On the Kodaira Dimension of the Siegel Modular Variety

by David Mumford

Let $\underline{A}_g$ represent the quotient of Siegel's upper half-space

Gothic H $\mathfrak{H}_g$ of rank g by the full integral symplectic group $Sp(2g, \mathbb{Z})$: this is known as Siegel's modular variety, or as the moduli space of g-dimensional principally polarized abelian varieties (called p.p.a.v. below). $\underline{A}_g$ has been shown to be a variety <u>of general type</u> (i.e., Kodaira dimension = dimension) for various g's: Freitag [F1] proved this first if $24|g$; Tai [T] proved this recently for all $g \geq 9$. On the other hand, $\underline{A}_g$ is known to be unirational for $g \leq 5$: Donagi [D] for $g = 5$, Clemens [C] for $g = 4$, classical for $g \leq 3$. The purpose of this paper is to refine Tai's result, showing:

<u>Theorem</u>: $\underline{A}_g$ <u>is of general type if</u> $g \geq 7$.

Note that this leaves only the Kodaira dimension of $\underline{A}_6$ still to be determined. We shall use results of Freitag and Tai in a crucial way, but the idea of the proof is a direct adaption of the proof [H-M] by Harris and the author that $\underline{M}_g$ is of general type if $g \geq 25$, g odd. In that proof the divisor D_k of curves which are k-fold covers of $\mathbb{P}^1$, $k = \frac{g+1}{2}$, is shown to be linearly equivalent to

$$nK\text{-(ample divisor)-(effective divisor)}.$$

Here we prove the same thing except that the role of D_k is taken by the components of N_o, where

$$N_k = \left[\text{locus of p.p.a.v. where } \dim(\text{sing. locus of } \Theta) \geq k.\right]$$

These sets N_k were introduced by Andreotti and Mayer [A-M] , and studied recently by Beauville [B] . I want to thank Beauville very much for stimulating discussions which led me to this result. At the same time, I would like to raise the question which seems very interesting to me: is there an explicit polynomial in theta constants, or other modular forms constructed from theta series (with quadratic forms and pluri-harmonic coefficients) whose zeroes give N_0 with suitable multiplicities? Although important steps are taken in this direction in Andreotti-Mayer [A-M] and Beauville [B], this is not answered because the "theta nulls" $C(r,\mu,z)$ are not in general modular forms — they are theta series whose coefficients are not pluri-harmonic; esp. you cannot form a modular form out of the $\partial^2\theta/\partial u_k^2$'s alone without using mixed derivatives $\partial^2\theta/\partial u_k \partial u_\ell$ too. Finally, I want to mention the related results of Stillman [S] (based on earlier ideas of Freitag [F2]) which prove A_g carries holomorphic (4g-6)-forms for $g \geq 7, g \neq 8$. These results are directly based on the use of theta series.

§1. A partial compactification of the Siegel modular variety.

Satake's compactification $\mathcal{A}_g^*$ of $\mathcal{A}_g$ consists, set-theoretically, in the union of (g+1)-strata:

$$\mathcal{A}_g^* = \mathcal{A}_g \amalg \mathcal{A}_{g-1} \amalg \cdots \amalg \mathcal{A}_0 .$$

The Kodaira dimension of $\mathcal{A}_g$ is based on pluri-canonical differentials on a desingularization $\tilde{\mathcal{A}}_g$ of $\mathcal{A}_g^*$. However, Tai has shown that a pluri-canonical differential form with "no poles above $\mathcal{A}_g \amalg \mathcal{A}_{g-1}$", is everywhere regular, so we do not have to study the full $\tilde{\mathcal{A}}_g$. We will make this precise in a minute. The space we want to work with is a blow-up of $\mathcal{A}_g \amalg \mathcal{A}_{g-1}$ first introduced by Igusa [I] and studied by the author [M] and by Namikawa [N]. To describe this space geometrically, let us define a rank 1 degeneration of a p.p.a.v. as follows: it is a pair $(\overline{G},D)$ where $\overline{G}$ is a complete g-dimensional variety and D is an ample divisor (i.e., $\overline{G}$ is to be the limit of a g-dimensional abelian variety and D the limit of its theta divisor). $\overline{G}$ is constructed as follows:

1) let B^{g-1} be a (g-1)-dimensional p.p.a.v., $\Xi \subset B$ its theta divisor

2) let G be an algebraic group which is an extension of B by $\mathbb{G}_m$:

$$0 \longrightarrow \mathbb{G}_m \longrightarrow G \longrightarrow B \longrightarrow 0.$$

3) Considering G as a $\mathbb{G}_m$-bundle over B, let $\tilde{G}$ be the associated $\mathbb{P}^1$-bundle:

$$\begin{array}{ccc} G & \subset & \tilde{G} \\ & {\scriptstyle \mathbb{C}_m}\searrow \quad \swarrow{\scriptstyle \pi}\ {\scriptstyle \mathbb{P}^1} & \\ & B & \end{array}$$

Then $\tilde{G}-G$ equals $\tilde{G}_0 \amalg \tilde{G}_\infty$, the union of 2 sections of $\tilde{G}$ over B.

4) Then $\overline{G}$ is to be the non-normal variety obtained by glueing $\tilde{G}_0, \tilde{G}_\infty$ with a translation by a point $b \in B$.

5) Note that on $\tilde{G}$

$$\begin{aligned} \tilde{G}_0-\tilde{G}_\infty &\equiv \pi^{-1}(E), \quad E \text{ algebraically equivalent to } 0 \text{ on } B \\ &\equiv \pi^{-1}(\Xi-\Xi_{b_1}), \text{ for a unique } b_1 \in B. \end{aligned}$$

Thus

$$\tilde{G}_0+\pi^{-1}(\Xi_{b_1}) \equiv \tilde{G}_\infty + \pi^{-1}(\Xi).$$

Let $\tilde{L} = \mathcal{O}_{\tilde{G}}(\tilde{G}_\infty+\pi^{-1}(\Xi))$. Via the Leray spectral sequence for π, we see that $h^0(\tilde{L}) = 2$ and that $\tilde{G}_0+\pi^{-1}(\Xi_{b_1})$, $\tilde{G}_\infty+\pi^{-1}(\Xi)$ span the linear system $|\tilde{L}|$. Then $|\tilde{L}|_{\tilde{G}_0} \cong \mathcal{O}_B(\Xi)$ and $\tilde{L}|_{\tilde{G}_\infty} \cong \mathcal{O}_B(\Xi_{b_1})$, so if b is chosen to be b_1 (and only then) the line bundle $\tilde{L}$ can be descended to a line bundle L on $\overline{G}$. Choose such an L and let

$$D = \text{the unique divisor in } |L|.$$

We now define

(1.1) $\overline{A}_g^{(1)} = \begin{cases} \text{coarse moduli space of p.p.a.v.}(A,\Theta) \text{ of} \\ \text{dimension } g \text{ and their rank 1 degenerations} \end{cases}$.

