On the Kodaira Dimension of the Moduli Space of Curves.
Harris, Joe; Mumford, David
pp. 23 - 86

NIEDERSÄCHSISCHE STAATS- UND UNIVERSITÄTSBIBLIOTHEK GÖTTINGEN

Terms and Conditions

The Göttingen State and University Library provides access to digitized documents strictly for noncommercial educational, research and private purposes and makes no warranty with regard to their use for other purposes. Some of our collections are protected by copyright. Publication and/or broadcast in any form (including electronic) requires prior written permission from the Goettingen State- and University Library.
Each copy of any part of this document must contain there Terms and Conditions. With the usage of the library's online system to access or download a digitized document you accept there Terms and Conditions.
Reproductions of material on the web site may not be made for or donated to other repositories, nor may be further reproduced without written permission from the Goettingen State- and University Library
For reproduction requests and permissions, please contact us. If citing materials, please give proper attribution of the source.

Contact:

Niedersächsische Staats- und Universitätsbibliothek
Digitalisierungszentrum
37070 Goettingen
Germany
Email: gdz@www.sub.uni-goettingen.de

Purchase a CD-ROM

The Goettingen State and University Library offers CD-ROMs containing whole volumes / monographs in PDF for Adobe Acrobat. The PDF-version contains the table of contents as bookmarks, which allows easy navigation in the document. For availability and pricing, please contact:
Niedersaechisische Staats- und Universitaetsbibliothek Goettingen - Digitalisierungszentrum
37070 Goettingen, Germany, Email: gdz@www.sub.uni-goettingen.de

Invent. math. 67, 23–86 (1982)

Inventiones mathematicae
© Springer-Verlag 1982

On the Kodaira Dimension of the Moduli Space of Curves

Joe Harris and David Mumford

The purpose of this paper is to prove that the moduli space $\mathcal{M}_g$ of curves of genus g over $\mathbf{C}$ is of general type if g is odd and $g \geqq 25$. Moreover, the Kodaira dimension is at least 0 if $g=23$. It appears that a variant of our technique, which is technically more difficult, will prove that $\mathcal{M}_g$ is of general type for all sufficiently large g. In order to keep this paper to a reasonable length, we are treating only the odd genus case here, postponing the even genus case to a later paper.

The result of this paper should be contrasted with an earlier weaker result [12] to the effect that $\mathcal{M}_g$ is of "log general type" in the sense that there are enough n-canonical forms[1] ω on $\mathcal{M}_g$-Sing$(\mathcal{M}_g)$ with logarithmic poles at infinity (i.e., if $\overline{\mathcal{M}}_g - \mathcal{M}_g = V(x_1)$, then $\omega = a(x)(dx_1 \wedge \ldots \wedge dx_{3g-3}/x_1)^{\otimes n}$) to separate points generically. By the results of Brylinski [3], this implies that for *all* g, moduli spaces $\mathcal{M}_g^{(\alpha)}$ of sufficiently high level α are of general type (the levels in question are non-abelian levels).

The proof is based on the analysis of a special divisor $D_k \subset \mathcal{M}_g$ with a very natural *geometric* meaning:

$$D_k = \left(\text{locus of curves } C \text{ which are } k\text{-fold covers of } \mathbf{P}^1 \text{ where } k = \frac{g+1}{2}\right).$$

We consider the closure $\overline{D}_k$ of D_k in $\overline{\mathcal{M}}_g$ and compute the divisor class of $\overline{D}_k$ in terms of the basic divisor classes

$$\lambda, \delta_0, \delta_1, \ldots, \delta_{[g/2]} \in \mathrm{Pic}(\mathcal{M}_g) \otimes \mathbf{Q}$$

introduced in [13]. Here λ is c_1 of the "Hodge bundle" L_λ, where L_λ has fibre $\Lambda^g H^0(\omega_C)$ over a curve C, and δ_i are the divisor classes of components Δ_i of $\overline{\mathcal{M}}_g - \mathcal{M}_g$. In particular, Δ_0 is the closure of the locus of *irreducible* singular curves. The final result of § 5 will be:

[1] By an n-canonical form on a smooth variety X^r, we mean a differential form locally expressed as

$$a(x_1, \ldots, x_r)(dx_1 \wedge \ldots \wedge dx_r)^{\otimes n}$$

0020-9910/82/0067/0023/$12.80

$$\text{Class of } \overline{D}_k \equiv \frac{(2k-4)!}{k!(k-2)!}\left\{6(k+1)\lambda - k\delta_0 - \sum_{\alpha=1}^{k-1} 3\alpha(2k-1-\alpha)\delta_\alpha\right\}.$$

On the other hand, using Grothendieck's Riemann-Roch formula, we calculate in § 2 the canonical class of $\overline{\mathcal{M}}_g$:

$$K_{\overline{\mathcal{M}}_g} \equiv 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \ldots - 2\delta_{[\frac{g}{2}]}.$$

Putting these together proves

$$c_k K_{\overline{\mathcal{M}}_g} \equiv 2\overline{D}_k + c_k\left(1-\frac{12}{k}\right)\lambda + \begin{pmatrix}\text{comb. of } \delta_1, \ldots, \delta_{[\frac{g}{2}]} \\ \text{with positive coefficients}\end{pmatrix}$$

if $c_k = (2k-4)!/(k-1)!(k-2)!$. It turns out also that the singularities of $\overline{\mathcal{M}}_g$ are sufficiently mild so that all divisors in $|nK_{\overline{\mathcal{M}}_g}|$ define n-canonical differential forms on $\overline{\mathcal{M}}_g$ without poles on the resolution of $\overline{\mathcal{M}}_g$. Thus if $|nK|_{\text{bir}}$ denotes the linear system of birationally holomorphic forms, we have

$$|nc_k K_{\overline{\mathcal{M}}_g}|_{\text{bir}} \supset (\text{eff. div.}) + \left|nc_k\left(1-\frac{12}{k}\right)\lambda\right|.$$

Since λ is ample on Satake's compactification $\overline{\mathcal{A}}_g$, the sections of $L_\lambda^{\otimes n}$ on $\overline{\mathcal{M}}_g$ define a birational morphism to $\mathbf{P}^N$. Therefore if $k=12$, $|c_k K_{\overline{\mathcal{M}}_g}|_{\text{bir}} \neq \phi$, and if $k>12$, $|nc_k K_{\overline{\mathcal{M}}_g}|_{\text{bir}}$ defines a birational map for $n \gg 0$.

The idea of this proof owes a great deal to *E.* Freitag and Y.-S. Tai, although most of the links to their work are not apparent. First and foremost, it was Freitag who saw beyond the classical picture according to which geometrically natural moduli spaces all seemed to be unirational, although their "higher level" variants were of general type: a picture largely based on the heavily studied case of $\mathcal{M}_1$, the moduli space of elliptic curves. Freitag showed that the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties is not unirational for $g \equiv 1\,(8)$, $g \geqq 17$ [4], and for $24|g$ [5]. Tai then showed that, in fact, $\mathcal{A}_g$ is of general type for all $g \geqq 9$ ([17] to be published). He introduced 2 important new techniques: the first was the proof that the singularities of $\mathcal{A}_g$ "didn't matter", except for the one along the 1st boundary component $\mathcal{A}_{g-1}$. In particular, Tai and, independently, Reid [15] found an important hypothesis on the action of a finite group G on $\mathbf{C}^r$ which implies that n-canonical differential forms on $\mathbf{C}^r/G$ have no poles on the resolution of $\mathbf{C}^r/G$. We use this criterion again here. Secondly, by means of the Hirzebruch Proportionality theorem, Tai proved that if g is large enough, there are Siegel modular forms of weight $n(g)$, vanishing at $\mathcal{A}_{g-1}$ to order $m(g)$ with $m(g)/n(g)$ *arbitrarily large.* Subsequent to this, E. Freitag and one of the authors discussed at length how these techniques could be extended to $\mathcal{M}_g$. They sought to use mixed holomorphic tensors on $\mathcal{A}_g$, i.e., differential forms which are not exterior forms, or symmetric forms, but sections of the full tensor power $(\Omega^1_{\mathcal{A}_g})^{\otimes n}$, so as to produce from theta series n-canonical differential forms on $\mathcal{M}_g$. This still looks quite hopeful. But what led directly to the present paper was the discovery of a particular holomorphic tensor on $\mathcal{A}_g$ which defines a map from the ring of

Siegel modular forms vanishing at $\mathscr{A}_{g-1}$ to sufficiently large order to the pluricanonical ring of $\mathscr{M}_g$. (It is still not clear whether or not all Siegel modular forms with this vanishing at $\mathscr{A}_{g-1}$ do not, by some fluke, also vanish on the locus of Jacobians, in which case this map is zero.) But the forms defined by this map must at least vanish on a certain divisor $D' \subset \mathscr{M}_g$:

$$D' = \left\{ \begin{array}{l} \text{locus of curves } C \text{ with a line bundle } L \text{ such that} \\ h^0(L) \geqq 2,\ h^0(\Omega^1_{\mathbf{C}} \otimes L^{-2}) \geqq 1 \end{array} \right\}.$$

If g is odd, the D above is one of the components of D'. In this roundabout way, Freitag's techniques pointed directly to the method of the present proof!

This paper suggests that there is much interest in a systematic investigation of the relations between the fundamental classes of the many subvarieties of $\mathscr{M}_g$ defined by geometric conditions, with the hope of establishing a calculus of such cycles. Such a study was in fact first undertaken by Steven Diaz, who is investigating the global geometry of the locus in $\overline{\mathscr{M}}_g$ of curves with abnormal Weierstrass points; his work has been extremely valuable in developing the techniques of this paper. Hopefully, a more complete picture of the geometry of $\mathscr{M}_g$ will emerge from such investigations; a great deal of work remains to be done. Another important question is whether $\mathscr{M}_g$ actually carries non-zero holomorphic *exterior* p-forms for some p. The analogous theory for $\mathscr{A}_g$ suggests that this *might* happen for $p=g$, $2g-1$ or $3g-3$ (cf. Anderson [1], Stillman [16]).

The paper is organized as follows. In § 1, we use the Reid-Tai criterion to prove that n-canonical forms on the open set $\overline{\mathscr{M}}_g^0$ parametrizing curves without automorphisms automatically are holomorphic on a resolution of $\overline{\mathscr{M}}_g$. In § 2, we compute $K_{\overline{\mathscr{M}}_g}$. In § 3 and § 6, we calculate the class of $\overline{D}_k$, using a compactification of the Hurwitz moduli scheme of k-fold coverings, developed in § 4, and using counts of the number of pencils of certain types on a generic curve of genus g, due to Griffiths and Harris, developed in § 5.

In order to indicate the significance of this result we would like to point out the following easy result:

Proposition. *Assume for some g that the Kodaira dimension of $\mathscr{M}_g$ is at least* 0. *Then if C is a generic curve of genus g* (i.e. *the corresponding point $[C] \in \mathscr{M}_g$ lies in no subvariety defined over* $\mathbf{Q}$), *and F is an algebraic surface containing C on which C moves in a non-trivial linear system, then F is birational to $C \times \mathbf{P}^1$.*

Proof. The Kodaira dimension being at least 0 means that $\mathscr{M}_g$ carries a n-canonical differential $a(x)\ (dx_1 \wedge \ldots \wedge dx_{3g-3})^{\otimes n}$ with no poles on a compact smooth model of $\mathscr{M}_g$. This implies that $\mathscr{M}_g$ is not "uniruled", i.e. there is no dominant rational map

$$\mathbf{P}^1 \times W^{3g-4} \to \mathscr{M}_g.$$

Therefore the images of all non-constant maps

$$\mathbf{P}^1 \to \mathscr{M}_g$$

lie on subvarieties of $\mathcal{M}_g$ defined over $\mathbf{Q}$. Therefore, given $C \in |C|$ on F, a pencil in $|C|$ defines a map from $\mathbf{P}^1$ to $\mathcal{M}_g$ through $[C]$, which must be constant, i.e. all $C' \in |C|$ are isomorphic to C. Since C has no automorphism, this means that F is birational to $C \times \mathbf{P}^1$. Q.E.D.

Corollary. *If g is odd, $g \geqq 23$, then a generic curve of genus g does not occur in a non-trivial linear system on any non-ruled surface.*

§ 1. Pluri-canonical Forms on $\overline{\mathcal{M}}_g$

If V^n is any quasi-projective variety, by k-canonical forms on V we understand holomorphic tensors ω given on the open set V_{reg} of smooth points of V by

$$\omega = a(x_1, \ldots, x_n)(dx_1 \wedge \ldots \wedge dx_n)^{\otimes k}$$

such that for one and hence all desingularizations

$$\pi\colon \tilde{V} \to V$$

of V, ω extends to a holomorphic tensor of this type on all of $\tilde{V}$. Pluri-canonical forms[2] refers to k-canonical forms for all k. If V is smooth, then $\mathcal{O}_V(nK)$ is the sheaf of n-canonical forms on V.

As in the introduction, $\overline{\mathcal{M}}_g$ stands for the coarse moduli space of stable curves of genus g: it is a normal projective variety. Related to $\overline{\mathcal{M}}_g$, we introduce the following further varieties:

$$\begin{array}{ll} & \overline{\mathcal{M}}_g \leftarrow \tilde{\mathcal{M}}_g = \text{desingularization of } \overline{\mathcal{M}}_g \\ & \bigcup \\ \text{open set of smooth pts. of } \mathcal{M}_g & = \overline{\mathcal{M}}_{g,\text{reg}} \\ & \bigcup \\ \text{open set of curves } C \text{ w/o automorphisms} & = \overline{\mathcal{M}}_g^0 \end{array}$$

The purpose of this section is to prove the following theorem:

Theorem 1. *If $g \geqq 4$, then for all n, every n-canonical form on $\overline{\mathcal{M}}_g^0$ extends to an n-canonical form on $\overline{\mathcal{M}}_g$. More precisely:*

$$\Gamma(\overline{\mathcal{M}}_g^0, \mathcal{O}_{\overline{\mathcal{M}}_g}(nK)) = \Gamma(\tilde{\mathcal{M}}_g, \mathcal{O}_{\tilde{\mathcal{M}}_g}(nK)).$$

Proof. Recall that locally $\overline{\mathcal{M}}_g$ can be described as follows: let C be a stable curve of genus g and let

$$\pi\colon \mathcal{C} \to \Delta^{3g-3}$$

[2] There is a general confusion of terminology for referring to these and other tensor forms which are not *exterior* forms. We suggest k-canonical forms as a good phrase to distinguish these from exterior k-forms or symmetric k-forms

be its local universal deformation space. $\mathrm{Aut}(C)$ is a finite group which operates on $\mathscr{C}$ and Δ^{3g-3}. Then a neighborhood of the point $[C]\in\overline{\mathscr{M}}_g$ defined by C is isomorphic as analytic space to

$$\Delta^{3g-3}/\mathrm{Aut}(C).$$

Moreover the action of a finite group on a smooth space can always be made linear in suitable coordinates, hence if $T_{0,\Delta}$ is the tangent space to Δ^{3g-3} at 0,

$$\Delta^{3g-3}/\mathrm{Aut}(C)\cong\text{neigh. of } 0 \text{ in } [T_{0,\Delta}/\mathrm{Aut}(C)].$$

On the other hand, $T_{0,\Delta}$ is the space of infinitesimal deformations of C which is well known to be

$$\mathrm{Ext}^1(\Omega^1_C, \mathscr{O}_C)$$

where Ω^1_C is the sheaf of Kähler differentials on C. By Serre duality,

$$\mathrm{Ext}^1(\Omega^1_C, \mathscr{O}_C)\cong H^0(\Omega^1_C\otimes\omega_C)^*$$

where ω_C is the dualizing sheaf on C[3]. Thus finally

$$(\text{neigh. of } [C] \text{ in } \overline{\mathscr{M}}_g)\cong(\text{neigh. of } 0 \text{ in } H^0(\Omega^1_C\otimes\omega_C)^*/\mathrm{Aut}(C)).$$

Note that in this description, the open set of points on the left which are in $\mathscr{M}^0_g$ is equal to the open set of points on the right where $\mathrm{Aut}(C)$ acts freely.

We have therefore particular cases of the problem:

V a vector space of dimension d

$G\subset GL(V)$ a finite group

$V_0\subset V$ open set where G acts freely

When do n-canonical forms on V_0/G extend holomorphically to a resolution $\widetilde{V/G}$ of V/G?

The following criterion is due independently to M. Reid [15] and Y.-S. Tai [17]:

Reid-Tai Criterion: In the above situation, for all $g\in G$, let g be conjugate to

$$\begin{pmatrix} \zeta^{a_1} & & 0 \\ & \ddots & \\ 0 & & \zeta^{a_d} \end{pmatrix}$$

where ζ is a primitive m^{th} root of 1, *and* $0\leqq a_i<m$. *If for all g and ζ,*

---

[3] If C is smooth, $\omega_C\cong\Omega^1_C$. But at double points given by $xy=0$, Ω^1_C is generated by dx, dy mod one relation $x\,dy+y\,dx=0$, and has the torsion submodule $\mathbf{C}\cdot(x\,dy)$, while ω_C is free on one generator given by the differentials $\frac{dx}{x}$ on $y=0$, $-\frac{dy}{y}$ on $x=0$

$$\sum_{i=1}^{d} a_i/m \geqq 1$$

then any n-canonical form on V_0/G extends holomorphically to $\widetilde{V/G}$.

In fact, we need a slight generalization of the Reid-Tai criterion. This says that for *any* G-action, certain n-canonical forms ω extend to $\widetilde{V/G}$, namely those which are holomorphic on V_0/G *and*, for all $g \in G$ for which $\sum a_i/m < 1$ (for some ζ) – call these bad g's – ω is also holomorphic on the divisors $E \subset \widetilde{V/G}$ mapping onto

$$\operatorname{Im}(\{x \in V \mid g\,x = x\} \to V/G).$$

Equivalently, we must assume ω to be holomorphic on the resolution of some open set $U \subset V/G$, where U contains V_0/G and U contains the image in V/G of the generic point η_g of $\{x \in V \mid g\,x = x\}$, for all *bad* g. We will give a proof in appendix 1 to this section.

