal{M}}_2)
 \end{aligned}$$

and because $ch_2(\mathbb{E}) = 0$, we get

$$\lambda_2 = \lambda_1^2/2.$$

From K on $\overline{\mathcal{C}}_2$, we get

$$\kappa_i \in A^i(\overline{\mathcal{M}}_2), \quad i = 1, 2, 3.$$

The calculations of §5 give us the relation:

$$(8.1) \quad \lambda_1 = \frac{1}{12} (\kappa_1 + \delta).$$

Here $\overline{M}_2 - M_2$ has 2 components Δ_0 and Δ_1 , Δ_0 the closure of the locus of irreducible singular curves, Δ_1 the locus of singular curves $C_1 \cup C_2$, $C_1 \cap C_2 =$ one pt., $p_a(C_1) = p_a(C_2) = 1$. By definition

$$\delta = [\Delta_0]_{\mathcal{Q}} + [\Delta_1]_{\mathcal{Q}}.$$

We shall write δ_0 for $[\Delta_0]_{\mathcal{Q}}$ and δ_1 for $[\Delta_1]_{\mathcal{Q}}$. We don't need (5.2) in codimension 3 but it provides an interesting check on the calculations later. It gives:

$$\frac{1}{6}\lambda_1^3 - \frac{1}{2}\lambda_1\lambda_2 = ch_3E = -\frac{1}{720}\left[\kappa_3 + \frac{1}{2} \cdot \sum_{h=0}^1 i_{h,*}((K_1 + K_2)^2 - 3K_1 \cdot K_2)\right]$$

or

$$60\lambda_1^3 = \kappa_3 + \frac{1}{2} \sum_{h=0}^1 i_{h,*}(K_1^2 - K_1K_2 + K_2^2).$$

In §10b we shall work out these terms numerically and check this.

We can refine the calculations of §6 by working out the boundary term too. It is easy to see that if C is a stable curve of genus 2, ω_C is generated by its global sections, unless $C = C_1 \cup C_2$, $C_1 \cap C_2 = \{P\}$, in which case $\Gamma(\omega_C)$ generates $m_P \cdot \omega_C$. Therefore, working over the whole of \tilde{C}_2 we get:

$$0 \rightarrow \mathcal{F} \rightarrow \pi^* \pi_* \omega_{\tilde{C}_2/\tilde{M}_2} \rightarrow I_{\Delta_1^*} \cdot \omega_{\tilde{C}_2/\tilde{M}_2} \rightarrow 0$$

(following the notation of §5). $I_{\Delta_1^*}$ has two generators at every point, so its projective dimension is 1, i.e., \mathcal{F} is locally free, hence invertible. Now use:

$$0 \rightarrow I_{\Delta_1^*} \cdot \omega \rightarrow \omega \rightarrow \omega \otimes \mathcal{O}_{\Delta_1^*} \rightarrow 0$$

and the fact that via residue, ω is trivial on the double cover Δ_1' of Δ_1^* , hence ω^2 is trivial on Δ_1^* . It follows that

$$c(\mathcal{F}) = \pi^*(1 + \lambda_1 + \lambda_2) \cdot (1 + K_{\tilde{C}_2/\tilde{M}_2})^{-1} \cdot c(\mathcal{O}_{\Delta_1^*}).$$

A useful lemma that we can use here is:

Lemma 8.2. *If $Y \subset X$ is a local complete intersection of codimension 2, and $i: Y \rightarrow X$ is the inclusion, then*

$$c(\mathcal{O}_Y) = 1 - i_*(c(I_Y/I_Y^2)^{-1}).$$

Proof. If $Y = D_1.D_2$ globally, then this formula is easily checked. But by the G-R-R,

$$c(\mathcal{O}_Y) = 1 + i_* (\text{univ. polyn. in } c_1(I/I^2), c_2(I/I^2))$$

and the universal polynomial must be $c(I/I^2)^{-1}$ because the two are equal whenever $Y = D_1.D_2$.