As first shown by Igusa, this space exists, is a quasi-projective variety, and is essentially the blow-up of the open set $\mathcal{A}_g \amalg \mathcal{A}_{g-1}$ in $\mathcal{A}_g^*$ along its boundary $\mathcal{A}_{g-1}$. $\overline{\mathcal{A}}_g^{(1)}$ is the union of $\mathcal{A}_g$ and a divisor Δ parametrizing the rank 1 degenerations. Via the map

$$(\overline{G},D) \longmapsto (B,\Xi)$$

the divisor Δ is seen to be fibred:

$$(1.2) \qquad \begin{array}{c} \Delta \\ \delta \big\downarrow \\ \mathcal{A}_{g-1} \end{array} \qquad \text{fibres } B/\mathrm{Aut}(B,\Xi) \quad .$$

Analytically, we may consider $\overline{\mathcal{A}}_g^{(1)}$ to represent precisely the degenerations of the abelian variety $A_{\Omega(t)}$ with period matrix $\Omega(t)$ when:

$$\left.\begin{array}{l} \operatorname{Im} \Omega_{11} \longrightarrow \infty \\ \text{and } \Omega_{ij},\ i>1 \text{ or } j>1, \text{ have finite limits} \end{array}\right\} \text{ as } t \longrightarrow 0 \quad .$$

Then $B = B_{\Omega^{(1)}}$, where $\Omega^{(1)}$ is the lower right block of the limit

$$\Omega(0) = \left(\begin{array}{c|c} i\infty & \omega \\ \hline {}^t\omega & \Omega^{(1)} \end{array}\right)$$

and b is the image of the vector $\vec{\omega} = (\Omega_{12}(0)\,\Omega_{13}(0),\cdots,\Omega_{1g}(0))$ in $B_{\Omega^{(1)}}$. To find D, we must translate $\Theta_{\Omega(t)} \subset A_{\Omega(t)}$ as $t \longrightarrow 0$.

Thus

$$\Theta_{\Omega(t)} = \left\{ \text{zeroes of } \ \theta(z,\Omega) = \sum_{n\in\mathbb{Z}^g} e^{\pi i\, {}^t n\Omega(t)n + 2\pi i\, {}^t n\cdot z} \right\}.$$

Translate $\Theta_{\Omega(t)}$ by $b(t)$, the image of $(\frac{\Omega_{11}(t)}{2},0,\cdots,0)$:

$$T_{b(t)}(\Theta_{\Omega(t)}) = \left\{ \text{zeroes of } \sum e^{\pi i(n_1^2-n_1)\Omega_{11}(t)} \cdot e^{\left[\pi i \sum_{i,j\neq 1,1} n_i n_j \Omega_{ij}(t) + 2\pi i\, {}^t n z\right]} \right\}.$$

Then $e^{\pi i(n_1^2-n_1)\Omega_n(t)} \longrightarrow 0$ unless $n_1 = 0$ or 1, hence the limit is

superscripted formula shown thus

$$\left\{ \text{zeroes of } \sum_{n_2,\cdots,n_g\in\mathbb{Z}} e^{\left[\pi i \sum_{i,j\geq 2} n_i n_j \Omega_{ij}(0) + 2\pi i \sum_{j\leq 2} n_j z_j\right]} \cdot \left(1 + e^{2\pi i z_1} \cdot e^{2\pi i \sum_{j\leq 2} n_j \Omega_{ij}}\right) \right\} \tag{1.13}$$

$$= \left\{ \text{zeroes of } \ \theta(z^{(1)},\Omega^{(1)}) + e^{2\pi i z_1} \cdot \theta(z^{(1)}+\omega,\Omega^{(1)}) \right\}$$

where $z^{(1)} = (z_2,\cdots,z_g)$ is the analytic coordinate on $B_{\Omega^{(1)}}$. Interpreting $e^{2\pi i z_1}$ as the algebraic coordinate in the fibre $\mathbb{G}_m$ of G, and Ξ as the zeroes of $\theta(z^{(1)},\Omega^{(1)})$, this is immediately seen to be D if L is suitably defined.

Next, let $\overline{A}_g^{(1),0}$ be the open set in $\overline{A}_g^{(1)}$ parametrizing those pairs (A,Θ) or $(\overline{G},D)$ whose automorphism group is the minimal one, $\{\pm 1\}$. More precisely, the only non-trivial automorphism of A (or $\overline{G}$) mapping Θ (resp. D) to itself is of the form $x \longmapsto -x+a$, some a*. Then $\overline{A}_g^{(1),0}$ is locally isomorphic

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* We have not normalized Θ and D to be symmetric. On the other hand, we have not fixed an origin either, so the pairs (A,Θ) and (A,Θ_c) are isomorphic by translation by c, and define the same point of $\overline{A}_g^{(1)}$.

to the universal deformation space of (A,Θ) (or $(\overline{G},D)$), hence is a smooth of dimension $g(g+1)/2$. Analytically, $\mathbb{A}_g^0$ is the open subset of $\mathbb{A}_g$ of points which are images of $\Omega \in \mathfrak{H}_g$ whose stabilizer in $Sp(2g,\mathbb{Z})$ are just $(\pm I)$. Likewise, using the analytic description of $\overline{\mathbb{A}}_g^{(1)}$ in Ash et al [A-M-R-S], $\overline{\mathbb{A}}_g^{(1),0}$ is the open subset of $\overline{\mathbb{A}}_g^{(1)}$ of points which are images of points in $\overline{(\mathfrak{H}_g/U_{\mathbb{Z}})_{\{\sigma_\alpha\}}}$ whose stabilizer in the normalizer of the first boundary component is just $(\pm I)$. (Compare Tai [T], §). This set includes, in particular, those $\overline{G}$ constructed from a $(B,\Xi) \in \mathbb{A}_{g-1}^0$ and a point $b \in B$ not of order 2. We are now in a position to state one of the main results of Tai's paper [T], in the form in which we need it:

Theorem 1.4 (Tai). If $g \geq 5$, then

a) $\operatorname{codim} (\overline{\mathbb{A}}_g^{(1)} - \overline{\mathbb{A}}_g^{(1),0}) \geq 2$

and

b) $\Gamma(\widetilde{\mathbb{A}}_g, \mathcal{O}(nK)) = \Gamma(\overline{\mathbb{A}}_g^{(1),0}, \mathcal{O}(nK))$, if $n \geq 1$.