Our main task is therefore to investigate, case-by-case, what are the eigenvalues of an automorphism ϕ of a stable curve C acting on $H^0(\Omega_C^1 \otimes \omega_C)$. We take the case of a smooth C first.

Proposition. *Let C be a smooth curve over* **C**, *and let ϕ be an automorphism of C of order n. Let ζ be any primitive n^{th} root of* 1, *and let the action ϕ on $H^0(\mathscr{O}_C(2K))^*$ be given by*

$$\begin{pmatrix} \zeta^{a_1} & & 0 \\ & \ddots & \\ 0 & & \zeta^{a_{3g-3}} \end{pmatrix} \quad \textit{where } 0 \leqq a_i < n.$$

Then either

$$\sum_{i=1}^{3g-3} (a_i/n) \geqq 1$$

or else (C, ϕ) is one of the following cases:

i) genus $C = 0$

ii) genus $C = 1$

iii) genus $C = 2$, $n = 2$, *ϕ is the hyperelliptic involution*

iv) genus $C = 2$, *C is a double cover of an elliptic curve and ϕ is the associated involution*

v) genus $C = 3$, *C is hyperelliptic and ϕ is the hyperelliptic involution.*

Proof. Let C_0 be the quotient of C by ϕ, let g_0 be the genus of C_0 and let $P_1, \ldots, P_\beta \in C_0$ be the branch points of the cover

$$\pi\colon\ C \to C_0.$$

Let m_i = order of branching of π over P_i: i.e., there are n/m_i points $Q_{i,\alpha} \in C$ over P_i and π is given locally at each $Q_{i,\alpha}$ by the m_i^{th} root of a suitable local parameter at P_i.

Consider the family of cyclic branched coverings

$$\pi'\colon\ C' \to C'_0$$

which you get by varying the moduli of C_0 and varying the branch points $P_i \in C_0$. The dimension of this family is $3g_0 - 3 + \beta$. This defines a subvariety $W \subset \mathscr{M}_g$ of codimension $3(g-g_0)-\beta$ containing C such that all $C' \in W$ admit automorphisms ϕ' deforming ϕ. On the other hand, if the local deformation space of C is $p\colon \mathscr{C} \to S$, S a germ of smooth $(3g-3)$-dimensional manifold, then ϕ acts naturally on $\mathscr{C}$ and S:

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\phi_2"] \arrow[d, "p"'] & \mathscr{C} \arrow[d, "p"] \\ S \arrow[r, "\phi_1"'] & S \end{tikzcd}$$

In suitable coordinates on S, ϕ_1 acts linearly. The subvariety W of $\mathscr{M}_g$ is just the image of the fixed point set of ϕ_1 on S. Therefore, the dimension of W is the dimension of the subspace of $T_{0,S} \cong H^0(\mathscr{O}_C(2K))$ fixed by ϕ, i.e., the number of a_i equal to 0. This proves:

$$\#\{i \mid a_i = 0\} = 3g_0 - 3 + \beta,$$

$$\#\{i \mid a_i \geqq 1\} = 3(g-g_0) - \beta.$$

Therefore

$$\sum_{i=1}^{3g-3} (a_i/n) \geqq \frac{3(g-g_0)-\beta}{n}.$$

But by Hurwitz's formula:

$$2g-2 = n(2g_0-2) + \sum_{i=1}^{\beta} \frac{n}{m_i}(m_i - 1).$$

Let's now assume $\sum (a_i/n) < 1$, hence in particular

$$n > 3(g-g_0) - \beta.$$

Combining this with Hurwitz's formula, you easily check:

$$\frac{2}{3} > \frac{2n-2}{n}(g_0-1) + \sum_{i=1}^{\beta} \left(1 - \frac{2}{3n} - \frac{1}{m_i}\right). \tag{$*$}$$

Note that as $n \geqq 2$, $m_i \geqq 2$,

$$1 - \frac{2}{3n} - \frac{1}{m_i} \geqq \frac{1}{6} > 0,$$

hence $(*)$ implies immediately that $g_0 \leqq 1$.

Moreover, the m_i cannot be chosen as arbitrary divisors of n because of the following:

Lemma. *Let* $M = \text{l.c.m.}\,(m_i)$. *For all primes* p *dividing* M, *let* $p^r | M$, $p^{r+1} \nmid M$, *and let*

$$I = \{i \,|\, p^r | m_i\}.$$

Then I *has at least* 2 *elements in it, and if* $p=2$, I *consists of an even number of* i*'s. Moreover, if* $g_0 = 0$, $n = M$.

Proof. The covering C of C_0 is defined by a surjective homomorphism:

$$\pi_1(C_0 - \{P_1, \ldots, P_\beta\}) \twoheadrightarrow \mathbf{Z}/n\mathbf{Z}.$$

This factors through $H_1(C_0 - \{P_1, \ldots, P_\beta\})$ because $\mathbf{Z}/n\mathbf{Z}$ is abelian. Let e_i be a small loop around P_i. Then $H_1(C_0 - \{P_1, \ldots, P_\beta\})$ is the direct sum of $H_1(C_0)$ and $\oplus \mathbf{Z} e_i$ modulo the one relation $\sum e_i = 0$. Let e_i be mapped to $\bar{e}_i \in \mathbf{Z}/n\mathbf{Z}$. It's easy to see that m_i is exactly the order of $\bar{e}_i$. But

a) $\sum_{i=1} \bar{e}_i = 0$ in $\mathbf{Z}/n\mathbf{Z}$

and

b) if $g_0 = 0$, hence $H_1(C_0) = (0)$, $\{\bar{e}_i\}$ generate $\mathbf{Z}/n\mathbf{Z}$.

With the notation in the lemma, if $m = M/p$

$$\sum_{i=1}^{\beta} m\bar{e}_i = 0 \quad \text{and} \quad m\bar{e}_i = 0 \text{ if } i \notin I.$$

Therefore $\# I \geqq 2$. And if $p=2$, $m\bar{e}_i$ is in the subgroup $\mathbf{Z}/2\mathbf{Z}$ of $\mathbf{Z}/n\mathbf{Z}$, hence $\# I$ is even. Also, if $g_0 = 0$, then $M = n$ by (b).

In particular, this shows that $\beta = 0$ or $\beta \geqq 2$, and if $n = 2$, then β is even. Using these restrictions on the m_i, a lengthy but straightforward calculation allows one to *list all the solutions* to $(*)$. The result is:

a) $g_0 = 1$, $\beta = 0$

b) $g_0 = 1$, $n = \beta = 2$, all $m_i = 2$

c) $g_0 = 0$, $n = 2$, $\beta = 2$, 4, 6 or 8, all $m_i = 2$

d) $g_0 = 0$, $\beta = 2$, any n

e) $g_0 = 0$, $\beta = 3$ with one of the following triples (m_1, m_2, m_3):

$(2, 4k, 4k)$	$n = 4k$
$(2, 2k, k)$, k odd	$n = 2k$
$(3, 3k, 3k)$	$n = 3k$
$(3, 3k, k)$, $3 \nmid k$	$n = 3k$
$(4, 4k, k)$, k odd, $5 \leqq k \leqq 13$	
$(4, 4k, 2k)$, $k = 3, 5, 7$	
$(4, 8k, 8k)$, $k = 1, 2$	
$(5, 5k, 5k)$, $k = 1, 2$	
$(5, 5k, k)$, $k = 6, 7, 8$	
$(7, 7, 7)$	

f) $g_0=0$, $\beta=4$ with one of the following quadruples (m_1, m_2, m_3, m_4):

$$\begin{array}{lll} (2,2,k,k), & k=3,5, & n=2k \\ (2,2,2k,2k), & k=2,3, & n=2k \\ (3,3,3,3), & & n=3 \\ (2,6,3,3), & & n=6. \end{array}$$

The Proposition asserts that a), b), c) and d) are in fact the only cases where $\sum a_i/n<1$. In cases e) and f) we need to look more closely and evaluate the a_i's.

For (f), we can describe the curves C which occur as follows:

for $(2,2,k,k)$, k odd, C and ϕ are

$$y^2=(x^k-1)(x^k-a)$$

$$\phi(x,y)=(\zeta_k x, -y),\ \zeta_k \text{ a primitive } k^{\text{th}} \text{ root of } 1$$

for $(2,2,2k,2k)$, C and ϕ are

$$y^2=x(x^k-1)(x^k-a)$$

$$\phi(x,y)=(\zeta_k x, \zeta_{2k} y),\ \zeta_{2k}^2=\zeta_k$$

for $(3,3,3,3)$, C is

$$y^2=(x^3-1)(x^3-a) \text{ again}$$

but ϕ is

$$\phi(x,y)=(\zeta_3 x, y),\ \zeta_3 \text{ a primitive } 3^{\text{rd}} \text{ root of } 1,$$

for $(2,6,3,3)$, C is the non-hyperelliptic curve of genus 3

$$y^3=(x^2-1)(x^2-a)$$

$$\phi(x,y)=(-x, \zeta_3 y).$$

From this description, one calculates the a_i's in this table:

(m_1, m_2, m_3, m_4)	(a_i/n)
$(2,2,3,3)$	$(0/6, 2/6, 4/6)$
$(2,2,4,4)$	$(0/4, 2/4, 2/4)$
$(2,2,5,5)$	$(\frac{0}{10}, \frac{1}{10}, \frac{2}{10}, \frac{4}{10}, \frac{4}{10}, \frac{6}{10}, \frac{6}{10}, \frac{8}{10}, \frac{9}{10})$
$(2,2,6,6)$	$(\frac{0}{6}, \frac{2}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{4}{6})$
$(3,3,3,3)$	$(0/3, 1/3, 2/3)$
$(2,6,3,3)$	$(0/6, 1/6, 2/6, 3/6, 4/6, 4/6)$

which confirms the Proposition in this case. (Note that the ζ_k and ζ_3 used to describe ϕ may be *any* primitive k^{th} or 3rd root of 1, hence one must not merely check

$$\sum a_i/n \geqq 1$$

but also check that for all $j \in \{1, \ldots, n-1\}$ relatively prime to n,

$$\sum \mathrm{res}_n(j \cdot a_i)/n \geqq 1$$

where $\mathrm{res}_n(k)$ is the residue in $\{0, 1, \ldots, n-1\}$ of $k \bmod n$.)

For (e), a similar check could be carried out, but because of its tediousness and the possibility of error, it seemed easier *and* much more convincing to write a computer program to calculate $\sum a_i/n$ one at a time for all cyclic covers of $\mathbf{P}^1$ with 3 branch points. This is reproduced in appendix 2 where a table of the minimum values of $\sum(a_i/n)$ for each $g \leqq 18$ is also given. On the other hand, the computer only checks finitely many cases and the 1st 4 types in (e) are infinite families. These are the curves

$$\begin{array}{ll} y^2 = x^k - 1, & \phi(x, y) = (\zeta_k x, -y),\ k \text{ odd}, \\ y^2 = x(x^{2k} - 1), & \phi(x, y) = (\zeta_{2k} x, \zeta_{4k} y) \\ y^3 = x(x^k - 1), & \phi(x, y) = (\zeta_k x, \zeta_{3k} y) \\ y^3 = x^2(x^k - 1), & \phi(x, y) = (\zeta_k x, \zeta_{3k}^2 y) \end{array}$$

for which the a_i are readily worked out explicitly. We omit this.

Using this analysis as a building block, we look next at automorphism of singular stable curves:

Theorem 2. *Let C be a stable (possibly singular) curve of arithmetic genus g and let ϕ be an automorphism of C of order n. Let ζ be any primitive n^{th} root of 1 and let the action of ϕ on $H^0(\Omega_C^1 \otimes \omega_C)$ be given by*

$$\begin{pmatrix} \zeta^{a_1} & & 0 \\ & \ddots & \\ 0 & & \zeta^{a_{3g-3}} \end{pmatrix}, \quad \textit{where} \quad 0 \leqq a_i < n.$$

We assume $g \geqq 4$.

Then either

$$\sum_{i=1}^{3g-3} a_i/n \geqq 1$$

or else (C, ϕ) belongs to one of the following cases:

(i) $C = C_1 \cup C_2$ *where* $\mathrm{genus}(C_1) = g - 1$,

C_2 is either elliptic or rational with one node, $C_1 \cap C_2 = \{P\}$, $n = 2$ and

$$\phi|_{C_1} = \textit{identity}$$

$$\phi|_{C_2} = \textit{inverse with respect to origin } P$$

(ii) *$C = C_1 \cup C_2$ as above, but C_2 is elliptic with $j(C_2) = 0$, $n = 6$ and*

$$\phi|_{C_1} = \textit{identity}$$

$$\phi|_{C_2} = \textit{one of the two automorphisms of } C_2 \textit{ of order 6 fixing } P.$$

(iii) *$C=C_1\cup C_2$ as above, but C_2 is elliptic with $j(C_2)=12^3$, $n=4$ and*

$\phi|_{C_1}=$ identity

$\phi|_{C_2}=$ one of the two automorphisms of C_2 of order 4 fixing P.

Proof. Note that we have proved the theorem for smooth curves C. Also note that the set of exceptions (i), (ii) and (iii) forms a closed set in the moduli space $\overline{\mathcal{M}}_g$. Therefore we can prove the theorem by induction on the number of double points $P\in C$, checking for each (C,ϕ) that either

a) $\sum(a_i/n)\geqq 1$ or

b) (C,ϕ) has a deformation (C',ϕ') with fewer double points.

This is because in any family $\mathscr{C}\to S$ with connected base and with an automorphism $\phi\colon \mathscr{C}\to\mathscr{C}$ over S of order n, the eigenvalues of ϕ on $H^0(\Omega^1_{C_s}\otimes\omega_{C_s})$ vary continuously and are n^{th} roots of 1, hence are constant.

Note next $\Omega^1_C\otimes\omega_C$ has the following local description: at smooth points $P\in C$ with local coordinate x, it is an invertible sheaf with generator $dx^{\otimes 2}$; at ordinary double points $P\in C$, if C is given locally by $x\cdot y=0$, it is generated by the differentials:

$$\omega_1=dx^{\otimes 2}/x,$$

$$\omega_2=dy^{\otimes 2}/y$$

subject to the relation

$$y\omega_1=x\omega_2.$$

Thus

$$y\omega_1=\frac{y\,dx^{\otimes 2}}{x}=\frac{x\,dy^{\otimes 2}}{y}$$

generates a submodule of dimension 1 over k (because $x(y\omega_1)=0$ and $y(y\omega_1)=y(x\omega_2)=0$), and mod this we have the direct sum of the sheaves of quadratic differentials on the 2 branches at P, with simple poles at P. This gives rise to the exact sequence:

$$0\to\bigoplus_{\substack{\text{double pts.}\\ P\in C}}(\mathrm{tor}_P)\to\Omega^1_C\otimes\omega_C\to\bigoplus_\alpha\mathscr{O}_{C_\alpha}(2K_{C_\alpha}+\textstyle\sum P_\beta)\to 0$$

where C_α are the normalizations of the components of C, and for each α, the $P_\beta\in C_\alpha$ are those points of C_α whose image in C are double points of C. Therefore:

$$0\to\bigoplus_{\substack{\text{double}\\ \text{pts. }P}}(\mathrm{tor}_P)\to H^0(\Omega^1_C\otimes\omega_C)\to\bigoplus_\alpha H^0(\mathscr{O}_{C_\alpha}(2K_\alpha+\sum_\beta P_\beta))\to 0.$$

The first step is to analyze the eigenvalues of ϕ on $\oplus(\mathrm{tor}_P)$. Say P is a double point of C and $\{P,\phi P,\phi^2P,\dots,\phi^{m-1}P\}$ are distinct, $\phi^mP=P$, where $m|n$. Then ϕ^m acts on tor_P and if this action is trivial, take $\bar{e}\in(\mathrm{tor}_P)$ and consider the element $e\in H^0(\Omega^1_C\otimes\omega_C)$ given by

$$e\in\bigoplus_Q(\mathrm{tor}_Q),\qquad e_Q=\begin{cases}\phi^i\bar{e} & \text{if } Q=\phi^iP\\ 0 & \text{if } Q\notin\{P,\phi P,\dots,\phi^{n-1}P\}.\end{cases}$$

Then $\phi e = e$. Thus dually, e defines a deformation of C to which ϕ lifts and in which the double points $\phi^i P$ disappear. By induction on the number of double points, this case is taken care of. Next suppose ϕ^m acts non-trivially on (tor_P). We then calculate all the eigenvalues of ϕ on

$$(\mathrm{tor}_P) \oplus \ldots \oplus (\mathrm{tor}_{\phi^{m-1}P}).$$

Let ζ be a primitive n^{th} root of 1. We must have

$$\phi^m \bar{e} = \zeta^{m\ell} \bar{e}, \qquad 1 \leqq \ell < \frac{n}{m}.$$

Let

$$e_a = \sum_{i=0}^{m-1} \zeta^{ia} \cdot (\phi^i \bar{e}).$$

Then

$$\begin{aligned}
\phi(e_a) &= \sum_{i=0}^{m-1} \zeta^{ia} \cdot (\phi^{i+1} \bar{e}) \\
&= \sum_{i=1}^{m} \zeta^{(i-1)a} (\phi^i \bar{e}) \\
&= \zeta^{-a} \sum_{i=1}^{m-1} \zeta^{ia} (\phi^i \bar{e}) + \zeta^{(m-1)a} \cdot \zeta^{m\ell} \cdot \bar{e} \\
&= \zeta^{-a} \cdot e_a
\end{aligned}$$

provided $m(a+l) \equiv 0 \pmod{n}$, i.e., $a \equiv -l \pmod{n/m}$. Therefore

$$\zeta^{\ell}, \zeta^{\ell + \frac{n}{m}}, \ldots, \zeta^{\ell + (m-1)\frac{n}{m}}$$

are the eigenvalues of ϕ on

$$(\mathrm{tor}_P) \oplus \ldots \oplus (\mathrm{tor}_{\phi^{m-1}P}).$$

The corresponding part of the sum $\sum a_i/n$ works out to be

$$\frac{\ell \cdot m}{n} + \frac{1}{m}(1 + 2 + \ldots + (m-1)) = \ell \cdot \frac{m}{n} + \frac{m-1}{2}.$$

In particular, if $m \geqq 3$, it follows that $\sum a_i/n \geqq 1$ already. If $\sum (a_i/n) < 1$, it follows that either

a) ϕ fixes all double points $P \in C$ or

b) ϕ fixes all but one pair $\{P, \phi P\}$ which are interchanged, and $n \geqq 6$.