As in §5, $\Delta'_1 \cong \overline{M}_{1,1} \times \overline{M}_{1,1}$, i.e., there is a 2-1 map:

$$i_1: \overline{M}_{1,1} \times \overline{M}_{1,1} \rightarrow \Delta'_1$$

and $i_1^*(I/I^2) = K_1 + K_2$. Thus

$$c(\mathcal{O}_{\Delta'_1}) = 1 - \frac{1}{2} \cdot i_{1,*}((1 - K_1 - K_2)^{-1}).$$

Thus

$$c(\mathcal{F}) = \pi^*(1 + \lambda_1 + \lambda_2) \cdot (1 - K + K^2 - K^3 + K^4) \cdot \left(1 - \frac{1}{2} \cdot i_{1,*}(1 + K_1 + K_2 + K_1 \cdot K_2)\right).$$

In particular,

$$0 = c_2(\mathcal{F}) = \pi^*\lambda_2 - K \cdot \pi^*\lambda_1 + K^2 - [\Delta'_1]_{\mathcal{Q}}.$$

Multiplying this by K and K^2 , we get even simpler formulae:

$$\begin{aligned} 0 &= K \cdot \pi^*\lambda_2 - K^2 \cdot \pi^*\lambda_1 + K^3 \\ 0 &= K^2 \cdot \pi^*\lambda_2 - K^3 \cdot \pi^*\lambda_1 + K^4. \end{aligned}$$

Taking π_* , this gives

$$(8.3) \quad \begin{aligned} \kappa_1 &= 2\lambda_1 + \delta_1 \\ \kappa_2 &= \kappa_1 \cdot \lambda_1 - 2\lambda_2 = \lambda_1 \cdot (\lambda_1 + \delta_1) \\ \kappa_3 &= \kappa_2 \lambda_1 - \kappa_1 \lambda_2 = \frac{1}{2} \lambda_1^2(\delta_1). \end{aligned}$$

Combining (8.1) and (8.3), we see that both κ_1 and λ_1 are expressible in terms of δ_0, δ_1 :

$$(8.4) \quad 10\lambda_1 = \delta_0 + 2\delta_1$$

$$(8.5) \quad 5\kappa_1 = \delta_0 + 7\delta_1.$$

As κ_1 is ample, this implies the well-known fact that M_2 is affine!

This relation (8.4) has a very simple analytic proof. Consider the modular form of weight 10 on Siegel's space \mathcal{H}_2 given by

$$f(Z) = \left[\prod_{a,b \text{ even}} \theta \begin{bmatrix} a \\ b \end{bmatrix} (0, Z) \right]^2$$

(Each θ has weight 1/2 and there are ten even a, b 's.) It vanishes on \mathcal{H}_2 precisely when

$$\gamma Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}, \quad \text{some } \gamma \in Sp(4, \mathbb{Z})$$

and then to order 2. At the principal cusp

$$\begin{pmatrix} i\infty & w \\ w & z \end{pmatrix}$$

it has the form

$$\begin{aligned} & (\text{unit}) \cdot \theta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^2 \cdot \theta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^2 \cdot \theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^2 \cdot \theta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^2 \\ & = (\text{unit}) \cdot (e^{\pi i(1/2)\Omega_{11}})^8 \\ & = \text{unit} \cdot e^{2\pi i\Omega_{11}} \end{aligned}$$

i.e., it vanishes to order 1. Thus f defines a section of $(\Lambda^2 \mathbb{E})^{\otimes 10}$ whose zeroes in \tilde{M}_2 are $2\tilde{\Delta}_1 + \tilde{\Delta}_0$. This reproves (8.4).

§9. Generators of $A(\overline{M}_2)$.

We use the exact sequence:

$$A(Y) \rightarrow A(X) \rightarrow A(X - Y) \rightarrow 0$$

($Y \subset X$ closed subvariety) to get generators of $A(\overline{M}_2)$. Recall that M_2 is known from Igusa's results [I] to be isomorphic to \mathbb{C}^3 modulo $\mathbb{Z}/5\mathbb{Z}$ acting by