This means that a pluri-canonical differential with no poles on $\overline{\mathbb{A}}_g^{(1),0}$ is everywhere regular on a full desingularization $\widetilde{\mathbb{A}}_g$ of $\mathbb{A}_g^*$.

The second result we need is the calculation of $\operatorname{Pic}(\mathbb{A}_g^0)$. This follows from the theory of Matsushima, Borel, Wallach and others on the low cohomology groups of discrete subgroups of Lie groups. In particular, the results of Borel [Bo] imply that for any subgroup $\Gamma \subset Sp(2g,\mathbb{Z})$ of finite index:

$$H^*(\Gamma,\mathbb{Q}) \cong \mathbb{Q}[C_2, C_6, C_{10},\ldots] \text{ , in degrees } \leq g-2 .$$

In particular:

$$H^2(\mathbb{A}_g,\mathbb{Q}) \cong H^2(Sp(2g,\mathbb{Z}),\mathbb{Q}) \cong \mathbb{Q} \quad \text{if } g \geq 4 .$$

An immediate corollary* is:

Theorem 1.5 (Borel et al): $\mathrm{Pic}(\mathcal{A}_g^0) \otimes \mathbb{Q} \cong \mathbb{Q}.\lambda$, if $g \geqq 4$, where λ is the line bundle on $\mathcal{A}_g^0$ defined by the co-cycle $\det(C\Omega+D)$.

Corollary 1.6: $\mathrm{Pic}(\overline{\mathcal{A}}_g^{(1),0}) \otimes \mathbb{Q} \cong \mathbb{Q}\lambda + \mathbb{Q}.\delta$

where δ is the divisor class of the boundary Δ.

In terms of these generators, a standard result is:

Proposition 1.7. $K_{\overline{\mathcal{A}}_g^{(1),0}} \equiv (g+1)\lambda - \delta$.

For a proof, see for instance Tai [T], §1. Another fairly standard result that we need is:

Proposition 1.8. Let (B,Ξ) be a $(g-1)$-dimension p.p.a.v. whose automorphism group is (± 1). Consider the 2-1 map

$$\phi: (B-B_2) \longrightarrow \overline{\mathcal{A}}_g^{(1),0}$$

defined by $\phi(b)$ = the pair $(\overline{G},D)$ constructed from (B,Ξ) with

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* If $\tilde{\mathcal{A}}_g$ is a smooth compactification of $\mathcal{A}_g^0$, then use:

$$\begin{array}{ccccc} \oplus\mathbb{Q}\delta_i & \longrightarrow & \mathrm{Pic}\tilde{\mathcal{A}}_g \otimes \mathbb{Q} & \overset{\text{res}}{\longrightarrow\!\!\!\!\rightarrow} & \mathrm{Pic}\mathcal{A}_g^0 \otimes \mathbb{Q} \\ \| & & \downarrow & & \downarrow \\ \oplus\mathbb{Q}\delta_i & \longrightarrow & H^2(\tilde{\mathcal{A}}_g,\mathbb{Q}) & \longrightarrow\!\!\!\!\rightarrow & H^2(\mathcal{A}_g^0,\mathbb{Q}) \end{array}$$

plus $H^2(\mathcal{A}_g^0,\mathbb{Q}) \cong H^2(\mathcal{A}_g,\mathbb{Q}) \cong \mathbb{Q}$.

glueing via b. Then

$$\phi^*(\mathcal{O}_{\bar{A}_g}(1),0(\Delta)) \cong \mathcal{O}_B(-2\Xi).$$

Proof: Let's construct over B the family of $(\bar{G},D)$'s made up with all possible b's. To do this, let P be the Poincaré bundle over $B\times B$, trivial on $e\times B$, $B\times e$. Then $P^* = P-(0\text{-section})$ serves as the universal family of G's. Let $\bar{P} \supset P$ be the associated $\mathbb{P}^1$-fibre bundle, and

$$\mathfrak{P} = \bar{P}/(b_1,b_2,0) \sim (b_1,b_1+b_2,\infty).$$

Then the projection on the first factor:

$$p_1 : \mathfrak{P} \longrightarrow B$$

is the universal family of $\bar{G}$'s. The deformation theory of such a $\bar{G}$ gives an exact sequence:

$$0 \longrightarrow H^1(\bar{G},\underline{T}^0(\mathcal{O}_{\bar{G}})) \longrightarrow T^1(\bar{G}) \longrightarrow H^0(\text{Sing }\bar{G}, \underline{T}^1(\mathcal{O}_{\bar{G}})) = H^0(B,N_0\otimes N_\infty)$$

where N_0,N_∞ are the normal bundles to the locus of double points of $\bar{G}$. For one $\bar{G}$, made up starting from a line bundle L over B, completed at ∞ and glued by translation by $b \in B$,

$$N_0 \otimes N_\infty \cong L \otimes T_b^*(L^{-1}).$$

Note that L must be algebraically equivalent to 0, hence $T_b^*L^{-1} \cong L^{-1}$, hence $N_0 \otimes N_\infty \cong \mathcal{O}_B$. Thus $H^0(B, N_0 \otimes N_\infty) \cong k$. This one-dimensional vector space represents the normal bundle to Δ in $\overline{\mathcal{A}}_g$ at the point $(\overline{G}, D)$. Doing this now for the whole family $\mathcal{P} \longrightarrow B$, $N_0 \otimes N_\infty$ is the line bundle on $B \times B$ given by

$$P \otimes T^*(P^{-1})$$
$$\text{where } T(x,y) = (x, x+y).$$

Then the normal bundle to Δ, pulled back to this family, is

$$p_{1,*}(P \otimes T^*(P^{-1}))$$

which is the same as the restriction of $P \otimes T^* P^{-1}$ to $B \times e$, i.e., $\delta^*(P^{-1})$, where $\delta(x) = (x,x)$. Since P, along the diagonal of $B \times B$ is $\mathcal{O}(2\Xi)$, this proves the Proposition. QED

§2. The divisor N_0 and its class in $\mathrm{Pic}(\overline{\mathcal{A}}_g^{(1)})$.

Andreotti-Mayer [A-M] defined the important subsets N_k in $\mathcal{A}_g$:

$$(2.1) \qquad N_k = \{(A,\Theta) \mid \mathrm{Sing}\,\Theta \neq \emptyset \text{ and } \dim(\mathrm{Sing}\,\Theta) \geq k\}.$$

Andreotti and Mayer prove by using the Heat equation for that $N_0 \subsetneq \mathcal{A}_g$, but it is not easy to estimate the dimension of N_k in general. Nowever, we are interested only in codimension 1 and we must at least check that none of the N_k, $k \geq 1$, have codimension 1 components. This follows by an elaboration of

Andreotti-Mayer's arguments using the heat equation:

Lemma 2.2. The codimension of N_1 (hence of $N_2, N_3, \cdots$) in $\underline{A}_g$ is greater than 1.