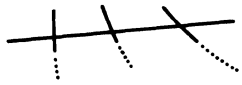
Moreover, if C has δ double points, the torsion eigenvalues contribute at least

$$\delta/n, \qquad \text{resp. } \delta/n + 1/2$$

to $\sum a_i/n$, in case a), resp. b).

The second step is to analyze the action of ϕ on the set of components of C. We shall prove that if for any α, $\phi C_\alpha \neq C_\alpha$ then either $\sum a_i/n \geqq 1$ or (C, ϕ)

has a deformation with fewer double points. To see this, consider that the image C'_α of C_α in C looks like:

(a) $C_\alpha = \mathbf{P}^1$, C'_α 

(b) $C_\alpha \cong \mathbf{P}^1$, C'_α

(c) $C_\alpha \cong \mathbf{P}^1$, C'_α

(d) $C_\alpha \cong \mathbf{P}^1$, C'_α

(e) C_α elliptic, C'_α

(f) If $g_\alpha = \text{genus}(C_\alpha)$, and $\delta_\alpha = \#\{\text{pts. } P_\beta \in C_\alpha \text{ mapping to double pts. of } C\}$, then

$$3g_\alpha - 3 + \delta_\alpha \geqq 2.$$

As in the analysis of double points, suppose $C_\alpha, \phi C_\alpha, \ldots, \phi^{m-1} C_\alpha$ are distinct and $\phi^m C_\alpha = C_\alpha$. Look at the action of ϕ on

$$W = \bigoplus_{k=0}^{m-1} H^0(\mathcal{O}_{\phi^k C_\alpha}(2K_{(\phi^k C_\alpha)} + \sum P_\beta)).$$

The same calculation given for double points shows that if

$$k = \dim H^0(\mathcal{O}_{C_\alpha}(2K_\alpha + \sum P_\beta))$$

and if the eigenvalues of ϕ^m here are $\zeta^{m\ell_1}, \ldots, \zeta^{m\ell_k}$, then the eigenvalues of ϕ on W are

$$\zeta^{\ell_i + j\left(\frac{n}{m}\right)}, \qquad 1 \leqq i \leqq k, \qquad 0 \leqq j < \frac{n}{m},$$

hence W gives *a* contribution to $\sum(a_i/n)$ at least equal to

$$k \cdot \frac{m-1}{2}.$$

Therefore if $\sum a_i/n < 1$, either $k=0$ or $k=1$ and $m=2$. In cases a), b), $k=0$, in cases c), d), e), $k=1$ and in case f), $k \geqq 2$. On the other hand, in cases c), d), e), if any double point moves, it also gives a contribution of $1/2$ to $\sum a_i/n$, hence together with the contribution from W, $\sum a_i/n \geqq 1$. But if all double points are fixed, $\phi(C'_\alpha)$ must be the second component through all double points P where C'_α meets another component of C, i.e., $C = C'_\alpha \cup \phi(C'_\alpha)$. In this case, one sees immediately that $g \leqq 3$. Turning to case a), at least one of the three double points on C'_α must be fixed, call it P_0, hence $\phi(C_\alpha)$ must be the second component of C through P_0. Therefore ϕ interchanges these 2 components and ϕ^2 fixes C_α. But ϕ^2 also fixes all double points, hence ϕ^2 is an automorphism

of $C'_\alpha \cong \mathbf{P}^1$ with 3 fixed points, hence $\phi^2|_{C_\alpha}=$identity. At P_0, let $x \cdot y=0$ be a local equation of C. Then $\phi^2|_{C_\alpha}=$identity implies that ϕ acts by:

$$\phi^*(x)=y,$$

$$\phi^*(y)=x.$$

But then ϕ fixes the torsion differential $y\,dx^{\otimes 2}/x$ (or alternately, one can argue that ϕ lifts to the universal deformation $xy=t$ of P_0 by $\phi^*x=y$, $\phi^*y=x$, $\phi^*t=t$). Therefore (C,ϕ) has a deformation in which P_0 disappears. Finally, in case (b), the double point of C'_α must move, hence the other double point of C on C'_α does *not* move, and the whole of C will be nothing but $C'_\alpha \cup \phi(C'_\alpha)$. Then $g=2$ which was excluded.

This reduces us to the case where ϕ fixes every component of C. But then every component C_α contributes to $\sum(a_i/n)$ the eigenvalues of ϕ on

$$H^0(\mathcal{O}_{C_\alpha}(2K_{C_\alpha}+\sum P_\beta)).$$

In most cases, by the Proposition, the eigenvalues of ϕ on $H^0(\mathcal{O}_{C_\alpha}(2K_{C_\alpha}))$ already give us $\sum(a_i/n) \geqq 1$. In particular, it follows that for all α, one of the following is true:

a) $\phi|_{C_\alpha}$ is the identity,

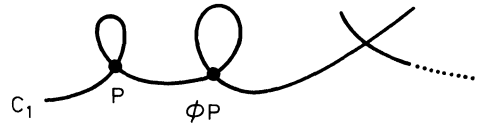
b) $C_\alpha \cong \mathbf{P}^1$,

c) C_α is elliptic,

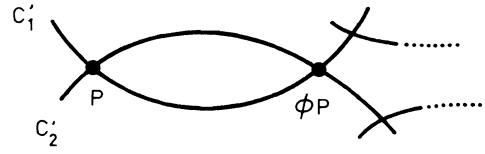
d) C_α is hyperelliptic of genus 2 or 3, $\phi|_{C_\alpha}=$hyperelliptic involution or

e) C_α has genus 2, and is a double cover of an elliptic curve with $\phi|_{C_\alpha}=$sheet interchange.

We can now argue that ϕ fixes all double points of C too. This goes as follows: note that ϕ^2 fixes all double points of C, hence ϕ^4 fixes all pairs consisting of a double point of C and a branch of C at P, i.e., all P_β in all C_α. Since in case (b), $\mathbf{P}^1$ has at least three P_β's on it, $\phi^4|_{C_\alpha}=$identity in this case. Moreover in case (c), $\phi|_{C_\alpha}$ either has a fixed point, hence has order $\leqq 6$, or is a translation. And if it is a translation, then $\phi^4|_{C_\alpha}$ is still a translation and also fixes the points P_β on C_α, i.e., $\phi^4|_{C_\alpha}=$identity. Therefore in all cases the order of ϕ on each C_α is 1, 2, 3, 4 or 6. Now suppose ϕ moves a double point. This can happen in 2 ways:



or



In the first case, $\mathrm{tor}_P \oplus \mathrm{tor}_{\phi P}$ contributes at least $1/2+2/n_1$ to $\sum a_i/n$ where n_1 $=$order of $\phi|_{C_1}$. If $n_1 \leqq 4$, we are done. If $n_1=6$, then C_1 is elliptic with $j(C_1)$ $=0$. Then we get an extra term from the action of ϕ on

$$H^0(\mathcal{O}_{C_1}(2K_{C_1})) \subset H^0(\mathcal{O}_{C_1}(2K_{C_1}+\sum P_\beta)).$$

If dz is the translation-invariant differential on C_1, then $dz^{\otimes 2}$ generates $\mathcal{O}(2K_{C_1})$ and ϕ acts by $z_1 \mapsto \zeta_6 z_1$. Thus we have a term of $1/3$ and

$$1/2+2/6+1/3>1.$$

In the second case, if $n_1=l\mathrm{cm}$ (order of ϕ on C_1, C_2), then again $\mathrm{tor}_P \oplus \mathrm{tor}_{\phi P}$ contributes at least $1/2+2/n_1$ to $\sum a_i/n$. If $n_1 \leqq 4$, we are done. If $n_1 \geqq 6$, then one of the C_i, say C_1, is again elliptic with $j(C_1)=0$. If $n_1=6$, we conclude as before. And if $n_1>6$, the order of ϕ on C_2 must be 4. But $P \in C_2$ satisfies $\phi P \neq P$, $\phi^2 P=P$, so C_2 cannot be $\mathbf{P}^1$ (because the map $z \to \sqrt{-1}\, z$ on $\mathbf{P}^1$ has no points of order exactly 2). Thus C_2 must be elliptic with $j(C_2)=12^3$. Then $dz^{\otimes 2} \in H^0(\mathcal{O}(2K_{C_2}))$ contributes $1/2$ to $\sum a_i/n$ and we are done. This completes the proof that ϕ fixes all double points of C.

We now go back to the list a)–e) and enumerate the possible components that C can have.

Case a). Any C_α with $\phi|_{C_\alpha}=$identity.

Case b). $C_\alpha=\mathbf{P}^1$. In suitable coordinates on C_α, $\phi|_{C_\alpha}$ is the map $z \mapsto \zeta z$. But all $P_\beta \in C_\alpha$ are either fixed by ϕ or of order 2. ϕ has only 2 fixed points and there are at least 3 P_β's on C_α. Thus at least one of the P_β, call it P_1, has order 2. But ϕ has a point of order exactly 2 only if ϕ itself has order 2: $\phi(z)=-z$. Now P_1 and ϕP_1 are mapped to the same double point P of C and because ϕ^2 is the identity on C_α, we see that (C, ϕ) near P is given by $xy=0$, $\phi^* x=y$, $\phi^* y=x$. As above, this means that (C, ϕ) can be deformed to eliminate P, so this case is taken care of.

Case c). C_α is elliptic.

Case c1). $\phi|_{C_\alpha}$ is a translation. Since $\phi^2 P_\beta=P_\beta$ for all β, $\phi|_{C_\alpha}$ has order 2 and $\phi^2=$identity. Then as in case (b), the double points of C which are images of these P_β can be deformed away and this case is taken care of.

Case c2). $\phi|_{C_\alpha}$ fixes $0 \in C_\alpha$ and has order 2, i.e., it is the inverse in the group structure on C_α. As $\phi^2|_{C_\alpha}=$identity, we are through as above if $\phi P_\beta \neq P_\beta$ for some β. Therefore we may assume all P_β are points of order 2. Now for any 2 points P_1, $P_2 \in C_\alpha$ of order 2,

$$H^0(\mathcal{O}_{C_\alpha}(P_1+P_2))$$

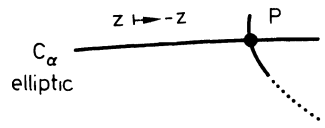
contains one even function, namely 1, and one odd function, in fact one with a simple pole at each P_i. Thus

$$H^0(\mathcal{O}_{C_\alpha}(2K_{C_\alpha}+\sum P_\beta))$$

contributes eigenvalues

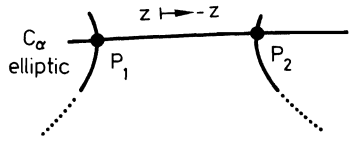
$$(0, 1/2, \ldots, 1/2)$$

to $\sum a_i/n$. Thus if there are 3 or 4 P_β's, we are through, and we are left with 2 cases for C'_α:



P fixed, C_α contributes 0 to $\sum a_i/n$

or

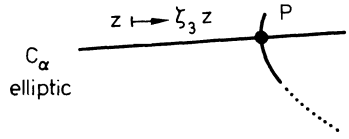


P_1, P_2 fixed, C_α contributes 1/2 to $\sum a_i/n$.

Case c3). $\phi|_{C_\alpha}$ fixes $0 \in C_\alpha$ and has order 3. Then $j(\alpha)=0$ and each P_β is a fixed point. There are 3 of these. If 2 or 3 of these occur as P_β, we are done by considering the eigenvalues of ϕ on $H^0(\mathcal{O}_{C_\alpha}(2K_\alpha+P_1+P_2))$. In fact, if dz is the translation-invariant differential and $P_1=0$, then locally the 2 sections here look like

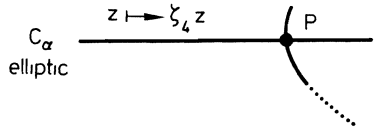
$$dz^{\otimes 2}, z\,dz^{\otimes 2}.$$

Under $z \mapsto \zeta_3 z$, these give a contribution $\frac{1}{3}+\frac{2}{3}$ to $\sum a_i/n$. This leaves the case of one P_β, or C'_α looking like:



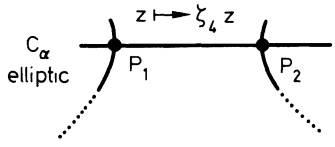
P fixed, C_α contributes 1/3 to $\sum a_i/n$.

Case c4). $\phi|_{C_\alpha}$ fixes $0 \in C_\alpha$ and has order 4. Then $j(C_\alpha)=12^3$, ϕ has 2 fixed points $0=P_0$ and P_1 and $\phi^2=$inverse has 2 more fixed points P_2 and ϕP_2. Thus $\{P_\beta\}$ could consist in $\{P_0\}$, $\{P_0, P_1\}$, $\{P_0, P_2, \phi P_2\}$, $\{P_0, P_1, P_2, \phi P_2\}$. ($\{P_2, \phi P_2\}$ is impossible because C'_α must meet the rest of C somewhere). The eigenvalues on $H^0(\mathcal{O}_{C_\alpha}(2K_{C_\alpha}+\sum P_\beta))$ are respectively (1/2), (1/2, 1/4), (1/2, 1/4, 3/4) in the first 3 cases, so the only cases where $\sum a_i/n<1$ could occur are the first 2 cases:



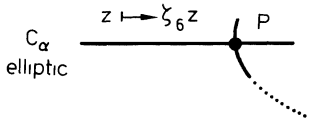
P fixed, C_α contributes 1/2 to $\sum a_i/n$

or



P_1, P_2 fixed, C_α contributes 3/4 to $\sum a_i/n$

Case c5). $\phi|_{C_\alpha}$ fixes $0 \in C_\alpha$ and has order 6. Then $j(C_\alpha)=0$, ϕ has only $0=P_0$ as fixed point, but ϕ^2 has 2 further fixed points P_1, ϕP_1. The set $\{P_\beta\}$ is either $\{P_0\}$ or $\{P_0, P_1, \phi P_1\}$ and the eigenvalues for these are $(\frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3}, \frac{1}{6})$. (In fact, $dz^{\otimes 2} \in H^0(\mathcal{O}_{C_\alpha}(2K_{C_\alpha}))$ is transformed under $z \mapsto \zeta_6 z$ by ζ_3; and a basis of $H^0(\mathcal{O}_{C_\alpha}(2K_{C_\alpha}+P_0+P_1+\phi P_1))$ looks locally like $dz^{\otimes 2}$, $\dfrac{dz^{\otimes 2}}{z}$ and $z^2 dz^{\otimes 2}$.) The only case to consider is therefore



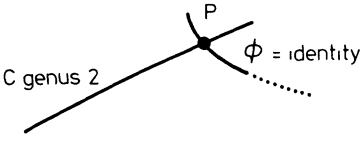
P fixed, C_α contributes 1/3 to $\sum a_i/n$.

Case d). C_α is hyperelliptic of genus 2 or 3, $\phi|_{C_\alpha}$ = hyperelliptic involution. The set $\{P_\beta\}$ must consist in at least one fixed point of ϕ, i.e., a Weierstrass point of C_α and possibly further Weierstrass points or pairs P, ϕP. But the dimension of the (-1)-eigenspace of ϕ on $H^0(\mathcal{O}(2K_C+\sum P_\beta))$ is always $\geqq 2$ if $g=3$ and P_1 is Weierstrass point. And it is 1 if $g=2$ and $\{P_\beta\}$ consists in a single Weierstrass point, $\geqq 2$ if $g=2$ and there are further P. So the only case not eliminated is

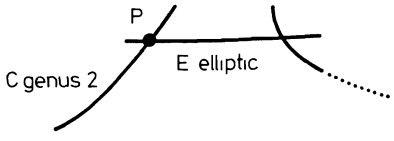
C_α of genus 2, P fixed, C_α contributes 1/2 to $\sum a_i/n$.

Case e). C of genus 2, double cover of elliptic curve, ϕ = sheet interchange. If P is a fixed point of ϕ, then the dimension of the (-1)-eigenspace of ϕ on $H^0(\mathcal{O}(2K_C+P))$ is 2, giving $(\frac{1}{2}, \frac{1}{2})$ to $\{a_i/n\}$. So this case is eliminated.

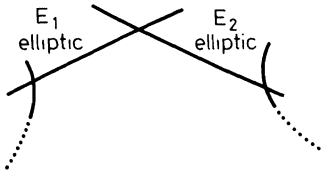
We are left with a problem of patching together the few curves of genus 1 and 2 above with a lot of C_α's where ϕ acts identically. In each curve so obtained, we must add up all the contributions to $\sum a_i/n$. One notes that the following lead to $\sum a_i/n \geqq 1$:

i) 

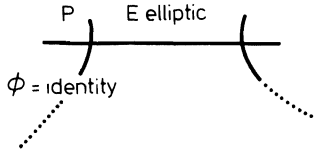
$\frac{1}{2}$ for C
$\frac{1}{2}$ for P

ii) 

$\frac{1}{2}$ for C
$\frac{1}{2}$ or $\frac{3}{4}$ for E

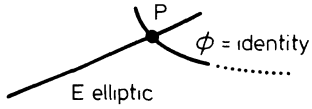
iii) 

$\frac{1}{2}$ or $\frac{3}{4}$ for E_1, E_2

iv) 

$\frac{1}{2}$ for E, $\frac{1}{2}$ for P or $\frac{3}{4}$ for E, $\geqq\frac{1}{4}$ for P

Excluding cases where $g\leqq 3$, it's easy to see that we are left with "the elliptic tails":



Finally, if $\phi|_E$ has order 3, one finds that E contributes 1/3, P 2/3 so this gives $\sum a_i/n\geqq 1$ too. Q.E.D.