$$(x, y, z) \mapsto (\zeta x, \zeta^2 x, \zeta^3 y).$$

Then

$$A(\mathbb{C}^3) \rightarrow A(M_2)$$

is surjective, hence $A_k(M_2) = (0)$, if $k < 3$. Thus

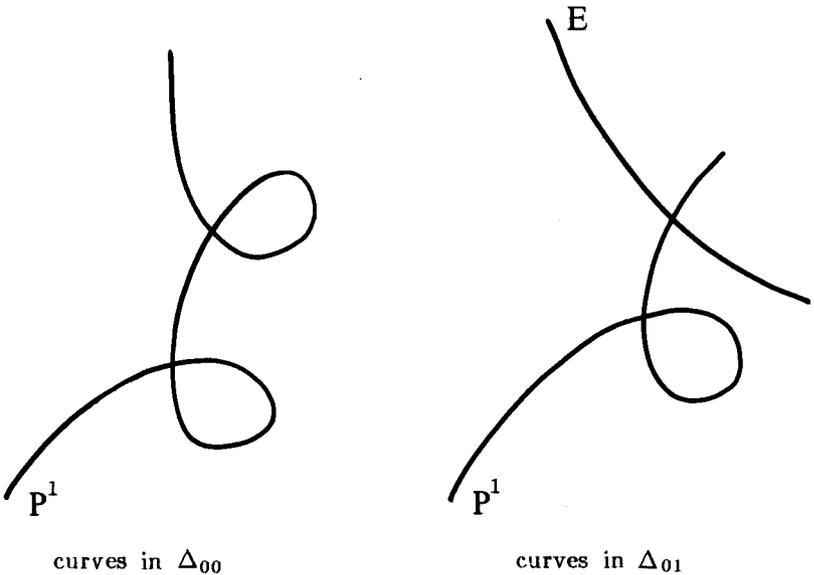
$$A_k(\Delta_0) \oplus A_k(\Delta_1) \rightarrow A_k(\overline{M}_2)$$

is surjective if $k < 3$. In particular, $A_2(\overline{M}_2)$ is generated by δ_0 and δ_1 .

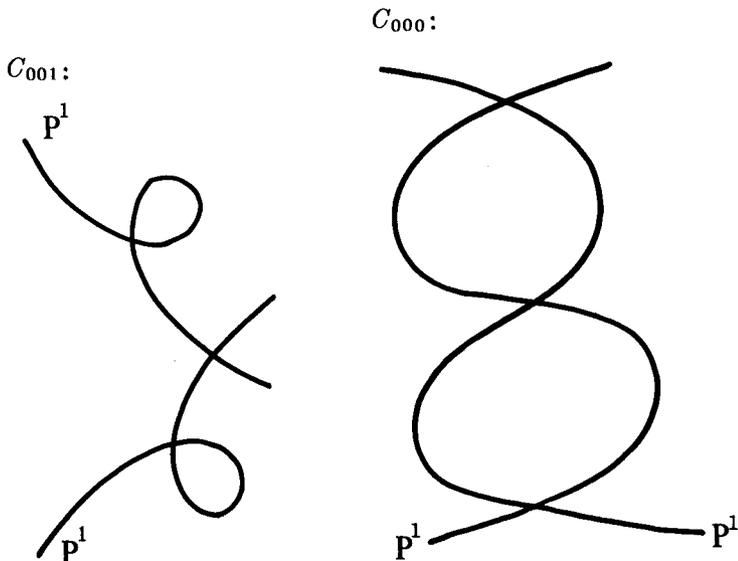
Define the dimension 1 subsets:

$$\Delta_{00} = \left\{ \begin{array}{l} \text{closure of curve in } \overline{M}_2 \text{ parametrizing} \\ \text{irreducible rational curves with 2 nodes} \end{array} \right\}$$

$$\begin{aligned} \Delta_{01} &= \Delta_0 \cap \Delta_1 \\ &= \text{Curve in } \overline{M}_2 \text{ parametrizing curves } C_0 \cup C_2, \\ &\quad \text{where } C_1 \cap C_2 = \{x\}, C_1 \text{ is elliptic or} \\ &\quad \text{rational with one node and} \\ &\quad C_2 \text{ is rational with one node} \end{aligned}$$



Note that Δ_{00} contains, besides the irreducible curves illustrated, the two reducible curves