Proof: We use the heat equation

$$(2\pi i)(1+\delta_{\alpha\beta})\frac{\partial \theta}{\partial \Omega_{\alpha\beta}} = \frac{\partial^2 \theta}{\partial z_\alpha \partial z_\beta} .$$

If the lemma were false, we could find a matrix $\overline{\Omega}$, a smooth analytic hypersurface $g(\Omega) = 0$ defined in a neighborhood of $\overline{\Omega}$ and containing $\overline{\Omega}$, and a vector-valued function

$$\vec{f}(\Omega,t) \in \mathbb{C}^g$$

defined in a neighborhood of $\overline{\Omega}$ and for $|t|$ small, such that

$$\left.\begin{array}{l} \theta(\vec{f}(\Omega,t),\Omega) \equiv 0 \\ \frac{\partial \theta}{\partial z_k}(\vec{f}(\Omega,t),\Omega) \equiv 0, \quad 1 \le k \le g \end{array}\right\} \text{ whenever } g(\Omega) = 0.$$

We may assume that for each Ω, $t \longmapsto \vec{f}(\Omega,t)$ is part of an algebraic curve $C_\Omega \subset A_\Omega$. Note that the lemma is obvious if $g = 2$ and if $g \ge 3$, then the codimension of the locus of non-simple abelian varieties is greater than 1. Therefore we can also assume that the abelian variety $A_{\overline{\Omega}}$ is simple. It follows that the set of differences $x-y, x,y \in C_{\overline{\Omega}}$ generates $A_{\overline{\Omega}}$, hence the set of differences $x-y, x,y \in C_\Omega$, generates A_Ω for Ω near $\overline{\Omega}$. Therefore, for no Ω near $\overline{\Omega}$ is there a vector $\vec{a}$ such that

$$\frac{\partial}{\partial t}(\vec{a}\cdot\vec{f}) \quad = \quad (\vec{a}\cdot\frac{\partial\vec{f}}{\partial t}) = 0, \quad \text{all } t.$$

We prove by induction on d that:

$(*)_d$ If $|\alpha| = d$, then $\left(\frac{\partial^\alpha\theta}{\partial z_1^{\alpha_1}\cdots\partial z_g^{\alpha_g}}\right)(\vec{f}(\Omega,t),\Omega) \equiv 0$ whenever $g(\Omega) = 0$.

Since $\theta(z,\overline{\Omega})$ does not vanish identically as a function of z, this is a contradiction. In fact, to prove this it will suffice to apply:

If $\eta(\Omega,z)$ satisfies the heat equation and

$$(**)\qquad \left.\begin{aligned} &\eta(\vec{f}(\Omega,t),\Omega) \equiv 0 \\ &\frac{\partial\eta}{\partial z_k}(\vec{f}(\Omega,t),\Omega) \equiv 0 \end{aligned}\right\} \text{ whenever } g(\Omega) = 0$$

then

$$\frac{\partial^2\eta}{\partial x_k\partial z_\ell}(\vec{f}(\Omega,t),\Omega) \equiv 0 \qquad \text{whenever } g(\Omega) = 0$$

to all the partial derivatives of θ in turn. To prove (**), differentiate the first relation with respect to Ω. We find that if $\omega_{k\ell}$ satisfies $\sum\omega_{k\ell}\partial g/\partial\Omega_{k\ell}(\Omega) = 0$, then $\Omega+\varepsilon\omega$ is tangent to the hypersurface $g(\Omega) = 0$, hence

$$\begin{aligned} 0 &= \eta(\vec{f}(\Omega+\varepsilon\omega,t),\Omega+\varepsilon\omega) \\ &= \varepsilon\left\{\sum_{k,a,b}\frac{\partial\eta}{\partial z_k}(\vec{f}(\Omega,t),\Omega)\cdot\frac{\partial f_k}{\partial\Omega_{ab}}\cdot\omega_{ab} + \sum_{a\leq b}\frac{\partial\eta}{\partial\Omega_{ab}}(\vec{f}(\Omega,t),\Omega)\cdot\omega_{ab}\right\} \\ &= \frac{\varepsilon}{4\pi i}\sum_{a,b}\frac{\partial^2\eta}{\partial z_a\partial z_b}(\vec{f}(\Omega,t),\Omega)\cdot\omega_{ab}\ . \end{aligned}$$

Therefore

$$\frac{\partial^2 \eta}{\partial z_a \partial z_b}(\vec{f}(\Omega,t),\Omega) = \phi(\Omega,t)\cdot(1+\delta_{ab})\cdot \frac{\partial g}{\partial \Omega_{ab}}(\Omega)$$

with some factor ϕ, for all Ω near $\bar{\Omega}$, all small t. Now differentiate the second relation in (**) with respect to t. We find:

$$\text{for all } a, \quad \sum_b \frac{\partial^2 \eta}{\partial z_a \partial z_b} (\vec{f}(\Omega,t),\Omega)\cdot\frac{\partial f_b}{\partial t}(\Omega,t) \equiv 0 \quad \text{whenever } g(\Omega) = 0.$$

If $\phi(\Omega,t) \equiv 0$ when $g(\Omega) = 0$, we are done. If not, we find by substitution that

$$\text{for all } a, \quad \sum_b (1+\delta_{ab})\frac{\partial g}{\partial \Omega_{ab}}(\Omega)\cdot\frac{\partial f_b}{\partial t}(\Omega,t) \equiv 0 \quad \text{whenever } g(\Omega) = 0,$$

i.e.,

(***) $$(\vec{c}(a)\cdot\frac{\partial \vec{f}}{\partial t}) = 0$$

where

$$c(a)_b = (1+\delta_{ab})\frac{\partial g}{\partial \Omega_{ab}}(\Omega).$$

For some a, $\vec{c}(a) \neq 0$ since $g(\Omega) = 0$ is a smooth hypersurface. But we saw that (***) did not occur, so thus completes the proof.

In the other direction, Beauville [$\mathcal{B}$], Remark 7.7 proved*:

<u>Proposition 2.3</u> (Beauville): N_0 <u>has codimension</u> 1 <u>in</u> $\underline{A}_g$.

*The result is stated only for g = 4; however the argument works without any modification for all g.

His proof also uses an elaboration of the techniques of Andreotti-Mayer — in this case their technique for deriving "explicit" equations for the N_k. (It might be thought that this Proposition could be proven from general principles, but I don't see how, without specific information, one could have excluded the possibilities that some component of some N_k, $k > 1$, was not in the closure of $N_0 - N_1$.)