By Reid-Tai's criterion, it now follows that an n-canonical form ω on $\overline{\mathscr{M}}_g^0$ is holomorphic on $\tilde{\mathscr{M}}_g$ provided it is holomorphic on the resolution of a neighborhood of the following 3 points in $\overline{\mathscr{M}}_g$:

a) the generic curve $C=C_1\cup C_2$, $C_1\cap C_2=\{P\}$, C_1 of genus $g-1$, C_2 elliptic,

b) the generic curve $C=C_1\cup C_2$, $C_1\cap C_2=\{P\}$, C_1 of genus $g-1$, C_2 elliptic with $j(C_2)=0$,

c) the generic curve $C=C_1\cup C_2$, $C_1\cap C_2=\{P\}$, C_1 of genus $g-1$, C_2 elliptic with $j(C_2)=12^3$.

In fact, we now take any smooth curve C_1 without automorphisms and a point $P\in C_1$ and consider the curve in $\overline{\mathscr{M}}_g$ parametrizing the stable curves $C_1\cup C_2$, where $C_1\cap C_2=\{P\}$ and C_2 is *any* elliptic curve or is a rational curve with node. We shall show that any ω holomorphic on $\overline{\mathscr{M}}_g^0$ is also holomorphic on the resolution of a neighborhood of this curve.

To do this, let C_0 be the curve obtained from C_1 by making P into an ordinary cusp $P_0\in C_0$, i.e.,

$$\mathscr{O}_{C_0}=k+m_{P,C}^2.$$

Consider the universal deformation space:

$$\mathscr{C}\to\Delta^{3g-3}$$

of C_0. Since any deformation of C_0 induces a deformation of a neighborhood of P_0, Δ^{3g-3} fibres over Δ^2, the base space of the universal deformation of a cusp. More precisely, there are coordinates $t_1,t_2,t_3,\ldots,t_{3g-3}$ on Δ^{3g-3} and coordinates $x,y,t_1,\ldots,t_{3g-3}$ on $\mathscr{C}$ near P_0 such that

i) C_0 is given by $y^2=x^3$, $t_i=0$, $|x|,|y|<\varepsilon$, near P_0,

ii) more generally, $\mathscr{C}$ is given by

$$y^2=x^3+t_1x+t_2,\qquad |x|,|y|<\varepsilon.$$

Outside of the codimension 2 set $V(t_1, t_2)$ (i.e., $t_1 = t_2 = 0$) $\mathscr{C}$ is a family of stable curves of genus g. Moreover, the fibres of $\mathscr{C}$ are all irreducible curves which either have one cusp, one node, or are smooth. Considering

$$\mathrm{Isom}_{\Delta^{3g-3}}(\mathscr{C}, \mathscr{C}) = \left\{ \begin{array}{c} \text{scheme of triples } s_1, s_2, \phi, \\ s_1, s_2 \in \Delta^{3g-3}, \\ \phi\colon C_{s_1} \xrightarrow{\simeq} C_{s_2} \text{ an isomorphism} \end{array} \right\}$$

we see that because C_0 has no automorphisms, all nearby curves in the family $\mathscr{C}$ have no automorphisms and occur only once in the family. Therefore we have a holomorphic map

$$f\colon \Delta^{3g-3} - V(t_1, t_2) \to \overline{\mathscr{M}}_g^0$$

which is injective, and therefore an isomorphism of $\Delta^{3g-3} - V(t_1, t_2)$ with an open subset of $\overline{\mathscr{M}}_g^0$. We want to study the singularity of f at $V(t_1, t_2)$. To do this, we want to convert $\mathscr{C}/\Delta^{3g-3}$ into a family of stable curves.

Let

$$S \to \Delta^{3g-3}$$

be the normalization of the blow up of the ideal (t_1^3, t_2^2). We shall show that f extends to an isomorphism $\tilde{f}$ of S with a suitable open set in $\mathscr{M}_g$. S is covered by 2 charts:

$$S_1 \text{ with coordinates } t_1,\ t_2,\ \frac{t_1^2}{t_2},\ \frac{t_1^3}{t_2^2},$$

$$S_2 \text{ with coordinates } t_1,\ t_2,\ \frac{t_2}{t_1},\ \frac{t_2^2}{t_1^3}.$$

Let $p_1\colon \tilde{S}_1 \to S_1$ be the normalization of S_1 in the 6-cyclic covering $u_2 = t_2^{1/6}$. Then $\tilde{S}_1$ is smooth with coordinates $u_1, u_2, t_3, \ldots, t_{3g-3}$, p_1 being given by

$$\begin{aligned} t_1 &= u_1 u_2^4, \\ t_2 &= u_2^6, \\ (t_1^2/t_2) &= u_1^2 u_2^2, \\ (t_1^3/t_2^2) &= u_1^3. \end{aligned}$$

Moreover, the group $\boldsymbol{\mu}_6$ of 6th roots of 1 acts on $\tilde{S}_1$ by

$$(u_1, u_2) \mapsto (\zeta^2 u_1, \zeta u_2)$$

so that $S_1 \cong \tilde{S}_1/\boldsymbol{\mu}_6$. Let $p_2\colon \tilde{S}_2 \to S_2$ be the normalization of S_2 in the 4-cyclic covering $u_1 = t_1^{1/4}$. Then $\tilde{S}_2$ is smooth with coordinates $u_1, u_2, t_3, \ldots, t_{3g-3}$, p_2 being given by

$$t_1 = u_1^4,$$
$$t_2 = u_2 u_1^6,$$
$$t_2/t_1 = u_2 u_1^2,$$
$$t_2^2/t_1^3 = u_2^2.$$

Moreover, the group $\boldsymbol{\mu}_4$ of 4th roots of 1 acts on $\tilde{S}_2$ by

$$(u_1, u_2) \mapsto (\zeta u_1, \zeta^2 u_2)$$

so that $S_2 = \tilde{S}_2/\boldsymbol{\mu}_4$.

Now pull the family $\mathscr{C} \to \Delta^{3g-3}$ back to $\tilde{S}_1$ and $\tilde{S}_2$. Over $\tilde{S}_1$, the family is given near P_0 by

$$y^2 = x^3 + u_1 u_2^4 x + u_2^6.$$

Let $\mathscr{C}_1$ be the normalization of the blow-up of $\mathscr{C} \times_{(\Delta^{3g-3})} \tilde{S}_1$ in the ideal (x, u_2^2). It is covered by the 2 charts

$$\mathscr{C}_{1,a}\colon\ y'^2 = x'^3 + u_1 x' + 1, \quad \text{where} \quad x' = x/u_2^2,\ y' = y/u_2^3$$

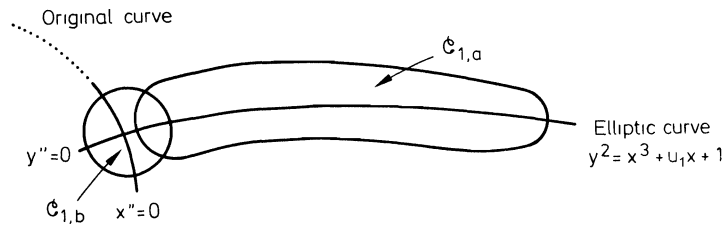
$$\mathscr{C}_{1,b}\colon\ x'' \cdot y'' = u_2, \quad \text{where} \quad x'' = u_2 x/y,\ |x''|, |y''| < \varepsilon,\ y'' = y/x$$

$$\left(\text{note that: } y''^2 = x\left(1 + u_1\left(\frac{u_2^2}{x}\right)^2 + \left(\frac{u_2^2}{x}\right)^3\right)\right.$$

$$x''^2 = \frac{(u_2^2/x)}{(1 + u_1(u_2^2/x)^2 + (u_2^2/x)^3)}$$

$$\left.\text{so that } x'',\ y'' \text{ are integrally dependent on } x,\ u_1,\ \frac{u_2^2}{x} \text{ when } \left|\frac{u_2^2}{x}\right| < \varepsilon.\right)$$

In particular, the fibre of $\mathscr{C}_1$ over points where $u_2 = 0$ looks like:



Moreover, $\mathscr{C}_1$ is a smooth variety, fibred in stable curves over $\tilde{S}_1$. The same works for the pull-back of $\mathscr{C}$ to $\tilde{S}_2$, which is given near P_0 by

$$y^2 = x^3 + u_1^4 x + u_2 u_1^6.$$

Let $\mathscr{C}_2$ be the normalization of the blow-up of $\mathscr{C} \times_{(\Delta^{3g-3})} \tilde{S}_2$ in the ideal (x, u_1^2). It is covered by the 2 charts:

$$\mathscr{C}_{2,a}\colon\ y'^2 = x_1'^3 + x' + u_2 \quad \text{where} \quad y' = y/u_1^3,\ x' = x/u_1^2,$$

$$\mathscr{C}_{2,b}\colon\ x''y'' = u_1, \quad \text{where} \quad x'' = \frac{x u_1}{y},\ y'' = \frac{y}{x},\ |x''|, |y''| < \varepsilon$$

$$\left(\text{note that } y''^2 = x\left(1 + \left(\frac{u_1^2}{x}\right)^2 + u_2\left(\frac{u_1^2}{x}\right)^3\right)\right.$$

$$x''^2 = \frac{u_1^2/x}{1 + (u_1^2/x)^2 + u_2(u^2/x)^3}$$

$$\left.\text{so that } x'',\ y'' \text{ are integrally dependent on } x,\ u_2,\ u_1^2/x \text{ when } |u_1^2/x| < \varepsilon.\right)$$

Thus $\mathscr{C}_2$ has the same form as $\mathscr{C}_1$, except that the elliptic "tail" is now:

$$y^2 = x^3 + x + u_2.$$

It follows that the map f extends to holomorphic maps $\tilde{f}_1\colon \tilde{S}_1 \to \overline{\mathscr{M}}_g$, $\tilde{f}_2\colon \tilde{S}_2 \to \overline{\mathscr{M}}_g$ in the diagram:

$$\begin{tikzcd}
\tilde{S}_1 \arrow[dr] \arrow[drrr, "\tilde{f}_1"] & & & \\
& \tilde{S}_2 \arrow[dl] \arrow[drr, "\tilde{f}_2"] & & \\
\Delta^{3g-3} & S \arrow[rr, dashed, "\tilde{f}"] & & \overline{m}_g \\
\supset & \cup & & \cup \\
& {[\Delta^{3g-3} - V(t_1, t_2)]} \arrow[rr, "f"'] & & \overline{m}_g^0
\end{tikzcd}$$

But the action of $\boldsymbol{\mu}_6$ on $\tilde{S}_1$ lifts to an action on $\mathscr{C}_1$:

$$(x', y') \mapsto (\zeta^4 x', \zeta^3 y'),$$

$$(x'', y'') \mapsto (\zeta x'', y'')$$

and the action of $\boldsymbol{\mu}_4$ on $\tilde{S}_2$ lifts to an action on $\mathscr{C}_2$:

$$(x', y') \mapsto (\zeta^2 x', \zeta y'),$$

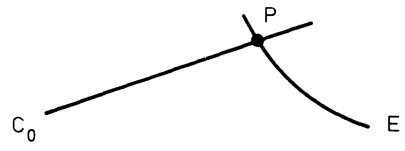
$$(x'', y'') \mapsto (\zeta x'', y'').$$

Thus $\tilde{f}_1$, $\tilde{f}_2$ factor through

$$\tilde{f}\colon S \to \overline{\mathscr{M}}_g.$$

Examining the fibres, we see that $\tilde{f}$ is injective, which proves:

Lemma. *S is isomorphic to a neighborhood in $\overline{\mathscr{M}}_g$ of the curve parametrizing the set of stable curves:*



where E is an arbitrary elliptic curve or rational curve.

It now follows from this lemma that any holomorphic tensor on $\overline{\mathcal{M}}_g^0$

a) restricts to $\Delta^{3g-3} - V(t_1, t_2)$

b) extends holomorphically to Δ^{3g-3}, since $V(t_1, t_2)$ has codimension 2

c) pulls back holomorphically to a resolution of S, hence

d) extends holomorphically to a resolution of $\overline{\mathcal{M}}_g$ over all points made up of a smooth curve C_0 without automorphisms joined at one point to an elliptic curve or nodal rational curve.

Together with the Reid-Tai criterion, this completes the proof of Theorem 1.

Appendix 1 to §1

Reid-Tai's Criterion. Let V be a vector space of dimension v, $G \subset GL(V)$ a finite group. For all $g \in G$ of order n, let the eigenvalues of g be $\zeta^{a_1}, \ldots, \zeta^{a_v}$, where ζ is a primitive n^{th} root of 1 and $0 \leqq a_i < n$. Moreover, let $V^0 \subset V$ be the open set where G acts freely, and let ω be an m-canonical differential on V^0/G. Then ω extends holomorphically to a resolution $\widetilde{V/G}$ of V/G if:

for all $g \in G$ such that $\sum a_i/n < 1$ for some choice of ζ, ω is holomorphic along all divisors $E \subset \widetilde{V/G}$ mapping onto the image in V/G of the fixed point set V^g.

Proof. We first reduce the result to the special case G cyclic. In fact, let p: $\tilde{V} \to \widetilde{V/G}$ be the normalization of $\widetilde{V/G}$ in the function field $\mathbf{C}(V)$. Then ω is holomorphic on $\widetilde{V/G}$ if ω has no poles on any divisor $E \subset \widetilde{V/G}$. But over each divisor E, the covering $\tilde{V} \to \widetilde{V/G}$ has a cyclic ramification subgroup, i.e., for all components E_1 of $p^{-1}E$, there is a cyclic subgroup $H \subset G$ fixing E_1 identically such that in an H-stable Zariski-open subset $U_E \subset \tilde{V}$ meeting E_1, p factors:

$$U_E \longrightarrow U_E/H \xrightarrow[p_E]{\text{étale}} \widetilde{V/G}.$$

In particular, ω is regular on E if $p_E^* \omega$ is regular on E_1. But if the criterion holds for G, it also holds for H, hence ω is regular on U_E/H, hence ω is regular on E.

Now assume G is cyclic, $g \in G$ is a generator and that $X_1, \ldots, X_v$ are coordinates such that $g^* X_i = \zeta^{a_i} X_i$. Then G is contained in the torus $\mathbf{G}_m^v$ of all diagonal automorphisms of V, and we may assume that the resolution $\widetilde{V/G}$ is $\mathbf{G}_m^v$-equivariant. Now $\widetilde{V/G}$ is a smooth equivariant partial compactification of the torus $\mathbf{G}_m^v/G$. By the theory of torus embeddings, each divisor E of $\widetilde{V/G} - (\mathbf{G}_m^v/G)$, hence each component E of $\widetilde{V/G} - V^0/G$, determines a monomorphism:

$$\lambda\colon \mathbf{G}_m \to \mathbf{G}_m^v/G$$

such that

a) if X^α is a character of $\mathbf{G}_m^v/G$ such that $X^\alpha \circ \lambda$ vanishes to order 1 at $0 \in \overline{\mathbf{G}}_m$, then X^α is a local equation almost everywhere for E,

b) the function field $\mathbf{C}(E)$ of E is generated by the restrictions of the characters X^α such that $X^\alpha \circ \lambda \equiv 1$.

c) $\lim_{t\to 0} \lambda(t)(a) \in E$ for all $a \in \mathbf{G}_m^v/G$.

The homomorphisms λ may be described by

$$\lambda^*(X_i) = t^{\ell_i}, \qquad 1 \leqq i \leqq v$$

where

$$(\ell_1, \ldots, \ell_v) \in \mathbf{Z}^v + \mathbf{Z}\left(\frac{a_1}{n}, \ldots, \frac{a_v}{n}\right),$$

and if λ is associated to $E \subset V/\widetilde{G}$, then $\ell_i \geqq 0$, $1 \leqq i \leqq v$.

Now let

$$\omega = \Big(\sum_{\alpha \geqq 0} c_\alpha X^\alpha\Big)(dX_1 \wedge \ldots \wedge dX_v)^{\otimes m}$$

be the given m-canonical differential. Write

$$\omega = \Big(\sum_{\alpha \geqq 0} c_\alpha X^{\alpha + m\varepsilon}\Big)\left(\frac{dX_1}{X_1} \wedge \ldots \wedge \frac{dX_v}{X_v}\right)^{\otimes m},$$
$$\varepsilon = (1, \ldots, 1).$$

Let $Y_i = X^{\beta_i}$, $1 \leqq i \leqq v$, be a basis of the character group of $\mathbf{G}_m^v/G$ such that $Y_1 \circ \lambda$ vanishes to order 1 at $t=0$, $Y_i \circ \lambda \equiv 1$, $2 \leqq i \leqq v$. Then

$$\frac{dY_1}{Y_1} \wedge \ldots \wedge \frac{dY_v}{Y_v} = (\text{const.}) \frac{dX_1}{X_1} \wedge \ldots \wedge \frac{dX_v}{X_v},$$

hence ω is holomorphic along E if and only if

$$c_\alpha \neq 0 \Rightarrow X^{\alpha + m\varepsilon} = Y_1^{r_1} \ldots Y_v^{r_v} \qquad \text{where } r_1 \geqq m.$$

But r_1 is just the order of vanishing of $X^{\alpha + m\varepsilon} \circ \lambda$ at $t=0$, i.e.,

$$r_1 = \langle \alpha + m\varepsilon, (\ell_1, \ldots, \ell_v) \rangle.$$

But then as $\alpha_i \geqq 0$, $\ell_i \geqq 0$, we get

$$r_1 \geqq m\left(\sum \ell_i\right).$$

Now

$$\ell_i = k_i + k a_i/n, \qquad \text{some } k, k_i \in \mathbf{Z}.$$

Note that

$$(g^k)^* X_i = \zeta^{k a_i} X_i = \zeta^{n k_i + k a_i} X_i = \zeta^{n \ell_i} X_i.$$

Let

$$n\ell_i \equiv a_i' (\text{mod } n), \qquad 0 \leqq a_i' < n$$

so that

$$(g^k)^* X_i = \zeta^{a_i'} X_i.$$

If

$$\sum a_i'/n \geqq 1,$$

it follows that

$$\sum \ell_i \geqq \sum \frac{a_i'}{n} \geqq 1,$$

hence $r_1 \geqq m$ as required. We are also done if $\ell_i \geqq 1$ for any i. Now assume $\sum \frac{a_i'}{n} < 1$ and $\ell_i < 1$, all i. Then g^k is one of the "bad" elements of G, so that by the assumption, ω is holomorphic on those E's over the fixed point set of $V^{(g^k)}$. But

$$\begin{aligned} V^{(g^k)} &= \{(X_1, \ldots, X_v) \mid X_i \neq 0 \Rightarrow a_i' = 0\} \\ &= \{(X_1, \ldots, X_v) \mid X_i \neq 0 \Rightarrow \ell_i \in \mathbf{Z}\} \\ &= \{(X_1, \ldots, X_v) \mid X_i \neq 0 \Rightarrow \ell_i = 0\}. \end{aligned}$$

By property (c) above, E maps onto $V^{(g^k)}$, so we are done for this E too. Q.E.D.