$\text{Int } \Delta_1 = (\Delta_1 - \Delta_{01})$ is the locus of curves $C_1 \cup C_2$ where $C_1 \cap C_2 = \{x\}$, C_1, C_2 smooth elliptic. It is isomorphic to $\text{Symm}^2 M_{1,1}$, i.e., to the product of the affine j -line by itself mod the involution interchanging the factors. Therefore it is coordinatized by $j(C_1) + j(C_2), j(C_1).j(C_2)$:

$$\text{Int } \Delta_1 \cong \mathbb{C}^2.$$

Moreover, $\text{Int } \Delta_0 = \Delta_0 - (\Delta_{00} \cup \Delta_{01})$ is the locus of irreducible elliptic curves with one node, i.e., the space $M_{1,2}$ of triples $(E, x_1, x_2), x_1, x_2 \in E, x_1 \neq x_2$ mod the involution interchanging the 2 points. Write all elliptic curves as

$$y^2 = x(x-1)(x-\lambda), \quad \lambda \neq 0, 1$$

and take $x_1 = \text{pt. at } \infty, x_2 = (x, y)$. Interchanging x_1, x_2 carries (x, y) to $(x, -y)$. So we get a surjective map

$$\{(x, \lambda) \mid \lambda \neq 0, 1\} \rightarrow \text{Int } \Delta_0.$$

Putting this together

$$\begin{aligned} A_k(\Delta_1 - \Delta_{01}) &= (0), \quad k < 2 \\ A_k(\Delta_0 - (\Delta_{00} \cup \Delta_{01})) &= (0), \quad k < 2. \end{aligned}$$

Thus $A_1(\overline{\mathcal{M}}_2)$ is generated by:

$$\delta_{00} = [\Delta_{00}]_{\mathbb{Q}}$$

and

$$\delta_{01} = [\Delta_{01}]_{\mathbb{Q}}.$$

Finally, $\overline{\mathcal{M}}_2$ is unirational so all points of $\overline{\mathcal{M}}_2$ are rationally equivalent and we have proven:

Proposition 9.1. *$\overline{\mathcal{M}}_2$ is the disjoint union of 7 cells:*

$$\overline{\mathcal{M}}_2 = \mathcal{M}_2 \coprod \text{Int } \Delta_0 \coprod \text{Int } \Delta_1 \coprod \text{Int } \Delta_{00} \coprod \text{Int } \Delta_{01} \coprod \{C_{000}\} \coprod \{C_{001}\}.$$

Correspondingly, $A^*(\overline{\mathcal{M}}_2)$ is generated by

- a) 1 in codimension 0,
- b) δ_0, δ_1 in codimension 1,
- c) δ_{00}, δ_{01} in codimension 2,
- d) the class $[x]$ of a point in codimension 3: call this p .

Note that by the results of §8, λ_1 and κ_1 are also generators in codimension 1. We shall see that all the above cycles are independent. This will follow as a Corollary once we work out the multiplication table for these cycles.

§10. Multiplication in $A^*(\overline{\mathcal{M}}_2)$

We shall prove:

Theorem 10.1. *The ring $A^*(\overline{\mathcal{M}}_2)$ has a \mathbb{Q} -basis consisting of 1, δ_0 ,*

$\delta_1, \delta_{00}, \delta_{01}, p$ and multiplication table:

$$\delta_0^2 = \frac{5}{3}\delta_{00} - 2\delta_{01}$$

$$\delta_0 \cdot \delta_1 = \delta_{01}$$

$$\delta_1^2 = -\frac{1}{12}\delta_{01}$$

$$\delta_0 \cdot \delta_{00} = -\frac{1}{4}p$$

$$\delta_0 \cdot \delta_{01} = \frac{1}{4}p$$

$$\delta_1 \cdot \delta_{00} = -\frac{1}{8}p$$

$$\delta_1 \cdot \delta_{01} = -\frac{1}{48}p$$

An easier way to describe the ring structure is via λ_1 . Using the identities $10\lambda_1 = \delta_0 + 2\delta_1$, we can describe the multiplication by:

$$a) \delta_0 \cdot \delta_1 = \delta_{01}$$

$$b) \delta_{00} \cdot \delta_1 = \frac{1}{8}p$$

$$c) \delta_{00} \cdot \lambda_1 = 0$$

$$d) \delta_1 \cdot \lambda_1 = \frac{1}{12}\delta_{01}$$

$$e) \delta_0 \cdot \lambda_1 = \frac{1}{6}\delta_{00}.$$

The reader can check that these are equivalent to the relations of the Theorem.