We want now to consider the closure $\overline{N}_0$ of N_0 in $\overline{\mathbf{A}}_g^{(1)}$, and to give multiplicities to its components. To do this, we would like to use the "universal family" of pairs (A,Θ), $(\overline{G},D)$ over $\overline{\mathbf{A}}_g^{(1)}$. However, even generically these pairs still have an automorphism group of order 2, so a universal family need not exist. However, $\overline{\mathbf{A}}_g^{(1)}$ admits a "covering" $U_\alpha \longrightarrow \overline{\mathbf{A}}_g^{(1)}$ such that over U_α there are flat, proper families

$$\begin{array}{c} D_\alpha \subset \overline{G}_\alpha \\ \Big\downarrow p \\ U_\alpha \end{array}$$

consisting of abelian varieties and rank 1 degenerations thereof, and such that p is locally the universal deformation space of its fibre $(\overline{G}_s, D_s)$. Outside $\Delta \cap U_\alpha$, $\overline{G}_\alpha$ will be smooth over U_α; over points of $\Delta \cap U_\alpha$, $\overline{G}_\alpha$ itself will still be smooth, but at the double points of the fibres, p will look like the universal local deformation space:

$$\hat{\mathcal{O}}_{\overline{G}_\alpha} \cong \mathbb{C}[[z_1, z_1', z_2, \cdots, z_{g-1}, t_2, \cdots, t_{g(g+1)/2}]]$$

$$\uparrow \cup$$

$$\hat{\mathcal{O}}_{U_\alpha} \cong \mathbb{C}[[t_1, t_2, \cdots, t_{g(g+1)/2}]]$$

$$t_1 = z_1 \cdot z_1' .$$

On $\overline{G}_\alpha$, define the subsheaf of the tangent sheaf T_{vert} to be the kernel:

$$0 \longrightarrow T_{vert} \longrightarrow T_{\overline{G}_\alpha} \longrightarrow p^* T_{U_\alpha} .$$

Note that T_{vert} is locally free of rank g (at double points of the fibres, T_{vert} is spanned by $z_1 \partial/\partial z_1 - z_1' \partial/\partial z_1'$, $\partial/\partial z_2, \cdots, \partial/\partial z_g$). Using a local equation $\delta = 0$ of D_α, and interpreting sections of T as derivations, define:

$$T_{vert} \xrightarrow{\alpha} \mathcal{O}(D_\alpha)/\mathcal{O}$$

$$D \longmapsto D\delta/\delta \qquad \text{(independent of } \delta\text{)}.$$

Let

$$\mathrm{Sing}_{vert}\, D_\alpha = \text{subscheme of } D_\alpha \text{ where } \alpha \text{ is zero.}$$

Thus $\mathrm{Sing}_{vert}\, D_\alpha$ is defined locally by g equations and has codimension at most g. Set-theoretically:

(2.4) $p(\mathrm{Sing}_{\mathrm{vert}}\, D_\alpha)$ = set of points whose fibres are of 3 types

1) fibre is (A,Θ), A abelian variety, and Θ singular

2) fibre is $(\overline{G},D)$ and D has a singularity in G

3) fibre is $(\overline{G},D)$ and the divisor $\overline{D} = D.(\overline{G}-G)$ on $\overline{G}-G$ is singular.

To see this at fibres of type $(\overline{G},D)$, at points of $\overline{G}-G$, expand δ in a power series in $z_1, z_1', z_2, \cdots, z_g$, t's: then the origin lies in $\mathrm{Sing}_{\mathrm{vert}}\, D_\alpha$ if and only if

$$\delta \in (z_1, z_1', \ z_i z_j \ (2 \le i,j \le g), \ t_i) \ ,$$

i.e., if and only if $\delta = 0$ is singular in $\mathbb{C}[[z_2,\cdots,z_g]]$. The sets $p(\mathrm{Sing}_{\mathrm{vert}}\, D_\alpha)$ patch together into a subset $\tilde{N}_0$ of $\overline{\mathbf{A}}_g^{(1)}$. (We shall see shortly that $\tilde{N}_0 = \overline{N}_0$.)

Let us work out which $(\overline{G},D)$ arise in cases (2) and (3). Let G be the extension:

$$0 \longrightarrow \mathbb{G}_m \longrightarrow G \longrightarrow B \longrightarrow 0.$$

Then $\overline{G}-G \cong B$ and $D.(\overline{G}-G)$ is the theta divisor of B, called Ξ at the beginning of this section. Thus if $\pi: \Delta \longrightarrow \mathbf{A}_{g-1}$ is the natural projection, case (3) contributes $\pi^{-1}(N_0(\mathbf{A}_{g-1}))$ to $\tilde{N}_0$. As for case (2), if translation by $b \in \mathbf{B}$ is used in glueing together $\overline{G}$, then a local equation of D at any point of G is of the form

$$f(x,z) = \delta_p(x) + z\cdot\delta_{p+b}(x+b)$$

Here δ_P (resp. δ_{P+b}) are local functions on B near p (resp. P+b) which define the non-zero section of $\mathcal{O}_B(\Xi)$ near P (resp. P+b), and z is a vertical coordinate on G in a local splitting $G \cong \mathbb{G}_m \times B$. (We may use the analytic equation (1.13) if we want.) Taking derivatives of f, we see that:

$f(x,z) = 0$ is singular at $x = P$, some $z \in \mathbb{C}^*$	$\Longleftrightarrow$	$P, P+b \in \Xi$ and either Ξ has the same tangent plane at P,P+b, or is singular at both pts.

Looking at points $(\overline{G},D)$ not already covered in case (3), this shows that $\tilde{N}_0'$ contains the set of pairs $(\overline{G},D)$ such that $\Xi \subset B$ is smooth and Ξ, Ξ_b are tangent somewhere. If Ξ is smooth, let

$$\gamma_B : \Xi \longrightarrow \mathbb{P}^{g-2}$$

be the "Gauss map" associating tc each $P \in \Xi$, the tangent plane $T_{P,\Xi}$, as a point of $\mathbb{P}(T^*_{0,B})$. Then Ξ and Ξ_b are tangent at P if and only if $\gamma_B(P) = \gamma_B(P+b)$. Thus for any principally polarized abelian variety (B,Ξ) with smooth Ξ we may define

$$\begin{aligned} c(B,\Xi) &= \text{locus of points } x-y, \text{ where } \gamma_B(x) = \gamma_B(y) \\ &= \text{locus of points } x \text{ such that } \Xi,\Xi_x \text{ are tangent somewhere.} \end{aligned}$$

Then in the description (2.4):

$$\tilde{N}_0 \cap \Delta \cong \left[\bigcup_{(\overline{G},D)} c(B,\Xi)\right] \cup \left[\delta^{-1}(N_0 \text{ for } \mathcal{A}_{g-1})\right].$$

Next, the method of Andreotti-Mayer-Beauville extends to rank 1 degenerations, to prove that $\tilde{N}_0$ is a divisor. For abelian varieties A, their technique is to map A to $\mathbb{P}^{2^g-1}$ by $|2\Theta|$, i.e., explicitly by the theta functions

$$\theta_\mu(z,\Omega) = \sum_{n\in\mathbb{Z}^g} e^{2\pi i\,{}^t(n+\mu)\Omega(n+\mu)+4\pi i\,{}^t(n+\mu).z} .$$

Call this ϕ: $A \longrightarrow \mathbb{P}^{2^g-1}$.