Appendix 2 to § 1

The following BASIC program which was run on the second author's Apple II looks one at a time at all cyclic covers of $\mathbf{P}^1$ branched in $0, 1, \infty$. These are all given by

$$y^n = x^a (x-1)^b.$$

We may assume, permuting the branch points if necessary, that

$$1 \leqq a \leqq \frac{n}{3}, \qquad a \mid n,$$

$$a \leqq b \leqq \frac{n-a}{2}, \qquad (a, b) = 1.$$

The program calculates the genus g which by Hurwitz' formula is:

$$g = \frac{n - (n, a) - (n, b) - (n, a+b)}{2} + 1.$$

It can be checked that $n \leqq 4g+2$. If $H^0(\mathcal{O}_C(2K))$ is decomposed into eigenspaces V_i, $0 \leqq i < n$, under ϕ: $(x, y) \to (x, e^{2\pi i/n} y)$, then the program calculates $e(i) = \dim V_i$ and for all j, $1 \leqq j \leqq n-1$, $(j, n) = 1$, it calculates $\sum_1^{3g-3} a_i$ for the automorphism ϕ^j: this is the variable z. The max and min over all j are called za, zb, and the max and min of $\sum a_i/n$ for each g for curves studied so far are kept in the arrays min(g), max(g).

```
]LIST

 1  G0=18: REM G0 IS THE MAXIMUM GENUS WHICH WILL
    BE CONSIDERED
10  DIM E(500): DIM MIN(100): DIM MAX(100)
20  FOR G=2 TO G0
30  MIN(G)=1000000
```

```
35  MAX(G)=0
40  NEXT G
100 FOR N=3 TO (4*G0)+2: REM N IS THE ORDER OF THE
    CYCLIC COVER
105 REM WE LOOK AT THE N-TH ROOT OF (X**A)*((X-1)**B)
110 FOR A=1 TO INT(N/3)
115 X=N: Y=A: GOSUB 1000: D1=X
120 IF (D1<A) THEN GOTO 510: REM WE WANT A TO DIVIDE N
130 FOR B=A TO INT((N-A)/2)
150 X=D1: Y=B: GOSUB 1000
160 IF X>1 THEN GOTO 500: REM WE WANT G.C.D. (A, B)=1
170 X=N: Y=B: GOSUB 1000: D2=X
180 X=N: Y=A+B: GOSUB 1000: D3=X
190 G=(N-D1-D2-D3)/2+1: REM G IS THE GENUS
    OF THE COVER
192 IF G<2 THEN GOTO 500
193 IF G>G0 THEN GOTO 500
200 FOR I=1 TO N-1
210 E(I)=INT((I*A-2*D1)/N)+INT((I*B-2*D2)/N)
    +INT((-I*(A+B)-2*D3)/N)+3
215 IF E(I)<0 THEN E(I)=0
230 NEXT I
245 REM E(I) IS THE DIMENSION OF THE I-TH EIGENSPACE
    IN THE QUADRATIC DIFFERENTIALS
250 FOR J=1 TO N-1: REM WE LOOK AT THE J-TH POWER
    OF THE AUTOMORPHISM
255 X=N: Y=J: GOSUB 1000
256 IF X>1 GOTO 320: REM WE WANT G.C.D. (N, J)=1
260 Z=0
270 FOR I=1 TO N-1
280 K=J*I-N*INT(J*I/N)
290 Z=Z+K*E(I): REM Z IS WHAT WE WROTE ABOVE AS
    THE SUM OF ASUBI
295 NEXT I
298 IF J>1 GOTO 300
299 ZA=Z: ZB=Z: GOTO 320
300 IF (ZA<Z) THEN ZA=Z: IF (ZB>Z) THEN ZB=Z
320 NEXT J
330 IF (ZB/N)<MIN(G) THEN MIN(G)=ZB/N
340 IF (ZA/N)>MAX(G)THEN MAX(G)=ZA/N
500 NEXT B
510 NEXT A
520 NEXT N
600 FOR G=2 TO G0
610 PRINT G; "MIN="; MIN(G); "MAX="; MAX(G)
620 NEXT G
630 STOP
```

```
1000  REM THIS SUBROUTINE STARTS WITH TWO INTEGERS X, Y
      AND CHANGES X TO THEIR G.C.D.
1005  IF Y<=X GOTO 1020
1010  Z=X: X=Y: Y=Z
1020  IF Y=0 THEN RETURN
1030  Z=X-Y*INT(X/Y)
1040  X=Y: Y=Z
1050  GOTO 1020
```

The program gave the following output; where the first number on each line is the genus, the second the min of $\sum a_i/n$, the third the max of $\sum a_i/n$:

```
 2  MIN=1.2 MAX=1.8
 3  MIN=2.35714285 MAX=3.64285715
 4  MIN=3.55555556 MAX=5.44444444
 5  MIN=4.77272727 MAX=7.22727273
 6  MIN=6 MAX=9
 7  MIN=7.23333333 MAX=10.7666667
 8  MIN=8.47058824 MAX=12.5294118
 9  MIN=9.71052632 MAX=14.2894737
10  MIN=10.952381 MAX=16.047619
11  MIN=12.1956522 MAX=17.8043478
12  MIN=13.44 MAX=19.56
13  MIN=14.6851852 MAX=21.3148148
14  MIN=15.9310345 MAX=23.0689655
15  MIN=17.1774194 MAX=24.8225806
16  MIN=18.4242424 MAX=26.5757576
17  MIN=19.6714286 MAX=28.3285714
18  MIN=20.9189189 MAX=30.0810811
```

Break in 630

The regular growth of min $\sum a_i/n$ and max $\sum a_i/n$ indicates that there is some simple proposition at work, but we have not investigated this. All we care about here is that $\min \sum a_i/n \geqq 1$.

§ 2. The Canonical Divisor Class on $\bar{\mathcal{M}}_g$

As in the previous section, $\bar{\mathcal{M}}_g^0$ is the open set of $\bar{\mathcal{M}}_g$ parametrizing curves without automorphisms. The components of $\bar{\mathcal{M}}_g - \bar{\mathcal{M}}_g^0$ of codimension 2 or more are simply the components of the singular locus of $\bar{\mathcal{M}}_g$. But $\mathcal{M}_g - \bar{\mathcal{M}}_g^0$ has one component of codimension 1: namely, the locus Δ_1 of curves with "elliptic tails" encountered in the previous section. We shall first calculate the canonical divisor $K_{\bar{\mathcal{M}}_g}$ on the open set $\bar{\mathcal{M}}_g^0$, and afterwards, indicate how to modify the calculation to give the canonical divisor $K_{\bar{\mathcal{M}}_g}$ on the full open set of smooth points of $\bar{\mathcal{M}}_g$.

As before, let

$$\pi\colon \bar{\mathscr{C}}_g^0 \to \bar{\mathscr{M}}_g^0$$

be the universal family of stable, automorphism-free curves. We follow the technique in [13], pp. 99–102 and apply Grothendieck's relative Riemann-Roch theorem to the morphism π. For all coherent sheaves $\mathscr{F}$ on $\bar{\mathscr{C}}_g^0$,

$$ch(\pi_! \mathscr{F}) = \pi_*(ch(\mathscr{F}) \cdot T(\Omega^1_{\mathscr{C}/\mathscr{M}})) \quad \text{in } A(\bar{\mathscr{M}}_g^0) \otimes \mathbf{Q}.$$

We wish to apply this for

$$\mathscr{F} = \Omega^1_{\mathscr{C}/\mathscr{M}} \otimes \omega_{\mathscr{C}/\mathscr{M}}.$$

In this case,

$$\pi_* \mathscr{F} \cong T^*_{\bar{\mathscr{M}}_g^0}, \qquad R^1 \pi_* \mathscr{F} = (0)$$

because the cotangent space to $\bar{\mathscr{M}}_g^0$ at every point x is canonically isomorphic to $H^0(C_x, \Omega^1_{C_x} \otimes \omega_{C_x}) = H^0(\pi^{-1}(x), \mathscr{F} \otimes \mathscr{O}_{\pi^{-1}x})$, and because $H^1(C_x, \Omega^1_{C_x} \otimes \omega_{C_x}) = (0)$. Therefore

$$K_{\bar{\mathscr{M}}_g^0} \equiv [ch(\pi_! \mathscr{F})]_1.$$

We now follow closely the calculations of [13], pp. 99–102:

$$\begin{aligned} K_{\bar{\mathscr{M}}_g^0} &= \pi_*(ch(\mathscr{F}) \cdot T(\Omega^1_{\mathscr{C}/\mathscr{M}}))_1 \\ &= \pi_* \left(\left(1 + c_1(\mathscr{F}) + \frac{c_1(\mathscr{F})^2}{2} - c_2(\mathscr{F})\right) \cdot \left(1 - \frac{c_1(\Omega^1)}{2} + \frac{c_1(\Omega^1)^2 + c_2(\Omega^1)}{12}\right)\right)_1 \\ &= \pi_* \left(\frac{c_1(\mathscr{F})^2}{2} - c_2(\mathscr{F}) - \frac{c_1(\mathscr{F}) \cdot c_1(\Omega^1)}{2} + \frac{c_1(\Omega^1)^2 + c_2(\Omega^1)}{12} \right). \end{aligned}$$

Moreover

$$\Omega^1_{\mathscr{C}/\mathscr{M}} = I_{\text{sing}} \cdot \omega_{\mathscr{C}/\mathscr{M}}$$

where I_{sing} is the ideal of the singular locus, hence

$$\begin{aligned} c_1(\Omega^1_{\mathscr{C}/\mathscr{M}}) &= c_1(\omega), \\ c_1(\mathscr{F}) &= 2c_1(\omega), \\ c_2(\Omega^1_{\mathscr{C}/\mathscr{M}}) &= [\operatorname{sing} \mathscr{C}], \\ c_2(\mathscr{F}) &= [\operatorname{sing} \mathscr{C}]. \end{aligned}$$

Therefore

$$(*) \qquad \begin{aligned} K_{\bar{\mathscr{M}}_g^0} &\equiv \pi_* \left(2c_1(\omega)^2 - [\operatorname{Sing} \mathscr{C}] - c_1(\omega)^2 + \frac{c_1(\omega)^2 + [\operatorname{Sing} \mathscr{C}]}{12} \right) \\ &= \frac{13}{12} \pi_*(c_1(\omega)^2) - \frac{11}{12} \pi_*([\operatorname{Sing} \mathscr{C}]). \end{aligned}$$

On the other hand, let

$$\lambda = c_1(\pi_* \omega_{\mathscr{C}/\mathscr{M}})$$

be the so-called Hodge divisor class on $\bar{\mathscr{M}}_g^0$. Let

$$\delta = \pi_*([\operatorname{Sing} \mathscr{C}])$$

be the divisor class of singular curves $\bar{\mathcal{M}}_g^0 - \mathcal{M}_g$. Then we proved in [13] that

$$\pi_*(c_1(\omega)^2) = 12\lambda - \delta.$$

Combining this with (*) and recalling that $\mathrm{Pic}(\bar{\mathcal{M}}_g^0)$ is torsion-free ([13], p. 102) we have proven.

Theorem 2. *On the smooth variety* $\bar{\mathcal{M}}_g^0$, $K_{\mathcal{M}_g} \equiv 13\lambda - 2\delta$.

Does this continue to hold on the bigger open set $\bar{\mathcal{M}}_{g,\mathrm{reg}}$ of all smooth points? The answer depends on how λ and δ are defined on $\bar{\mathcal{M}}_{g,\mathrm{reg}}$. Recall from [13] that to deal with the problems posed by curves with automorphisms, one has 2 approaches:

a) one can introduce the group of line bundles on the *moduli functor*, i.e., for all flat proper families $\pi\colon C \to S$ of stable curves, a line bundle $L(\pi)$ on S and for all Cartesian diagrams

$$\begin{tikzcd} C_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & C_2 \arrow[d, "\pi_2"] \\ S_1 \arrow[r, "g"'] & S_2 \end{tikzcd}$$

between such families, an isomorphism

$$L(\pi_1) \cong g^* L(\pi_2)$$

with obvious compatibility requirements. We call this $\mathrm{Pic}_{\mathrm{fun}}(\bar{\mathcal{M}}_g)$

b) one can introduce the locally closed subscheme H_g of a suitable Hilbert scheme parametrizing stable curves in a fixed projective space $\mathbf{P}^{\nu-1}$. With H_g, one has

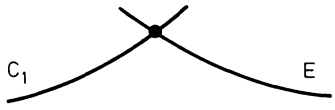
$$p\colon Z_g \to H_g\text{: a universal family of curves}$$
$$(PGL(\nu) \text{ acting on } H_g).$$

Then one considers the group $\mathrm{Pic}(H_g)^{PGL(\nu)}$ of isomorphism classes of line bundles on H_g invariant under $PGL(\nu)$.

As shown in [13], $\mathrm{Pic}_{\mathrm{fun}}(\bar{\mathcal{M}}_g) \cong \mathrm{Pic}(H_g)^{PGL(\nu)}$ and $\mathrm{Pic}(\bar{\mathcal{M}}_g)$ itself is a subgroup of these of finite index. Moreover, $\mathrm{Pic}_{\mathrm{fun}}(\bar{\mathcal{M}}_g)$ is torsion free, so the convenient way to relate these groups is to think of them all as lattices in the same $\mathbf{Q}$-vector space

$$\mathrm{Pic}(\bar{\mathcal{M}}_g) \otimes \mathbf{Q} \cong \mathrm{Pic}_{\mathrm{fun}}(\bar{\mathcal{M}}_g) \otimes \mathbf{Q} \cong \mathrm{Pic}(H_g)^{PGL(\nu)} \otimes \mathbf{Q}.$$

In the big group $\mathrm{Pic}(H_g)^{PGL(\nu)}$, we *define* λ to be $c_1(p_* \omega_{Z/H})$ and δ to be the class of the divisor on H_g of singular curves. Now if L is a line bundle on H_g with $PGL(\nu)$ acting equivariantly on it, then for every curve C with automorphism ϕ of order n, ϕ is induced by $\phi' \in PGL(\nu)$ fixing the point $[C] \in H_g$ defined by C. Then ϕ' acts on the fibre $L_{[C]}$ of the line bundle by an nth root of 1. If all these roots of 1 are trivial, L descends to a line bundle on $\bar{\mathcal{M}}_g = H_g / PGL(\nu)$. For example, if C is



C_1 of genus $g-1$, E elliptic, let $g\colon C \to C$ be defined by $g|_{C_1} = \mathrm{id.}$, $g|_E = -1$. For a line bundle L to descend to $\mathrm{Pic}(\bar{\mathcal{M}}_{g,\mathrm{reg}})$, we only need that this g acts by $+1$ on $L_{[C]}$.

As an example, take the Hodge line bundle

$$L_\lambda = \Lambda^g p_*(\omega_{Z_g/H_g}).$$

g acts on $\Lambda^g H^0(\omega_C)$ by -1, hence L_λ and the Hodge divisor class $\lambda = c_1(\Lambda^g p_* \omega_{Z/H})$ do *not* descend to $\bar{\mathcal{M}}_{g,\mathrm{reg}}$, but the square of the bundle, or twice the divisor class do. Thus the divisor class λ lies in

$$\tfrac{1}{2}\,\mathrm{Pic}(\bar{\mathcal{M}}_{g,\mathrm{reg}}).$$

Next, if L_{δ_0} is the line bundle on H_g defined by the divisor of irreducible singular curves, and L_{δ_i}, $1 \leqq i \leqq \left[\frac{g}{2}\right]$ are the line bundles on H_g defined by the divisors of curves with double points separating them into pieces of genus i, $g-i$, then the line bundle L_δ of all singular curves breaks up

$$L_\delta = \bigotimes_{i=0}^{[\frac{g}{2}]} L_{\delta_i}.$$

The fibre of L_{δ_i} over a point $[C]$ is

$$\bigotimes_P \Lambda^2(m_p/m_p^2)^*$$

where P runs over all double points of C of type i (i.e., non-separating double points for $i=0$, separating with pieces of genus i, $g-i$ for $i \geqq 1$). One checks that $L_{\delta_0}, L_{\delta_2}, \ldots, L_{\delta_{[g/2]}}$ all descend to $\bar{\mathcal{M}}_{g,\mathrm{reg}}$ and are, in fact, the line bundles defined by the divisors Δ_0, $\Delta_2, \ldots, \Delta_{[\frac{g}{2}]} \subset \bar{\mathcal{M}}_{g,\mathrm{reg}}$ of singular curves of various types. But the automorphism g of the curve C with elliptic tail acts by (-1) on the fibre of L_{δ_1} over $[C]$: hence $L_{\delta_1}^{\otimes 2}$ descends to $\bar{\mathcal{M}}_{g,\mathrm{reg}}$ and is the line bundle defined by the divisor $\Delta_1 \subset \bar{\mathcal{M}}_{g,\mathrm{reg}}$ of curves with elliptic tails. Thus in terms of divisor classes on $\bar{\mathcal{M}}_{g,\mathrm{reg}}$:

$$\delta_1 = \tfrac{1}{2}\Delta_1; \qquad \delta_i = \Delta_i, \qquad i \neq 1$$

$$\delta = \Delta_0 + \tfrac{1}{2}\Delta_1 + (\Delta_2 + \ldots + \Delta_{[\frac{g}{2}]}).$$

Note incidentally that $L_\lambda \otimes L_{\delta_1}$ also descends to $\bar{\mathcal{M}}_{g,\mathrm{reg}}$, hence $\lambda + \frac{1}{2}\Delta_1 \in \mathrm{Pic}(\bar{\mathcal{M}}_{g,\mathrm{reg}})$. On the other hand, the equality of divisor classes on H_g:

$$c_1(p_*(\Omega^1_{Z/H} \otimes \omega_{Z/H})) \equiv 13\lambda - 2\delta$$

is proven by exactly the same proof used to prove the same equality in $\mathrm{Pic}(\bar{\mathcal{M}}_g^0)$. The last step is the claim:

Lemma. $K_{(\mathcal{M}_{g,\mathrm{reg}})} \equiv c_1(p_*(\Omega^1_{Z/H} \otimes \omega_{Z/H})) - \delta_1$.