Relations (a) and (b) are proper intersections of cycles and are proved by the explicit formula of §3: thus (a) follows because the lifts of δ_0, δ_1 to the universal deformation space of a curve $C \in \Delta_{01}$ are smooth divisors meeting transversely in the smooth curve lifting Δ_{01} . And for (b), $\Delta_{00} \cap \Delta_1$ is the one curve C_{001} whose automorphism group has order 8. In the universal deformation space of C_{001}, Δ_{00} and Δ_1 lift to a smooth curve and surface meeting transversely, so

$$\delta_{00} \cdot \delta_1 = [C_{001}]_{\mathcal{Q}} = \frac{1}{8}p.$$

c) is an immediate consequence of the general theory of Knudsen [K] or of the fact that δ_{00} is blown down to a point in the Satake compactification \mathcal{A}_2^* of \mathcal{A}_2 . To prove (d), consider

$$i_1: \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1} \rightarrow \Delta_1 \subset \overline{\mathcal{M}}_2.$$

We check that $i_1^*(\lambda_1 - \frac{1}{12}\delta_0) = 0$, hence $(\lambda_1 - \frac{1}{12}\delta_0) \cdot \delta_1 = 0$. But

$$i_1^*(\lambda_1^{(2)}) = p_1^*(\lambda_1^{(1)}) + p_2^*(\lambda_1^{(1)})$$

where, for the sake of clarity, we write

$$\begin{aligned} \lambda_1^{(2)} &= \text{the class } \lambda_1 \text{ in } A^*(\overline{\mathcal{M}}_2) \\ \lambda_1^{(1)} &= \text{the class } \lambda_1 \text{ in } A^*(\overline{\mathcal{M}}_{1,1}). \end{aligned}$$

This is simply because on curves $E_1 \cup E_2$, E_i elliptic, $E_1 \cap E_2 = \text{one point}$,

$$\Gamma(E_1 \cup E_2, \omega_{E_1 \cup E_2}) \cong \Gamma(E_1, \omega_{E_1}) \oplus \Gamma(E_2, \omega_{E_2}).$$

Moreover,

$$i_1^*(\delta_0^{(2)}) = p_1^*(\delta_0^{(1)}) + p_2^*(\delta_0^{(1)}).$$

But in $A^*(\overline{\mathcal{M}}_{1,1})$, the relation

$$\lambda_1 = \frac{1}{12}\delta$$

holds. This is well known and is just the specialization to genus 1 of the theory of §5. Or else it may be seen using the elliptic modular form Δ of

weight 12 with a simple pole at the cusp. Finally, to prove (e), consider:

$$i_0: \overline{M}_{1,2} \rightarrow \Delta_0 \subset \overline{M}_2.$$

Then $i_{0,*}(1_{\overline{M}_{1,2}}) = 2\delta_0$. One should be careful here to note that the presence of automorphisms generically on \overline{M}_2 does not affect this: in fact

$$\begin{aligned} i_{0,*}(1_{\overline{M}_{1,2}}) &= i_{0,*}([\overline{M}_{1,2}]) \\ &= [\Delta_0] \quad (\text{because } \overline{M}_{1,2} \rightarrow \Delta_0 \text{ is birational}) \\ &= 2\delta_0 \quad (\text{because } \text{Aut}(C) = \mathbb{Z}/2\mathbb{Z}, [C] \in \Delta_0 \text{ generic}). \end{aligned}$$

Therefore

$$\begin{aligned} \lambda_1 \cdot \delta_0 &= \frac{1}{2} \lambda_1 \cdot i_{0,*}(1_{\overline{M}_{1,2}}) \\ &= \frac{1}{2} i_{0,*}(i_0^*(\lambda_1)). \end{aligned}$$