They define a linear subspace $L_\Omega \subset \mathbb{P}^{2^g-1}$ of codimension g+1 by

$$(2.5) \qquad \begin{aligned} &\sum \theta_\mu(0,\Omega)\cdot X_\mu = 0 \\ &\frac{\partial^2\theta_\mu}{\partial z_i^2}(0,\Omega)\cdot X_\mu = 0, \quad 1 \le i \le g \end{aligned}$$

and prove

$$(2.6) \qquad \phi^{-1}(L_\Omega) = \text{Sing } \Theta,$$

hence

$$(A,\Theta) \in N_0 \iff L_\Omega \cap \phi(A) \neq \emptyset$$

$\iff$ Chow form of $\phi(A)$ varieties at Plücker Coord of L_Ω

Now if $\text{Im } \Omega_{11} \longrightarrow \infty$, the limit of $\phi(A)$ is $\phi(\bar{G})$, where ϕ is defined by the 2^g "theta functions"

$$\left.\begin{aligned} \theta_\mu\left(z^{(1)},\Omega^{(1)}\right) + u^2\theta_\mu\left(z^{(1)}+\omega,\Omega^{(1)}\right) \\ u\theta_\mu\left(z^{(1)}+\tfrac{1}{2}\omega,\Omega^{(1)}\right) \end{aligned}\right\} \qquad \mu\in \tfrac{1}{2}\mathbb{Z}^{g-1}/\mathbb{Z}^{g-1}$$

(where, as above, G is a $\mathbb{C}_m^*$-bundle over B, $\Omega^{(1)}$ = period matrix of B, $\bar{G}$ is glued via ω, $z^{(1)}$ is the coordinate on B, u the coordinate on $\mathbb{C}_m$). The basic theta identity on which the proof of (2.6) is based becomes

$$\text{(2.7)}\quad \begin{aligned}&[\theta(x+y)+uw\theta(x+y+\omega)]\cdot[\theta(x-y)+\tfrac{u}{w}\theta(x-y+\omega)] = \\ &\sum_{\mu\in\frac{1}{2}\mathbb{Z}^{g-1}/\mathbb{Z}^{g-1}} [\theta_\mu(x)+u^2\theta_\mu(x+b)]\cdot\theta_\mu(y)+uw\theta_\mu(x+\tfrac{\omega}{2})\cdot[\theta_\mu(y+\tfrac{\omega}{2})+\tfrac{1}{w^2}\theta_\mu(y-\tfrac{\omega}{2})].\end{aligned}$$

The limit of L_Ω is the linear space

$$\text{(2.8)}\quad \begin{aligned}&\sum \theta_\mu(0,\Omega^{(1)})\cdot X_\mu + 2\sum \theta_\mu(\tfrac{\omega}{2},\Omega^{(1)})\cdot Y_\mu = 0\\ &\sum \frac{\partial^2\theta_\mu}{\partial z_i^2}(0,\Omega^{(1)})\cdot X_\mu + 2\sum \frac{\partial^2\theta_\mu}{\partial z_i^2}(\tfrac{\omega}{2},\Omega^{(1)})\cdot Y_\mu = 0\\ &\sum \theta_\mu(\tfrac{\omega}{2},\Omega^{(1)})\cdot Y_\mu = 0 .\end{aligned}$$

(The last equation comes from the 2^{nd} derivative of (2.7) with respect to $w\,\partial/\partial w$; these equations are not the exact analogs of the (2.5) because, in passing to the limit, we have renormalized the origin.) Then it follows from (2.7) exactly as in Andreotti-Mayer-Beauville that

$$\phi^{-1}(L_\Omega) = \left(\begin{array}{c}\text{singularities of } D \text{ in } G \text{ plus singularities}\\ \text{of } \bar{D}\cdot(\bar{G}-G) \text{ in } \bar{G}-G\end{array}\right)$$

hence

$$\left(\begin{array}{c}\text{Chow form of } \phi(\bar{G}) \text{ is}\\ \text{zero at } L_\Omega\end{array}\right) \iff (\bar{G},D)\in \hat{N}_0.$$

This proves that $\tilde{N}_0$ is a divisor.

On the other hand, it is clear that for all B, $c(B,\Xi) \subsetneqq B$ and for generic B,Ξ is smooth: hence $\tilde{N}_0 \cap \Delta \subsetneqq \Delta$. Thus $\tilde{N}_0$ must be the closure $\bar{N}_0$ of N_0. Incidentally, this proves that $c(B,\Xi)$ is always a divisor in B. At the same time, we can now give multiplicities to the components of $\bar{N}_0$. I think the Andreotti-Mayer-Beauville equation gives artificially large multiplicities, and want, instead, to assign multiplicities via the local description of $\bar{N}_0$ in U_α as $p(\mathrm{Sing}_{vert} D_\alpha)$. Let $\bar{N}_0'$ be the maximal open set of points of $\bar{N}_0$ such that for all α

$$p: \ \mathrm{Sing}_{vert} D_\alpha \longrightarrow (\bar{N}_0 \cap U_\alpha)$$

is <u>finite</u> over $\bar{N}_0'$. Because N_1 has codimension at least 2, $\bar{N}_0'$ is dense in $\bar{N}_0$. Then <u>over $\bar{N}_0'$</u>

$$\dim(\mathrm{Sing}_{vert} D_\alpha) = \dim N_0$$

hence

$$\operatorname{codim}(\mathrm{Sing}_{vert} D_\alpha) = g+1 = \#\text{ of equations defining } \mathrm{Sing}_{vert} D_\alpha$$

hence $\mathrm{Sing}_{vert} D_\alpha$ is Cohen-Macauley. Therefore, over $\bar{N}_0'$, $p_*(\mathcal{O}_{\mathrm{Sing}_{vert} D_\alpha})$ has a locally free resolution:

$$0 \longrightarrow \mathcal{E}_1 \xrightarrow{\ f\ } \mathcal{E}_0 \longrightarrow p_*(\mathcal{O}_{\mathrm{Sing}_{vert} D_\alpha}) \longrightarrow 0$$

and det f gives a local equation for $\bar{N}_0' \cap D_\alpha$, and this assigns multiplicities to $\bar{N}_0$.