Proof. If $f: Z_g \to \bar{\mathcal{M}}_g$ is the canonical map, then on the open set $f^{-1}(\bar{\mathcal{M}}_g^0)$, there is a canonical isomorphism:

$$\alpha\colon f^*(\Omega^{3g-3}_{\bar{\mathcal{M}}_g}|_{\bar{\mathcal{M}}_g^0}) \xrightarrow{\approx} \Lambda^{3g-3} p_*(\Omega^1_{Z/H} \otimes \omega_{Z/H})|_{f^{-1}\bar{\mathcal{M}}_g^0}.$$

If $\Delta_1' \subset H_g$ is the divisor of curves with elliptic tails, *a priori* α has a zero or pole on Δ_1' of some order ℓ, and on all of $f^{-1}(\bar{\mathcal{M}}_{g,\mathrm{reg}})$

$$f^*(\Omega^{3g-3}_{\bar{\mathcal{M}}_g}), \qquad \Lambda^{3g-3} p_*(\Omega^1_{Z/H} \otimes \omega_{Z/H})(\ell \Delta_1')$$

are isomorphic. To compute ℓ, let $C = C_1 \cup E$ be a curve with elliptic tail such that C_1 has no automorphisms, $j(E) \neq 0, 12^3, \infty$. Let (Δ_t^{3g-3}) be the base space of the universal deformation of C, with coordinates $t_1, t_2, \ldots, t_{3g-3}$ where the automorphisms $g\colon C \to C$ acts by

$$g^* t_1 = -t_1, \qquad g^* t_i = t_i, \qquad 2 \leqq i \leqq 3g-3.$$

($t_1 = 0$ being the locus of singular curves). Then $\bar{\mathcal{M}}_g$ near $[C]$ is just $\Delta_t^{3g-3}/\{e, g\}$, which is the polycylinder (Δ_s^{3g-3}) with coordinates $s_1 = t_1^2$, $s_2 = t_2, \ldots, s_{3g-3} = t_{3g-3}$. Then $ds_1 \wedge \ldots \wedge ds_{3g-3}$ is a local basis of $\Omega^{3g-3}_{\bar{\mathcal{M}}_g}$, while $\Omega^{3g-3}_{\Delta_t}$ is locally the same as $\Lambda^{3g-3} p_*(\Omega^1_{Z/H} \otimes \omega_{Z/H})$: thus $dt_1 \wedge \ldots \wedge dt_{3g-3}$ is a local basis of the latter. But

$$ds_1 \wedge \ldots \wedge ds_{3g-3} = 2t_1(dt_1 \wedge \ldots \wedge dt_{3g-3})$$

hence the map from $\Omega^{3g-3}_{\bar{\mathcal{M}}_g}$ to $\Lambda^{3g-3} p_*(\Omega^1_{Z/H} \otimes \omega_{Z/H})$ has a simple zero along the locus $t_1 = 0$. This proves that

$$f^*(\Omega^{3g-3}_{\bar{\mathcal{M}}_g}|_{\mathcal{M}_{g,\mathrm{reg}}}) \cong \Lambda^{3g-3} p_*(\Omega^1_{Z/H} \otimes \omega_{Z/H})(-\Delta_1')|_{f^{-1}(\mathcal{M}_{g,\mathrm{reg}})}.$$

Taking chern classes, this proves the lemma.

This proves:

Theorem 2 *bis. On the smooth variety* $\bar{\mathcal{M}}_{g,\mathrm{reg}}$,

$$\begin{aligned} K_{\mathcal{M}_g} &\equiv 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \ldots - 2\delta_{[\frac{g}{2}]} \\ &\equiv 13\lambda - 2\Delta_0 - \tfrac{3}{2}\Delta_1 - 2\Delta_2 - \ldots - 2\Delta_{[\frac{g}{2}]}. \end{aligned}$$

§ 3. The Class of the Divisor D_k, I

For the rest of this article, we assume that the genus g under consideration is odd, and let

$$g = 2k - 1.$$

As in the Introduction, we introduce as the fundamental point of this proof the divisor:

$D_k \subset \mathcal{M}_g$,
$D_k = \{$locus of curves C which are k-fold covers of $\mathbf{P}^1\}$.

Thus if $g=3$, $k=2$, $D_2 \subset \mathcal{M}_3$ is the hyperelliptic locus. And if $g=5$, $k=3$, $D_3 \subset \mathcal{M}_5$ is the trigonal locus. It is well known that $\dim D_k = 3g-4$, hence D_k is a divisor. (This can be checked by the usual dimension count, considering the number of branch points for a generic covering $\pi\colon C \to \mathbf{P}^1$.) The purpose of this section is to prove*:

Theorem 3. *For all $g=2k-1$, there is a rational number a_k such that on $\mathcal{M}_g^0$:*

$$[D_k] \equiv a_k \lambda.$$

Corollary. *If $\bar{D}_k \subset \overline{\mathcal{M}}_g$ is the closure of D_k, then there are also integers $n_{k,\ell}$, $0 \leqq l \leqq [g/2]$ such that on $\overline{\mathcal{M}}_{g,\mathrm{reg}}$*

$$[\bar{D}_k] \equiv a_k(\lambda + \tfrac{1}{2}[\Delta_1]) + \sum_{\ell=0}^{[\frac{g}{2}]} n_{k,\ell}[\Delta_\ell].$$

The Corollary follows because $[\bar{D}_k] - a_k(\lambda + \frac{1}{2}[\Delta_1])$ is an integral divisor class on $\overline{\mathcal{M}}_{g,\mathrm{reg}}$, trivial on the open set $\mathcal{M}_{g,\mathrm{reg}}$, hence is an integral combination of the components Δ_ℓ of $\overline{\mathcal{M}}_g - \mathcal{M}_g$.

To prove the theorem, we shall apply Porteous' formula ([2] or [9]) and the Riemann-Roch theorem again. As above, let

$$\pi\colon \mathcal{C}_g^0 \to \mathcal{M}_g^0$$

be the universal family of curves. Let

$$\mathcal{C}_g^{0,n} = \mathcal{C}_g^0 \times_{\mathcal{M}_g^0} \cdots \times_{\mathcal{M}_g^0} \mathcal{C}_g^0 \qquad (n \text{ factors}).$$

Let

$$p_i\colon \mathcal{C}_g^{0,n} \to \mathcal{C}_g^0 \quad \text{be the } i^{\text{th}} \text{ projection,}$$
$$\pi_n\colon \mathcal{C}_g^{0,n} \to \mathcal{M}_g^0 \quad \text{be the canonical map.}$$

Let

$$\Delta_{ij} \subset \mathcal{C}_g^{0,n}$$

be the $(i,j)^{\text{th}}$ diagonal, and let

$$K_i \in \mathrm{Pic}(\mathcal{C}_g^{0,n})$$

stand for the divisor class

$$p_i^*(c_1(\Omega^1_{\mathcal{C}_g^0/\mathcal{M}_g^0})).$$

We are interested in the Zariski-closed subset $Z \subset \mathcal{C}_g^{0,k}$ defined by:

$$Z = \left\{(P_1, \ldots, P_k) \in \mathcal{C}_g^{0,k} \,\middle|\, h^0\left(o_C\left(\sum_1^k P_i\right)\right) \geqq 2\right\}.$$

* See note at end of article.

D_k is, by definition, $\pi_k(Z) \subset \mathscr{M}_g$. For simplicity in what follows, we drop the g and 0 in $\mathscr{C}_g^{0,k}$. To compute Z, consider

$$R^0 p_* \mathcal{O}_{\mathscr{C}^{k+1}} \left(\sum_{j=1}^{k} \Delta_{j,k+1} \right)$$

and

$$R^1 p_* \mathcal{O}_{\mathscr{C}^{k+1}} \left(\sum_{j=1}^{k} \Delta_{j,k+1} \right)$$

where

$$p\colon \mathscr{C}^{k+1} \to \mathscr{C}^k$$

is the projection onto the 1st k factors. Since $\mathscr{C}^k$ is an integral scheme and for generic $P_1, \ldots, P_k$, $H^0\left(\mathcal{O}_C\left(\sum_1^k P_i\right)\right)$ consists only in constants, the 0th direct image above is just $\mathcal{O}_{\mathscr{C}^k}$.

Consider the exact sequence on $\mathscr{C}_g^{0,k+1}$:

$$0 \to \mathcal{O}_{\mathscr{C}^{k+1}} \to \mathcal{O}_{\mathscr{C}^{k+1}}(\sum \Delta_{j,k+1}) \to \mathcal{O}_{\mathscr{C}^{k+1}}(\sum \Delta_{j,k+1})/\mathcal{O}_{\mathscr{C}^{k+1}} \to 0.$$

Taking higher direct images, we find:

$$0 \to \underbrace{p_* \mathcal{O}_{\mathscr{C}^{k+1}}(\sum \Delta_{j,k+1})/\mathcal{O}_{\mathscr{C}^{k+1}}}_{\text{loc. free } rk\, k} \xrightarrow{\alpha} \underbrace{R^1 p_* \mathcal{O}_{\mathscr{C}^{k+1}}}_{\text{loc. free } rk\, g} \to R^1 p_* \mathcal{O}_{\mathscr{C}^{k+1}}(\sum \Delta_{j,k+1}) \to 0$$

Since the 2 locally free sheaves, after $\otimes \mathbf{k}(z)$, $z \in \mathscr{C}^k$, give

$$H^0(\mathcal{O}_C(\sum P_j)/\mathcal{O}_C) \xrightarrow{\alpha(z)} H^1(\mathcal{O}_C)$$

with kernel $H^0(\mathcal{O}_C(\sum P_j))/\mathbf{C}$, it follows that

$$Z = \{z \in \mathscr{C}^k \mid rk\, \alpha(z) \leqq k-1\}.$$

By Porteous' formula, this implies

$$[Z] = c_{g-k+1}(R^1 p_* \mathcal{O}_{\mathscr{C}^{k+1}}(\sum \Delta_{j,k+1})).$$

Here $[Z]$ will be the class of Z counted with some multiplicity: since the result we seek is just that $[D_k] \in \mathbf{Q} \cdot \lambda$, this does not matter**. Since $p_* \mathcal{O}_{\mathscr{C}^{k+1}}(\sum \Delta_{j,k+1}) = \mathcal{O}_{\mathscr{C}^k}$,

$$[Z] = c_{g-k+1}(-p_! \mathcal{O}_{\mathscr{C}^{k+1}}(\sum \Delta_{j,k+1})).$$

By Grothendieck's Riemann-Roch:

$$\begin{aligned}
[Z] &= \text{polyn. in } ch_\ell(p_! \mathcal{O}_{\mathscr{C}^{k+1}}(\sum \Delta_{j,k+1})) \\
&= \text{polyn. in } p_*(ch(\mathcal{O}_{\mathscr{C}^{k+1}}(\sum \Delta_{j,k+1})) \cdot Td(\Omega^1_{\mathscr{C}^{k+1}/\mathscr{C}^k}))_\ell \\
&= \text{polyn. in classes of the form} \\
&\quad p_*\left(\text{polyn. in } \sum_{j=1}^{k} [\Delta_{j,k+1}] \text{ and } K_{k+1}\right).
\end{aligned}$$

** In fact, it is not hard to see that the multiplicity is one

To see what this can be, note the easy identities:

$$[\Delta_{j_1,k+1}]\cdot[\Delta_{j_2,k+1}]=[\Delta_{j_1,k+1}]\cdot p^*[\Delta_{j_1,j_2}]$$
$$[\Delta_{j,k+1}]^2=-[\Delta_{j,k+1}]\cdot p^*(K_j)$$
$$[\Delta_{j,k+1}]\cdot K_{k+1}=[\Delta_{j,k+1}]\cdot p^*(K_j)$$
$$p_*([\Delta_{j,k+1}]\cdot p^*(a))=a, \qquad \text{any } a\in A(\mathscr{C}^{k+1})$$
$$p_*(K^\ell_{k+1})=\pi_k^*(\pi_{1*}(K^\ell_{\mathscr{C}/\mathscr{M}})).$$

The second of these is because the self-intersection of $\Delta_{j,k+1}$ is c_1 of its normal bundle, and the normal bundle is the restriction to $\Delta_{j,k+1}$ of $p_j^*\Omega^1_{\mathscr{C}/\mathscr{M}}$. The last comes from the Cartesian diagram

$$\begin{tikzcd}
 & \mathscr{C}^{k+1} \arrow[dl, "p_{k+1}"'] \arrow[dr, "p"] & \\
\mathscr{C} \arrow[dr] & & \mathscr{C}^k \arrow[dl] \\
 & \mathscr{M} &
\end{tikzcd}$$

It follows that

$$[Z]=\text{polyn. in } [\Delta_{j_1,j_2}]\text{'s, } [K_j]\text{'s, } \pi_k^*(\pi_{1,*}(K^\ell_{\mathscr{C}/\mathscr{M}})).$$

Now $[D_k]$ is *not* $\pi_{k,*}([Z])$. Indeed $\pi_{k,*}([Z])=0$ because if a curve has one $\sum P_i$ moving in a pencil, it of course has ∞^1 such cycles. To chop $[Z]$ down in dimension, ask that the first point P_1 in the cycle $\sum P_i$ be a member of a fixed canonical divisor. This gives us

$$(2g-2)(k-1)!\,[D_k]=\pi_{k,*}([Z]\cdot K_1).$$

Therefore

$$[D_k]=\text{polyn. in } \pi_{k,*}(\text{polyn. in } [\Delta_{j_1,j_2}]\text{'s, } [K_j]\text{'s, } \pi_k^*(\pi_{1,*}(K^\ell))).$$

Now factor π_k:

$$\mathscr{C}^k\to\mathscr{C}^{k-1}\to\mathscr{C}^{k-2}\to\ldots\to\mathscr{C}\to\mathscr{M},$$

and take images of the above polynomial one at a time. Using the previous identities, it is clear that under each projection

$$p_\ell\colon \mathscr{C}^\ell\to\mathscr{C}^{\ell-1}$$

$p_{\ell,*}$ carries any polyn. in $[\Delta_{j_1,j_2}]$'s, $[K_j]$'s and $\pi_\ell^*(\pi_{1,*}(K^\ell))$ into a polynomial of the same type. Finally, projecting to $\mathscr{M}$ itself we deduce

$$[D_k]=\text{polyn. in } \pi_{1,*}(K^\ell)\text{'s.}$$

But $[D_k]$ is a divisor, so this just means

$$[D_k]=\text{multiple of } \pi_{1,*}(K^2).$$

By [13], however,

$$\pi_{1,*}(K^2)=12\lambda. \quad \text{Q.E.D.}$$

It should be pointed out that this argument establishes more generally that for any integers $k, a_1, \ldots, a_n \geqq 2g^{n \geqq 1}$ and $g=2k+n-\Sigma a_i$, the divisor $\bar{D}_{\underline{a},k}$ in $\mathscr{M}_g$ defined as the closure of

$$D_{\underline{a},k}=\left\{C\in\mathscr{M}_g \,\middle|\, \begin{array}{c}\exists \text{ divisor } D=\Sigma\, a_i p_i+D_0\in C_k \\ h^0(O_C(D))\geqq 2\end{array}\right\}$$

is similarly linearly equivalent to a linear combination of λ and the $[\Delta_i]$. Indeed, Diaz has used this set-up to explicitly calculate the coefficient of λ in the expression for the class of the divisor $D_{g-1,g-1}$ of curves with a Weierstrass point p with $h^0(C, O_C((g-1)p))\geqq 2$; he finds that in $\mathscr{M}_g^0$, this coefficient equals $g^2(3g-1)(g-1)/2$.***

§ 4. Parametrization of $\bar{D}_k$

It is not obvious which stable curves are in the closure of D_k, especially which *reducible* stable curves. To have a way to enumerate the points of $\bar{D}_k$ as well as to determine the tangent plane to smooth branches of $\bar{D}_k$, we introduce a new moduli space, which will be a compactification of what is usually called the Hurwitz scheme. Recall that the Hurwitz scheme in its simplest form parametrizes the family of k-sheeted coverings of $\mathbf{P}^1$ with b ordinary branch points:

$$H_{k,b}=\left\{\begin{array}{l}\text{moduli space of the data} \\ \qquad \pi\colon C\to\mathbf{P}^1 \text{ of degree } k \\ \qquad P_1,\ldots,P_b\in\mathbf{P}^1 \text{ distinct} \\ C \text{ smooth curve, } \pi \text{ with one ordinary branch} \\ \text{point over each } P_i\text{, otherwise unbranched}\end{array}\right\}.$$

(It is usual to treat the $\{P_i\}$ as a cycle $\sum P_i$, but for our purposes we wish to order them.) By the usual theory (see [6]), $H_{k,b}$ is itself a finite étale cover of the space of sequences $\{P_i\}$:

$$H_{k,b}\to[(\mathbf{P}^1)^b-\bigcup_{i<j}\Delta_{ij}]/PGL(2)\cong(\mathbf{P}^1)^{b-3}-S$$

where $S=(\bigcup_{i<j}\Delta_{ij})\cup\bigcup_i p_i^{-1}(\{0,1,\infty\})$ (normalizing $P_{b-2}=0$, $P_{b-1}=1$, $P_b=\infty$). By Hurwitz's formula, the genus g of C is given by

$$2g-2=-2k+b$$

and we have a diagram

*** See note at end of article

$$\begin{tikzcd} & H_{k,b} \arrow[dl] \arrow[dr, "\sigma"] & \\ {[(\mathbf{P}^1)^{b-3}-S]} & & \mathfrak{m}_g \end{tikzcd}$$

In particular, if $g=2k-1$, $D_k=\sigma(H_{k,b})$. We want to compactify $H_{k,b}$ in such a way that σ extends to a morphism

$$\sigma\colon \overline{H}_{k,b} \to \overline{\mathscr{M}}_g.$$

We do this by means of the theory of F. Knudsen [10] of b-pointed curves with fine structure. Knudsen has introduced a smooth projective compactification

$$P_b \supset [(\mathbf{P}^1)^{b-3}-S]$$

which as a moduli space can be described as follows:

i) a stable b-pointed curve is a reduced, connected curve C with at most ordinary double points, plus b smooth distinct points $P_1, \ldots, P_b \in C$ such that every smooth rational component E of C contains at least 3 points which are either P_i's or double points of C

ii) $P_b = \left\{\begin{matrix}\text{set of } b\text{-pointed stable curves } C \text{ with } p_a(C)=0 \\ \text{up to isomorphism}\end{matrix}\right\}$

iii) the open dense set $(\mathbf{P}^1)^{b-3}-S$ is the set of $(C, P_1, \ldots, P_b)$ in P_b where C is irreducible.