Now let

$$\pi: \overline{M}_{1,2} \rightarrow \overline{M}_{1,1}$$

be the natural projection. Note that $\overline{M}_{1,1}$ is the j -line and $\overline{M}_{1,2}$ is the universal family over the j -line of elliptic curves mod automorphisms. Then

$$i_0^*(\lambda_1^{(2)}) = \pi^*(\lambda_1^{(1)})$$

by Knudsen's theory. This corresponds to the fact that if E' is elliptic with one node P , and E is the normalization of E' , then there is a canonical sequence

$$0 \rightarrow \Gamma(\omega_E) \rightarrow \Gamma(\omega_{E'}) \xrightarrow{res} k(P) \rightarrow 0$$

hence $\Lambda^2(\Gamma(\omega_{E'})) \cong \Gamma(\omega_E)$. Therefore

$$\begin{aligned} \lambda_1 \cdot \delta_0 &= \frac{1}{2} i_{0,*} \left(\pi^*(\lambda_1^{(1)}) \right) \\ &= \frac{1}{24} i_{0,*} \left(\pi^*(\delta^{(1)}) \right). \end{aligned}$$

But $\pi^*(\delta^{(1)}) = [\tilde{\Delta}], [\tilde{\Delta}] \subset \overline{M}_{1,2}$ being the closure of the locus of triples (C, x_1, x_2) , C a rational curve with a node, x_1, x_2 distinct smooth points of C . $\tilde{\Delta}$ maps birationally to Δ_{00} in \overline{M}_2 , and the automorphism group of

the generic rational curve with 2 nodes is $(\mathbb{Z}/2\mathbb{Z})^2$, hence:

$$\begin{aligned} \lambda_1 \cdot \delta_0 &= \frac{1}{24} [\Delta_{00}] \\ &= \frac{1}{6} \delta_{00}. \end{aligned}$$

Q.E.D.

§10b. A Check

An interesting check that these Q -stack-theoretic calculations are OK is to evaluate all terms in the identity

$$60 \lambda_1^3 = \kappa_3 + \frac{1}{2} \sum_{h=0}^1 i_{h,*} (K_1^2 - K_1 K_2 + K_2^2)$$

obtained in §8. Using Theorem 10.1, one finds

$$60 \lambda_1^3 = \frac{1}{48} p.$$

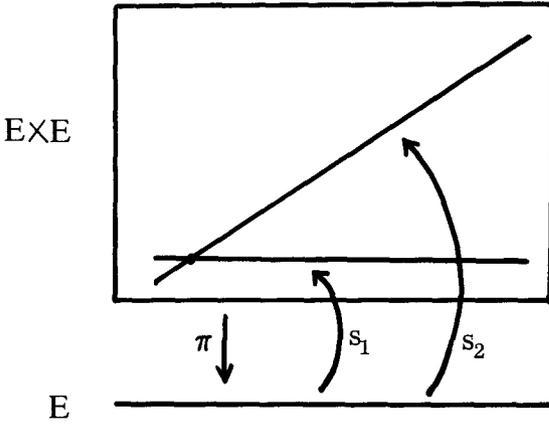
Using (8.3) plus theorem 10.1, one finds