Next, we want to break $\bar{N}_0$ up into 2 pieces: the first piece is

(2.9) $$\theta_{null} = \left\{ (A,\Theta) \;\middle|\; \begin{array}{l} \text{if } \Theta \text{ is normalized to be symmetric about } e, \\ \text{then } \Theta \text{ has a singularity at a point of order } 2 \end{array} \right\} .$$

It is easy to see that:

$$\theta_{null} \cap \Delta = \left[\bigcup_{\text{all } \overline{G},D} 2_B(\Xi) \right] \cup \left[\delta^{-1}(\theta_{null} \text{ for } \underset{\sim}{A}_{g-1}) \right]$$

where we note that (assuming Ξ is symmetric too) $c(B,\Xi)$ contains the "obvious" component:

$$2_B(\Xi) = \{2x \mid x \in \Xi\}$$

because $\gamma(-x) = \gamma(x)$, all $x \in \Xi$.

If a symmetric Θ has a singularities at a point x not of order 2, it is also singular at $-x$. Thus $\overline{N}_0$ breaks up:

$$\overline{N}_0 = \theta_{null} + 2.\overline{N}_0^*$$

where all multiplicities in the 2nd piece are divisible by 2. We can now state the main result of this paper:

<u>Theorem (2.10)</u>: <u>The divisor classes of</u> $\overline{N}_0$, θ_{null}, $\overline{N}_0^*$ <u>are given by</u>:

$$[\overline{N}_0] = \left(\frac{(g+1)!}{2} + g!\right)\lambda - \frac{(g+1)!}{12}\delta$$

$$[\theta_{null}] = 2^{g-2}(2^g+1)\lambda - 2^{2g-5}.\delta$$

$$[\overline{N}_0^*] = \left[\frac{(g+1)!}{4} + \frac{g!}{2} - 2^{g-3}(2^g+1)\right]\lambda - \left[\frac{(g+1)!}{24} - 2^{2g-6}\right]\delta .$$

Here is a table for low degrees:

g	$[\overline{N}_0]$	$[\theta_{null}]$	$[N_0^*]$	slope
2	$5\lambda-\frac{1}{2}\delta$	$5\lambda-\frac{1}{2}\delta$	0	—
3	$18\lambda-2\delta$	$18\lambda-2\delta$	0	—
4	$84\lambda-10\delta$	$68\lambda-8\delta$	$8\lambda-\delta$	8
5	$480\lambda-60\delta$	$264\lambda-32\delta$	$108\lambda-14\delta$	7.71
6	$3{,}240\lambda-420\delta$	$1{,}040\lambda-128\delta$	$1{,}100\lambda-146\delta$	7.53
7	$25{,}200\lambda-3{,}360\delta$	$4{,}128\lambda-512\delta$	$10{,}536\lambda-1{,}424\delta$	7.40

Note that the figures imply $\overline{N}_0^* = \emptyset$ for $g = 2,3$ as is well known. We also see that the divisor class of $\overline{N}_0^*$ is the same as that of the Jacobian locus for $g = 4$, confirming Beauville's results. The last column, "slope", refers to the ratio of the coefficient of λ to the coefficient of δ. As soon as this drops below the same ratio for K, $\mathcal{A}_g$ is of general type:

<u>Corollary (2.11)</u>. $\frac{(g+1)!}{12} K_{\overline{\mathcal{A}}_g}(1) \equiv [\overline{N}_0] + g!(g^2-4g-17)\lambda$.

<u>Proof</u>: Combine 1.7 and 2.10.

<u>Corollary (2.12)</u>. <u>If</u> $g \geq 7$, $\mathcal{A}_g$ <u>is of general type</u>.

<u>Proof</u>: Combine 1.4 and 2.11.

§3. Proof of the Theorem.

Now how are we going to prove the Theorem? The formula for $[\theta_{null}]$ is immediate, because we know the modular form that cuts out this divisor, viz.:

$$f(\Omega) = \prod_{\substack{a,b \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z} \\ {}^t(2a).(2b)\ \text{even}}} \theta[{a \atop b}](0,\Omega)$$

where

$$\theta[{a \atop b}](0,\Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i\, {}^t(n+a)\Omega(n+a) + 2\pi i\, {}^t(n+a).b} .$$

Each θ is a modular form of weight 1/2 and there are $2^{g-1}(2^g+1)$ "even" pairs a,b so f has weight $2^{g-2}.(2^g+1)$, and this is the coefficient of λ. On the other hand, if $\operatorname{Im}\Omega_{11} \longrightarrow \infty$, we see that if $a_1 = 0$, $\lim \theta[{a \atop b}] = 1$, while if $a_1 = \frac{1}{2}$, $\theta[{a \atop b}]$ is divisible by

$$e^{\pi i \Omega_{11}/4}$$

hence it goes to zero. The equation of Δ is $e^{2\pi i\Omega_{11}} = 0$, and there are 2^{2g-2} "even" pairs a,b with $a_1 = \frac{1}{2}$ (take any $a_2, b_2, \cdots, a_g, b_g$, set $a_1 = \frac{1}{2}$ and make b_1 zero or one-half to force a,b to be even). Thus f goes to zero like

$$(e^{2\pi i\Omega_{11}})^{(2^{2g-5})}$$

when $\operatorname{Im}\Omega_{11} \longrightarrow \infty$, hence the coefficient of δ.

It remains to prove the formula for $[\overline{N}_0]$. The value of the coefficient of λ follows from:

Proposition 3.1: Let

$$\begin{array}{ccc} \mathfrak{X} & \supset & \mathfrak{D} \\ \varepsilon \uparrow \downarrow p & & \\ C & & \end{array}$$

be a family of p.p.a.v. over a complete curve C such that every theta divisor $\mathfrak{D}_t$ has only a finite number of singularities and the generic $\mathfrak{D}_\eta$ is smooth. Let this family define the morphism

$$\varphi: \quad C \longrightarrow \mathcal{A}_g.$$

Then

$$\varphi^* N_0 \equiv \left(\frac{(g+1)!}{2} + g!\right)\varphi^*\lambda + \text{torsion}.$$

(Note that such a family exists because codim $N_1 \geq 2$ and because in Satake's compactification, the whole boundary has codim ≥ 2). The coefficient of δ, on the other hand follows from:

Proposition 3.2: Let (A,Θ) be a p.p.a.v. Then the divisor class of $c(B,\Theta)$ is given by:

$$c(B,\Theta) \equiv \frac{(g+2)!}{6} \cdot \Theta$$

together with Proposition 1.8.