In fact, Knudsen describes P_b as an explicit blow-up of $(\mathbf{P}^1)^{b-3}$ along an ideal sheaf with support S.

We next describe a functor that will be coarsely represented by the sought-for scheme $\overline{H}_{k,b}$:

Definition. $\overline{\mathscr{H}}_{k,b}$ is the functor which associates to a scheme S the set of isomorphism classes of the following data:

i) a stable b-pointed curve $(D; P_1, \ldots, P_b)$ of genus 0 over S,

ii) an admissible covering $\pi\colon C \to D$.

By an *admissible covering*, we mean that C/S itself is a proper flat family of reduced connected curves with at most ordinary double points, that π is étale except at unique smooth points $Q_i\colon S \to C$, one over each $P_i\colon S \to D$, where it has ordinary branching (i.e., $\pi\colon C \to D$ is analytically just $u=x^2$, x coordinate on C over S, u coordinate on D over S), and except over the double points of D/S. For each $s \in S$ and each point x of C_s over a double point y of a fibre D_s, C_s has an ordinary double point and locally C, D and π are described by:

$$\begin{aligned} &C\colon\ xy=a, \quad a \in \hat{o}_s, \quad x, y \text{ generate } \hat{m}_{x,C} \\ &D\colon\ uv=a^p, \qquad\quad u, v \text{ generate } \hat{m}_{y,D} \\ &\pi\colon\ u\ \ =x^p, \ v=y^p \end{aligned}$$

for some p. (This definition generalizes Beauville's admissible double coverings, used to compactify the space of double coverings of a curve of genus g.)

Theorem 4. *The functor $\overline{\mathscr{H}}_{k,b}$ is coarsely represented by a scheme $\overline{H}_{k,b}$ finite over P_b, i.e., there is a morphism*

$$\overline{\mathscr{H}}_{k,b} \to \overline{H}_{k,b}$$

bijective on **C**-*valued points, which is universal for morphisms from $\overline{\mathscr{H}}_{k,b}$ to schemes. Moreover, $\overline{H}_{k,b}$ represents $\overline{\mathscr{H}}_{k,b}$ on the open set of coverings $\pi\colon C \to D$ such that C has no automorphism $\alpha\colon C \to C$ with $\pi \circ \alpha = \pi$ except the identity.*

Proof. The functor $\overline{\mathscr{H}}_{k,b}$ and hence the scheme $\overline{H}_{k,b}$, if it exists, both lie over P_b, so the problem is local over P_b. Therefore we may cover P_b by suitable open sets and make the construction separately over each. Take a point $[D_0] \in P_b$. It may happen that every component of D_0 has at least one of the b-points P_i on it. But if not, choose further points $P_{b+1}, \ldots, P_c \in D_0$ so we have one P_i in each component and in some neighborhood $U \subset P_b$ of $[D_0]$, choose smooth disjoint sections of the universal curve $\mathscr{D} \to P_b$ through these points. (This is possible because there are birational morphisms $\mathscr{D} \to \mathbf{P}^1 \times P_b$ which over $[D_0]$ take any one of the components of $\mathscr{D}$ isomorphically to $\mathbf{P}^1$, collapsing the rest to points; and $\mathbf{P}^1 \times P_b$ has a section through any point of $\mathbf{P}^1 \times [D_0]$ – see Knudsen [10].) Choose U small enough so that for all $[D] \in U$, the sections $P_1, \ldots, P_c$ meet every component of D. Moreover, choose U's well enough so we may find a local coordinate t_i on the fibres of $\mathscr{D}$ over U, $t_i = 0$ on the section $P_i\colon U \to \mathscr{D}$ to 1st order. Then for all $[D] \in U$, if $\pi\colon C \to D$ is any admissible cover, $\mathscr{O}_C\left(\sum_1^c \pi^{-1} P_i\right)$ is ample on C. For some n, $\mathscr{O}_C\left(n \sum_1^c \pi^{-1} P_i\right)$ is very ample for all $\pi\colon C \to D$.

Next define a "rigidified" version of $\overline{\mathscr{H}}_{k,b}$ over U:

$$\begin{aligned}
\overline{\mathscr{H}}^U_{k,b} = {} & \text{functor of families of admissible coverings} \\
& \pi\colon C \to D,\ [D] \in U \textit{ plus} \text{ orderings of } \pi^{-1}(P_i)\text{:} \\
P_{1,1}, \ldots, P_{1,k-1} \in C, & \quad \text{the points over } P_1 \in D,\ P_{1,1} \text{ ramified,} \\
\cdots & \quad \cdots \qquad\qquad \cdots \\
P_{b,1}, \ldots, P_{b,k-1} \in C, & \quad \text{the points over } P_b \in D,\ P_{b,1} \text{ ramified,} \\
P_{b+1,1}, \ldots, P_{b+1,k} \in C, & \quad \text{the points over } P_{b+1} \in D \\
\cdots & \quad \cdots \\
P_{c,1}, \ldots, P_{c,k} \in C, & \quad \text{the points over } P_c \in D
\end{aligned}$$

plus a choice of square root

$$\sqrt{t_i} \in \hat{\mathscr{O}}_{P_{i,1},C}, \qquad 1 \leqq i \leqq b.$$

Note that by changing these choices in the obvious way, the finite group

$$G = (\mathbf{Z}/2\mathbf{Z} \times \Sigma_{k-2})^b \times (\Sigma_k)^{c-b}$$

acts on $\overline{\mathscr{H}}^U_{k,b}$, where Σ_ℓ = permutations of ℓ letters. Via this action $\overline{\mathscr{H}}_{k,b}(\operatorname{Spec}\mathbf{C}) \supset \overline{\mathscr{H}}^U_{k,b}(\operatorname{Spec}\mathbf{C})/G$. We shall show that $\overline{\mathscr{H}}^U_{k,b}$ is representable. To do this, note that the projection

$$\pi\colon C \to D$$

and the choice of $t_i \in m_{P_i, D}$, $t \notin m^2_{P_i, D}$, defines isomorphisms:

$$\hat{o}_{P_i,1,C} \xrightarrow{\approx} \hat{o}_{P_i,D}[\sqrt{t_i}] \xrightarrow{\approx} \mathbf{C}[[\sqrt{t_i}]] \qquad \text{if } 1 \leqq i \leqq b$$

or

$$\hat{o}_{P_i,j,C} \xrightarrow{\approx} \hat{o}_{P_i,D} \xrightarrow{\approx} \mathbf{C}[[t_i]] \qquad \text{if } j>1 \text{ or } i>b.$$

Therefore, if N is large enough, we get an injection:

$$H^0\left(C, o_C\left(n\sum_1^c \pi^{-1} P_i\right)\right) \longrightarrow \mathbf{C}^{k(N+1)c}$$
$$f \longmapsto (\ldots, a_{ijk}, \ldots)$$

if, near P_{ij}, f is expanded:

$$f = \sum_{k=0}^{N} a_{ijk} t_i^{-n+k} + \ldots$$

or

$$f = \sum_{k=0}^{2N+1} a_{i1k} t_i^{-n+k/2} + \ldots \qquad \text{if } j=1,\ i \leqq b.$$

In other words, for each admissible π: $C \to D$, we get canonically both a subspace $V \subset \mathbf{C}^{k(N+1)^c}$ and an embedding

$$C \subset \mathbf{P}(V).$$

The dimension of V is given by the Riemann-Roch theorem on C as $nkc-g+1$. Reversing this process, let G be the Grassmannian of $(nkc-g+1)$-dimensional subspaces of $\mathbf{C}^{k(N+1)c}$, let $\mathscr{V} \to G$ be the universal vector bundle, and let H be the Hilbert scheme of $\mathbf{P}(\mathscr{V})$ over G of curves in $\mathbf{P}(\mathscr{V})$ with Hilbert polynomial $nkcX-g+1$. By the given ordering of the coordinates in $\mathbf{C}^{k(N+1)c}$, we get maps:

$$\lambda_{ij}\colon\ V \subset \mathbf{C}^{k(N+1)c} \xrightarrow[\substack{\text{via}\\ \text{coord}\\ a_{ijk}}]{} \left\{\begin{matrix}\text{V. Sp. of series}\\ \sum_{k=0}^{N} a_k t_i^{-n+k}\end{matrix}\right\} \xrightarrow[\substack{\text{mult. by}\\ t_i^n}]{} \mathbf{C}[[t_i]]/(t_i^{N+1})$$

hence over a suitable open subset of G, we get canonical embeddings

$$\phi_{ij}\colon\ \operatorname{Spec} \mathbf{C}[[t_i]]/(t_i^{N+1}) \hookrightarrow \mathbf{P}(V).$$

(Replace t_i by $\sqrt{t_i}$ if $j=1$, $i \leqq b$.) Let $H_1 \subset H$ be the locally closed subscheme of points z where ϕ_{ij} exist and $\operatorname{Im}\phi_{ij}$ is a subscheme of the curve C_z in $\mathbf{P}(\mathscr{V})$ defined by z. Let $H_2 \subset H_1$ be the locally closed subset of connected reduced curves C_z with at most ordinary double points with $\operatorname{Im}\phi_{ij}$ being disjoint smooth points $P_{i,j}$ of C_z, with C_z embedded by a complete linear system and with $o_{C_z}(1) \cong o_{C_z}\left(2n\sum_1^b P_{i1} + n\sum_{\text{rest}} P_{ij}\right)$. Let $H_3 \to H_2 \times U$ be the Hilbert scheme representing morphisms π: $C_z \to D$, $[D] \in U$, and let $H_4 \subset H_3$ be the locally closed set where π is finite of degree k over each component and

$$\begin{tikzcd} & C_z \arrow[d, "\pi"] \\ \operatorname{Spec} \mathbb{C}[[t_i]]/(t_i^{N+1}) \arrow[ur, "\phi_{ij}"] \arrow[r, "\text{Canonical map image } P_i"'] & D \end{tikzcd}$$

commutes (modify this for $\phi_{i1}, \ldots, \phi_{ib}$ in the obvious way). Then H_4 will represent $\overline{\mathscr{H}}^U_{k,b}$!

It follows that H_4/G coarsely represents the open subfunctor of $\overline{\mathscr{H}}_{k,b}$ of π: $C \to D$ with $[D] \in U$, and represents it where G acts freely. But a fixed point of $g \in G$ means a covering π: $C \to D$ such that there is an isomorphism:

$$\begin{tikzcd} C \arrow[rr, "\alpha"] \arrow[dr, "\pi"'] & & C \arrow[dl, "\pi"] \\ & D & \end{tikzcd}$$

which permutes the finite sets $\pi^{-1}(Q_i)$ and/or acts by $\sqrt{t_i} \to -\sqrt{t_i}$ at the ramified points $P_{i,1}$. Glueing together the schemes H_4/G over various open sets $U \subset P_b$, we construct $\overline{H}_{k,b}$.

It is clear that the morphism $\overline{H}_{k,b} \to P_b$ is finite to one. To see that it is finite, we check that the functor $\overline{\mathscr{H}}_{k,b}$ satisfies the weak valuative criterion for properness, hence $\overline{H}_{k,b}$ is proper over $\mathbf{C}$. This means that given

$$\pi\colon\ C \to D$$

over the field $\mathbf{C}((t))$, then taking a suitable root $t^{1/n}$ of t, π extends to a family of admissible coverings:

$$\pi_n\colon\ \mathscr{C}_n \to \mathscr{D}_n$$

over the ring $\mathbf{C}[[t^{1/n}]]$. In fact, by Knudsen's results, P_b is complete, hence $D \subset \mathscr{D}_1$, $\mathscr{D}_1$ a b-pointed stable curve over $\mathbf{C}[[t]]$. Let $\mathscr{C}_1$ be the partial normalization of $\mathscr{D}_1$ in the fraction field of C defined by

$$\mathscr{O}_{\mathscr{C}_1} = (\text{functions integral over } \mathscr{O}_{\mathscr{D}}, \text{ generically in } \mathscr{O}_C).$$

$\mathscr{C}_1$ may be ramified over one of the components of the fibre $(\mathscr{D}_1)_0$ over $t=0$. But replacing t by $t^{1/n}$, by Abhyankar's lemma, this no longer happens. This gives us $\mathscr{C}_n \to \mathscr{D}_n$ which, by the purity of the branch locus, is ramified only over the sections P_i: $\operatorname{Spec} k[[t]] \to \mathscr{D}_n$ and the singular points of the fibres of $\mathscr{D}_n$. I claim that $\mathscr{C}_n$ must be an admissible covering. There are 2 points to check: what happens over an ordinary double point of $(\mathscr{D}_n)_0$ which lifts to the generic fibre $(\mathscr{D}_n)_\eta$, and what happens over those that don't. In the first case, $\mathscr{D}_n$ has the local equation

$$u \cdot v = 0$$

and $\mathscr{C}_n$ is a covering of the smooth branch $u=0$ plus a covering of the smooth branch $v=0$. So if $(\mathscr{C}_n)_\eta$ has p-fold branching on the curve $u=v=0$ on the surface

$u=0$ and on the curve $u=v=0$ on the surface $v=0$, $\mathscr{C}_n$ must do the same. In the second case, $\mathscr{D}_n$ has the local equation

$$u \cdot v = s^m \qquad (s = t^{1/n})$$

for some m, i.e., $\mathscr{D}_n$ has a singularity of type A_{m-1}. The universal cover of $\mathscr{D}_n$, ramified only at the origin, is then

$$x' \cdot y' = s, \qquad x' = u^{1/m}, \qquad y' = v^{1/m}.$$

Thus $\mathscr{C}_n$ must be given by

$$x \cdot y = x^k, \qquad x^\ell = u, \qquad y^\ell = v$$

for some factorization $k\ell = m$, and this is admissible. Q.E.D.

The scheme $\overline{H}_{k,b}$ maps on the one hand to P_b by considering only D in the cover π: $C \to D$, and on the other hand to $\overline{\mathscr{M}}_g$ by considering only C:

$$\begin{tikzcd} & \overline{H}_{k,b} \arrow[dl] \arrow[dr] & \\ P_b & & \overline{\mathfrak{m}}_g \end{tikzcd}$$

(C is not necessarily stable, but one may contract unnecessary smooth rational components with only 1 or 2 double points, obtaining a stable curve – see Knudsen [10].) If $g = 2k-1$, it is clear that the divisor $\overline{D}_k \subset \overline{\mathscr{M}}_g$ is the image of the morphism $\overline{H}_{k,b} \to \overline{\mathscr{M}}_g$.

We want to discuss the local structure of $\overline{H}_{k,b}$. More precisely, what family pro-represents the infinitesimal deformations of a particular admissible covering π: $C \to D$? Let $\{Q_i\}_{1 \leqq i \leqq d}$ be the double points of D and let $\pi^{-1}(Q_i) = \{Q_{i,1}, \ldots, Q_{i,r(i)}\}$. Say the covering π is ramified with order $p(i,j)$ at $Q_{i,j}$. Now let

$$\begin{tikzcd} \mathfrak{D} \arrow[r] & \operatorname{Spec} \mathbb{C}[[t_1, \ldots, t_{b-3}]] \arrow[l, bend left, "p_i"] \end{tikzcd}$$

be the universal deformation of the b-pointed curve D. If u_i, v_i are coordinates at $Q_i \in D$ so that D is given by $u_i v_i = 0$, then if the t_i are suitably chosen, we may assume that $\mathscr{D}$ is given locally near Q_i by the equation

$$u_i v_i = t_i.$$

In order to lift the covering π: $C \to D$ to a covering π: $\mathscr{C} \to \mathscr{D}$, Grothendieck's theory ([8] or [14]) tells us that we must merely give $\mathscr{C}$ locally near all non-étale points of π: $C \to D$. Over each of the b sections P_i of $\mathscr{D}/\mathbf{C}[[t_1, \ldots, t_{b-3}]]$, let s_i be a function on $\mathscr{D}$ vanishing to 1st order on the image of P_i. Then near

P_i, the covering $\pi\colon \mathscr{C} \to \mathscr{D}$ is given by adjoining $\sqrt{s_i}$. At each Q_{ij}, however, we must define $\mathscr{C}$ by

$$x_{ij}\, y_{ij} = t_{ij}$$

and π by

$$u_i = x_{ij}^{p(i,j)}, \qquad v_i = y_{ij}^{p(i,j)}.$$

In order to do this, t_i must equal $t_{ij}^{p(i,j)}$. The universal way to do this is to take our parameter space to be:

$$\operatorname{Spec} \mathscr{O}_{[\pi\colon C \to D]},$$

where

$$\mathscr{O}_{[\pi\colon C \to D]} = \mathbf{C}[[t_1, \ldots, t_{b-3}, t_{1,1}, \ldots, t_{i,j}, \ldots, t_{d,r(d)}]]/(t_{ij}^{p(i,j)} - t_i, \text{ all } i,j).$$

Over this base, we have now defined a covering $\pi\colon \mathscr{C} \to \mathscr{D}$. The complete local rings of the scheme $\overline{H}_{k,b}$ are, as usual, the rings of invariants of $\mathscr{O}_{[\pi\colon C \to D]}$ under the finite group of automorphisms $\alpha\colon C \to C$ such that $\pi \circ \alpha = \pi$.