$$\kappa_3 = \frac{1}{1152} p.$$

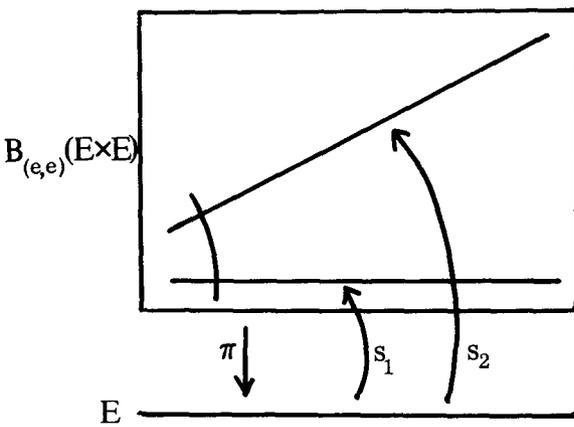
To calculate

$$i_{0,*} (K_1^2 - K_1 K_2 + K_2^2)$$

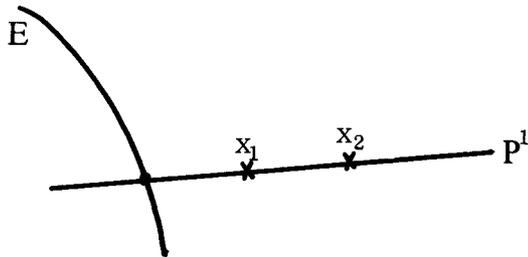
let $\pi: \overline{\mathcal{M}}_{1,2} \rightarrow \overline{\mathcal{M}}_{1,1}$ be the natural map and in $\tilde{\mathcal{M}}_{1,2}$ consider the points $\pi^{-1}([E])$, i.e., representing (E, x_1, x_2) , with E fixed. Up to automorphisms of E , x_1 can be normalized to be the identity. Letting x_2 vary, we parametrize this subset of $\tilde{\mathcal{M}}_{1,2}$ by E itself, and describe the universal family of triples (E, x_1, x_2) as $E \times E$ over E with x_1 being given by $s_1(x) = e(x)$, x_2 by the diagonal $s_2(x) = (x, x)$:



However, this allows $x_1 = x_2$ over $e \in E$, where we should have instead $E \cup \mathbb{P}^1$, $x_1, x_2 \in \mathbb{P}^1 - E \cap \mathbb{P}^1$, $x_1 \neq x_2$. Thus we must blow up $(e, e) \in E \times E$, getting:



This is now a family of 2-pointed stable elliptic curves. The conormal bundle K_i to s_i is $\mathcal{O}_E(+e)$. Thus on $\overline{\mathcal{M}}_{1,2}$, the invertible \mathcal{Q} -sheaf $\mathcal{O}(K_i)$, restricted to the fibres over $\overline{\mathcal{M}}_{1,1}$, is $\mathcal{O}_{\overline{\mathcal{M}}_{1,2}}(+\Sigma_1)$ where $\Sigma_1 \subset \overline{\mathcal{M}}_{1,2}$ is the locus of 2-pointed stable curves:



Therefore $K_1 \equiv K_2 \equiv \pi^*(A) + \Sigma_1$, for some \mathcal{Q} -divisor class A on $\overline{\mathcal{M}}_{1,1}$. But *along* Σ_1 , a canonical coordinate can be put on \mathbb{P}^1 making $E \cap \mathbb{P}^1 = \{\infty\}$, $x_1 = 0$, $x_2 = 1$, hence $\mathcal{O}_{\overline{\mathcal{M}}_{1,2}}(K_i)$, restricted to Σ_1 , is trivial. Now $\pi^* \mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(\lambda_1) \cong \omega_{\overline{\mathcal{M}}_{2,1}/\overline{\mathcal{M}}_{1,1}}$, hence the conormal bundle to Σ_1 is $\mathcal{O}(\lambda_1)$. This proves:

$$K_1 \equiv K_2 \equiv \pi^*(\lambda_1) + \Sigma_1.$$

Therefore

$$\begin{aligned} K_1^2 - K_1 K_2 + K_2^2 &= \pi^* \lambda_1 \cdot \Sigma_1 \\ &= \frac{1}{12} \pi^* \delta \cdot \Sigma_1 \\ &= \frac{1}{24} p \end{aligned}$$

hence

$$\frac{1}{2} i_{0,*}(K_1^2 - K_1 K_2 + K_2^2) = \frac{1}{48} p.$$

Finally to calculate

$$i_{1,*}(K_1^2 - K_1 K_2 + K_2^2)$$

note that on $\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1}$, $K_1^2 = K_2^2 = 0$, and K_i is the pull back from $\overline{\mathcal{M}}_{1,1}$ of λ_1 . Since $\lambda_1 = \frac{1}{24}p$,

$$\begin{aligned} \frac{1}{2}i_{1,*}(K_1^2 - K_1K_2 + K_2^2) &= \frac{1}{2}i_{1,*}\left(-\frac{1}{24} \cdot \frac{1}{24}p\right) \\ &= -\frac{1}{1152}p. \end{aligned}$$

This checks!

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Professor David Mumford
Department of Mathematics
Harvard University
Cambridge, Massachusetts 02138