To prove 3.1, we use the exact sequence

$$T_{\mathfrak{X}/C} \longrightarrow \mathcal{O}_{\mathfrak{X}}(\mathfrak{D})/\mathcal{O}_{\mathfrak{X}} \longrightarrow \mathcal{O}_{\mathrm{Sing}_{\mathrm{vert}}\mathfrak{D}} \otimes \mathcal{O}_{\mathfrak{X}}(\mathfrak{D}) \longrightarrow 0$$

used to define multiplicities for N_0. It follows that $\mathrm{Sing}_{\mathrm{vert}}\mathfrak{D}$ is the scheme of zeroes of a section of

$$\Omega^1_{\mathfrak{X}/C}(\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{D}}$$

hence

$$\varphi^* N_0 = p_*(c_g(\Omega^1_{\mathfrak{X}/C}(\mathfrak{D})) . \mathfrak{D}) .$$

But if $\mathfrak{E} = p_*(\Omega^1_{\mathfrak{X}/C})$, then the bundle $\Omega^1_{\mathfrak{X}/C}$, being trivial on each fibre of $\mathfrak{X}$ over C, is isomorphic to $p^*\mathfrak{E}$. Moreover, by definition of λ,

$$\varphi^*\lambda = c_1(\mathfrak{E}).$$

Thus

$$\begin{aligned}\varphi^* N_0 &= p_*\left(c_g(p^*\mathfrak{E} \otimes \mathcal{O}_{\mathfrak{X}}(\mathfrak{D})) . \mathfrak{D}\right) \\ &= p_*\left((\mathfrak{D}^g + \mathfrak{D}^{g-1} . c_1(p^*\mathfrak{E})) . \mathfrak{D}\right) \\ &= p_*(\mathfrak{D}^{g+1}) + p_*(\mathfrak{D}^g) \cdot c_1(\mathfrak{E}) \quad .\end{aligned}$$

Now on each fibre $\mathfrak{D}$ is Θ and $(\Theta^g) = g!$, so the second term is $g!\varphi^*(\lambda)$. To compute the first, we apply the Grothendieck-Riemann-Roch theorem to $\mathcal{O}_{\mathfrak{X}}(\mathfrak{D})$. Note that

$$p_*(\mathcal{O}_{\mathfrak{X}}(\mathfrak{D})) \cong \mathcal{O}_C$$

$$R^i p_*(\mathcal{O}_*(\mathfrak{D})) = (0), \ i \geq 1.$$

Thus

$$\begin{aligned} 1 &= ch(p_!(\mathcal{O}_{\mathfrak{X}}(\mathfrak{D}))) \\ &= p_*(ch\,\mathcal{O}_{\mathfrak{X}}(\mathfrak{D}).Td(\Omega^1_{\mathfrak{X}/C})) \\ &= p_*(e^{\mathfrak{D}}.p^*(1-\frac{c_1(\mathfrak{E})}{2})), \quad \text{mod torsion.} \end{aligned}$$

In codimension 1 on C, this says

$$0 = p_*(\frac{\mathfrak{D}^{g+1}}{(g+1)!}) - \frac{c_1(\mathfrak{E})}{2} \cdot p_*(\frac{\mathfrak{D}^g}{g!})$$

or

$$p_*(\mathfrak{D}^{g+1}) = \frac{(g+1)!}{2} c_1(\mathfrak{E}) \quad \text{mod torsion.}$$

This proves 3.1.

To prove 3.2, it suffices to establish the numerical equivalence of the 2 divisors. Namely, this will prove Theorem 2.10, and then Theorem 2.10 will imply Prop. 3.2 as an equality of divisor classes. Let $C \subset A$ be any curve. We shall calculate $(C.c(B,\Theta))$. Consider the map

$$C \times \Theta \xrightarrow{\ m\ } A$$

$$m(x,y) = x + y .$$

Then $m^{-1}(\Theta)$ is the locus of pairs (x,y) where $x+y \in \Theta$, i.e., $x = y'-y$, where $x \in C, y, y' \in \Theta$. The differential of m gives us a map

$$dm: \ p_2^* T_\Theta \otimes \mathcal{O}_{m^{-1}\Theta} \longrightarrow T_A \otimes \mathcal{O}_\Theta \longrightarrow N_{\Theta,A}$$

whose zeroes are exactly the points (x,y) such that not only is $x = y'-y$, $y,y' \in \Theta$, but also $T_{y,\Theta} = T_{y',\Theta}$, i.e., $x \in \underline{D}$. Now the above dm can be thought of as a section of

$$p_2^*\Omega_\Theta^1 \otimes m^*(N_{\Theta,A}) \otimes \underline{O}_{m^{-1}(\Theta)}$$

hence

$$(C.\underline{D}) = c_{g-1}\left(p_2^*\Omega_\Theta^1 \otimes m^*(\underline{O}(\Theta)) \otimes \underline{O}_{m^{-1}\Theta}\right).$$

Let $\theta_1 = \text{pt.}\times\Theta$, $\theta_2 = m^{-1}\Theta$ be these divisor classes (mod numerical equivalence) on $C\times\Theta$. Then

$$(C.\underline{D}) = c_{g-1}\left(p_2^*\Omega_\Theta^1 \otimes \underline{O}(\theta_2)\right).\theta_2.$$

Using

$$0 \longrightarrow \underline{O}(-\Theta)/\underline{O}(-2\Theta) \longrightarrow \Omega_A^1\big|_\Theta \longrightarrow \Omega_\Theta^1 \longrightarrow 0,$$

we see that

$$c(\Omega_\Theta^1) = (1-\Theta)^{-1}\big|_\Theta = (1+\Theta+\Theta^2+\cdots\cdots)\big|_\Theta .$$

Thus

$$(C.\underline{D}) = \theta_1^{g-1}.\theta_2 + \theta_1^{g-2}.\theta_2^2 + \cdots\cdots + \theta_2^g .$$

But now

$$(\theta_1^k\cdot\theta_2^{g-k})_{C\times\Theta} = (m(\theta_1^{k+1})\cdot \Theta^{g-k})_A$$
$$= ((C \dotplus \Theta^{k+1}).\Theta^{g-k})_A$$

if $\dotplus$ is Pontryagin product. By symmetry of Θ, this is

$$= (C.(\Theta^{k+1} \dotplus \Theta^{g-k}))_A$$
$$= (C.(k+1)(g-k)(g-1)!\Theta)_A .$$

Thus

$$(C.\underline{D}) = (C.\Theta)(g-1)!\sum_{k=0}^{g-1}(k+1)(g-h)$$
$$= \frac{(g+2)!}{6}(C.\Theta).$$

<u>QED</u>

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