Unfortunately, some of the rings $\mathscr{O}_{[\pi\colon C \to D]}$ are rather messy, esp. not integrally closed. Suppose, however, that for all double points $Q_i \in D$, there is *at most one* $Q_{ij} \in C$ over Q_i such that $p(i,j) > 1$. Let it be the point $Q_{i,1}$. Then

$$\mathscr{O}_{[\pi\colon C \to D]} = \mathbf{C}[[t_{1,1}, \ldots, t_{d,1}, t_{d+1}, \ldots, t_{b-3}]]$$

and if, furthermore, C has no automorphisms over D, $\overline{H}_{k,b}$ is smooth at the point defined by $\pi\colon C \to D$. For further applications it seems that the normalization of $\overline{H}_{k,b}$ is probably more useful than $\overline{H}_{k,b}$ itself. Note that the integral closure of $\mathscr{O}_{[\pi\colon C \to D]}$ is a semi-local ring whose local rings are all regular.

As an application of the surjective map

$$\overline{H}_{k,b} \xrightarrow{\lambda} \overline{D}_k \subset \overline{\mathscr{M}}_g, \qquad \begin{array}{l} g = 2k-1 \\ b = 2g+2k-2 \end{array}$$

we can describe set-theoretically at least part of $\overline{D}_k$ fairly easily. The first part of the following theorem can be proven easily without use of $\overline{H}_{k,b}$, but the second is harder:

Theorem 5. *Let C be a stable curve of genus $g = 2k-1$, $[C] \in \overline{\mathscr{M}}_g$ the corresponding point. Then*

a) *if C is irreducible, $[C] \in \overline{D}_k$ if and only if there exists a torsion-free rank 1 sheaf $\mathscr{F}$ on C such that*

$$h^0(\mathscr{F}) \geqq 2$$
$$\chi(\mathscr{F}) = 2-k.$$

b) *if $C = C_1 \cup C_2$, C_i irreducible, $C_1 \cap C_2 = \{p\}$, then $[C] \in \overline{D}_k$ if and only if there are torsion-free rank 1 sheaves $\mathscr{F}_1$ on C_1, $\mathscr{F}_2$ on C_2 and an integer ℓ such that:*

$$h^0(\mathscr{F}_1) \geqq 2, \qquad h^0(\mathscr{F}_2) \geqq 2,$$
$$h^0(\mathscr{F}_1(-\ell p)) \geqq 1, \qquad h^0(\mathscr{F}_2(-\ell p)) \geqq 1,$$
$$\chi(\mathscr{F}_1) + \chi(\mathscr{F}_2) = 3-k+\ell.$$

Proof. We first prove (a). By the theory of the compactification of $\mathrm{Pic}(C)$, the space of pairs $(C, \mathscr{F})$, C irreducible, $\mathscr{F}$ torsion-free rank 1 is proper over the moduli space of such C's, and the conditions in (a) define a closed subset of this space. Therefore the set of irreducible C such that such an $\mathscr{F}$ exists is closed in the space of such C's. Moreover, if C is smooth and $[C] \in D_k$, let π: $C \to \mathbf{P}^1$ be the given covering of degree k, $\mathscr{F} = \pi^* o(1)$. Then

$$\chi(\mathscr{F}) = \deg \pi - g + 1 = k - (2k-1) + 1 = 2 - k.$$

This proves that the C's satisfying the conditions in (a) contains the set of irreducible C's such that $[C] \in \overline{D}_k$. Conversely, let C satisfy the conditions of (a). We will check that $[C]$ lifts to a point of $\overline{H}_{k,b}$, hence $[C] \in \mathrm{Im}(\overline{H}_{k,b} \to \overline{\mathscr{M}}_g) = \overline{D}_k$. To see this, let C' be the desingularization of C. We shall embed C' in a curve C'' with ordinary double points so that collapsing rational curves of C'' we get back to C and we construct at the same time an admissible covering

$$\pi\colon\ C'' \to P, \qquad P \text{ a } b\text{-pointed stable rational curve.}$$

We start by choosing $s_1, s_2 \in \Gamma(\mathscr{F})$ and letting $\mathscr{F}' \subset \mathscr{F}$ be the subsheaf generated by s_1, s_2. Let A be the set of double points of C. At each $z \in A$, $\mathscr{F}'_z$ is isomorphic either to $o_{z,C}$ or to $m_{z,C}$. Call these subsets $A_1, A_2 \subset A$. Let C^* be the partial desingularization of C obtained by separating the branches of C at the $z \in A_2$ only. Then there is an invertible sheaf $\mathscr{F}^*$ on C^* such that if

$$C' \xrightarrow{f} C^* \xrightarrow{g} C$$

are canonical maps, $\mathscr{F}' = g_*(\mathscr{F}^*)$. The function s_1/s_2 defines a morphism

$$\pi^*\colon\ C^* \to \mathbf{P}^1$$

hence

$$\pi = \pi^* \circ f\colon\ C' \to \mathbf{P}^1.$$

Note that

$$\begin{aligned}
\# A_2 + \deg(\pi) &= \# A_2 + \deg(\pi^*) \\
&= \# A_2 + \deg(\mathscr{F}^*) \\
&= \# A_2 + \chi(\mathscr{F}^*) + p_a(C^*) - 1 \\
&= \chi(\mathscr{F}') + (\# A_2 + p_2(C^*)) - 1 \\
&= \chi(\mathscr{F}') + p_a(C) - 1 \\
&\leqq \chi(\mathscr{F}) + 2k - 2 \\
&= k.
\end{aligned}$$

Now π: $C'' \to P$ will be built up starting from π: $C' \to \mathbf{P}^1$. Let $S \subset \mathbf{P}^1$ be

a) the multiple branch points of π

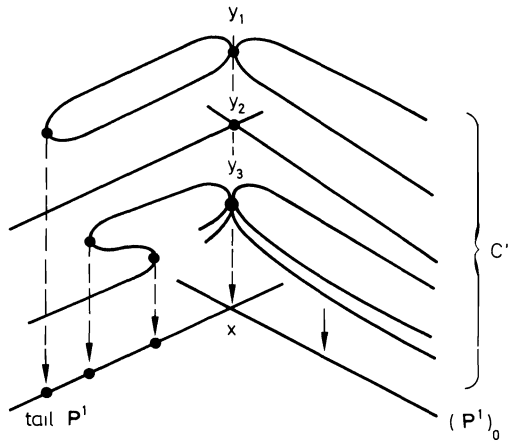
b) the images of the points $z_{1,i}, z_{2,i} \in C'$ over the double points $z_i \in C$

c) the images of $k - \# A_2 - \deg(\pi)$ further generic points $w_i \in C'$.

Then P is the original $\mathbf{P}^1$, henceforth called $(\mathbf{P}^1)_0$, with a "tail" $\mathbf{P}^1$ glued at each point of S.

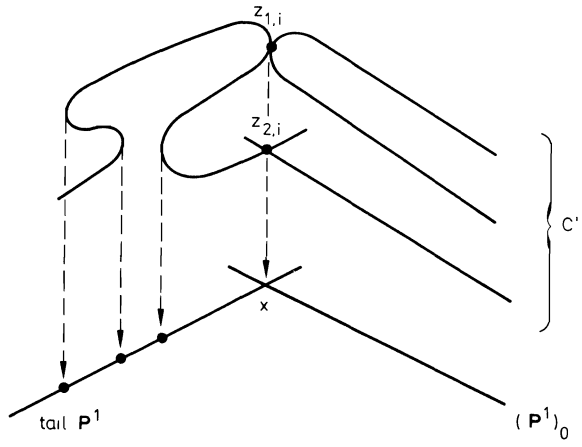
Firstly, if $x \in (\mathbf{P}^1)_0$ is a multiple branch point, let $\pi^{-1}(x) = \sum n_i y_i$. For each i, add to C' a copy of $\mathbf{P}^1$ glued to C' at y_i and mapped to the tail at x by a generic map $\mathbf{P}^1 \to \mathbf{P}^1$ of degree n_i, with n_i-fold branching at y_i:

Examples: $n_1 = 2,\ n_2 = 1,\ n_3 = 3$

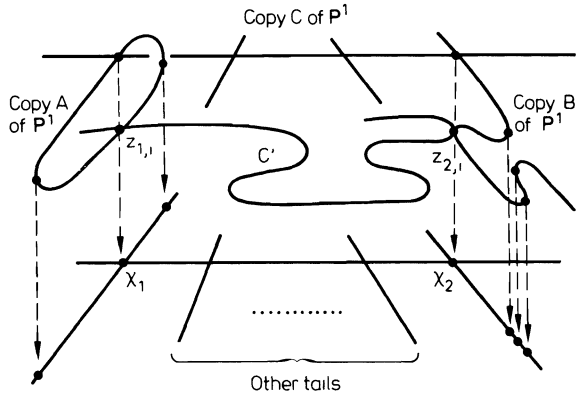


Secondly, if $x \in (\mathbf{P}^1)_0$ is the image of $z_{1,i}$ *and* $z_{2,i}$ (this happens if $z_i \in A_1$), let π be ramified to order ℓ_1 at $z_{1,i}$, ℓ_2 at $z_{2,i}$. Then add one copy of $\mathbf{P}^1$ to C' meeting C' at $z_{1,i}$ and $z_{2,i}$, and lying over the tail at x with degree $\ell_1 + \ell_2$ by a generic map $\mathbf{P}^1 \to \mathbf{P}^1$ ramified to order ℓ_1 at $z_{1,i}$, ℓ_2 at $z_{2,i}$:

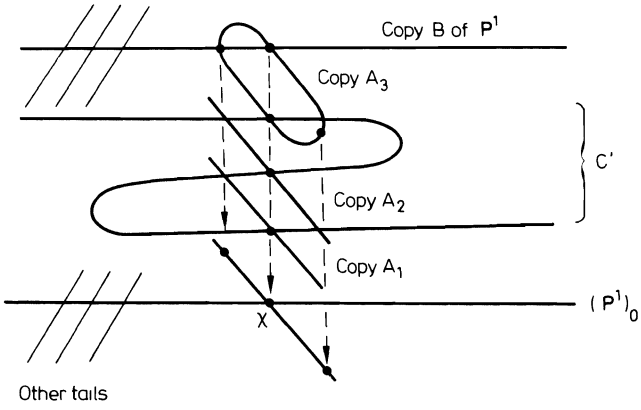
Example: $\ell_1 = 2,\ \ell_2 = 1$



(At other points of C' in $\pi^{-1}(x)$, add further $\mathbf{P}^1$'s as in the 1st step.) Thirdly, if $x_1, x_2 \in (\mathbf{P}^1)_0$, $x_1 \neq x_2$ are the images of $z_{1,i}$ and $z_{2,i}$ (this happens if $z_i \in A_2$), let π be ramified to order ℓ_1 at $z_{1,i}$, ℓ_2 at $z_{2,i}$ again. This time, however, add 3 copies of $\mathbf{P}^1$ to C', copy A over the tail at x_1, copy B over the tail at x_2 and copy C over $(\mathbf{P}^1)_0$ as follows:



$\ell_1=1$, $\ell_2=2$ is illustrated. Note that copy C goes isomorphically to $(\mathbf{P}^1)_0$, copy B goes to the tail at x_1 with degree ℓ_1+1, generic except for ℓ_1-fold ramification at $z_{1,i}$, copy C goes to the tail at x_2 with degree ℓ_2+1, generic except for ℓ_2-fold ramification at $z_{2,i}$. Both copy A and B meet copy C at their remaining (unramified) point over x_1 (resp. x_2). Finally, more $\mathbf{P}^1$'s are joined to copy C to cover all the other tails added to $(\mathbf{P}^1)_0$. Fourthly, at the $k-\#A_2-\deg\pi$ generic points x added to S, the following "plumbing fixture" is thrown in to soup up the degree of π to k:



Thus $\deg\pi$ $\mathbf{P}^1$'s are added over the tail at x, all but one isomorphic to it, one of degree 2 over it. To the new point over x introduced by this last $\mathbf{P}^1$, a copy B of $\mathbf{P}^1$ isomorphic to $(\mathbf{P}^1)_0$ is joined, and to it further $\mathbf{P}^1$'s over the other tails.

After some reflection, the reader will see that the resulting C'' is an admissible cover of P of degree k, and arithmetic genus $g=2k-1$, which blows down to the original C when the extra $\mathbf{P}^1$'s are collapsed to points. This completes the proof of part (a).

To check part (b), let $\overline{D}_k^*$ be the locus of $[C]$'s satisfying the conditions of parts (a) or (b). We show first that $\overline{D}_k^*$ is closed in the open set of $\overline{\mathcal{M}}_g$ of curves with at most 2 components, meeting at most once, hence in this open set, $\overline{D}_k$

$\subset \overline{D}_k^*$, and second that points of $\overline{D}_k^*$ lift to $\overline{H}_{k,b}$ hence $\overline{D}_k \supset \overline{D}_k^*$. (The latter will use almost the same construction to that we just gave.) It is clear that the locus of curves satisfying (b) is closed in the locus of curves $C_1 \cup C_2$, $C_1 \cap C_2 = \{p\}$. Thus $\overline{D}_k^*$ is a constructible subset of $\overline{\mathscr{M}}_g$. Let $\mathscr{C}/\operatorname{Spec} R$ be a family of stable curves over a valuation ring whose generic member $C^{(\eta)}$ is smooth and in D_k and whose special member $C^{(0)}$ is of the form $C_1^{(0)} \cup C_2^{(0)}$, $C_1^{(0)} \cap C_2^{(0)} = \{p\}$, $C_i^{(0)}$ irreducible. Let $\mathscr{F}_\eta$ be the invertible sheaf on $C^{(\eta)}$ of degree k with $h^0(\mathscr{F}_\eta) \geqq 2$. Then $\mathscr{F}_\eta$ extends to a torsion-free, reflexive, rank 1 sheaf on $\mathscr{C}$ in an infinite number of ways parametrized by an integer k. If $\mathscr{F}/\mathscr{C}$ is one of these, then $\mathscr{F}(-k\,C_1^{(0)}) \cong \mathscr{F}(+k\,C_2^{(0)})$ are the others. Now the local ring $\hat{o}_{\mathscr{C},p}$ is isomorphic to

$$\mathbf{C}[[x, y, t]]/(x\,y - t^n)$$

for some n, where $t\,R$ is the maximal ideal of R and $(x, t) =$ ideal of $C_1^{(0)}$, $(y, t) =$ ideal of $C_2^{(0)}$. Then the group $\operatorname{Pic}(\operatorname{Spec} \hat{o}_{\mathscr{C},p} - \{m\})$ is cyclic of order n, being given by the restrictions to the punctured spectrum $\operatorname{Spec} \hat{o}_{\mathscr{C},p} - \{m\}$ of the ideal sheaves (x, t^k), $0 \leqq k \leqq n-1$. If $\mathscr{F}$ is isomorphic, as invertible sheaf on $\operatorname{Spec} \hat{o}_{\mathscr{C},p} - \{m\}$ to (x, t^k), then $\mathscr{F}(k\,C_1^{(0)})$ is invertible at p. Replacing $\mathscr{F}$ by this, we may as well assume that $\mathscr{F}$ itself is invertible at p, hence so are all the sheaves in the sequence

$$\ldots, \mathscr{F}(-n\,C_1^{(0)}), \mathscr{F}, \mathscr{F}(n\,C_1^{(0)}), \mathscr{F}(2n\,C_1^{(0)}), \ldots.$$

Restricting $\mathscr{F}$ to $C^{(0)}$, we get a pair of torsion free, rank 1 sheaves $\mathscr{F}_1$ on $C_1^{(0)}$, $\mathscr{F}_2$ on $C_2^{(0)}$, invertible at p and "glued" there to give a sheaf on $C^{(0)}$. Restricting the above twists of $\mathscr{F}$, we get the pairs $\mathscr{F}_1(k\,p)$, $\mathscr{F}_2(-k\,p)$, glued to a sheaf $\mathscr{F}_k$ on $C^{(0)}$. By upper semi-continuity of h^0, it follows that $h^0(\mathscr{F}_k) \geqq 2$. Fix $k_1 > k_2$ by the hypothesis

$$h^0(\mathscr{F}_1(k\,p)) \geqq 2 \quad \text{if } k \geqq k_1,$$
$$h^0(\mathscr{F}_1(k\,p)) \geqq 1 \quad \text{if } k \geqq k_2.$$

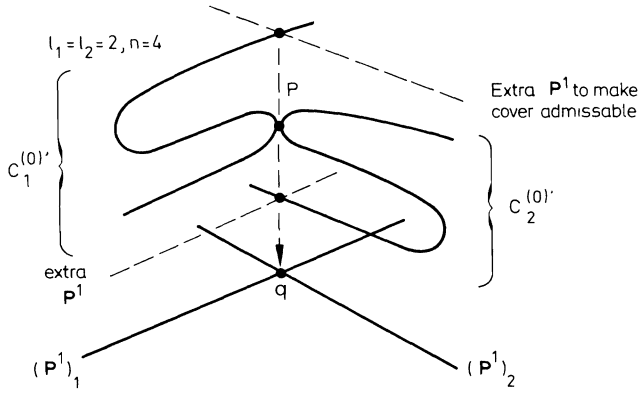
Then $h^0(\mathscr{F}_{k_1}) \geqq 2$ implies $h^0(\mathscr{F}_2(-k_1 p)) \geqq 1$ and $h^0(\mathscr{F}_{k_2}) \geqq 2$ implies $h^0(\mathscr{F}_2(-k_2 p)) \geqq 2$. If $\mathscr{F}_1' = \mathscr{F}_1(k_1 p)$, $\mathscr{F}_2' = \mathscr{F}_2(-k_2 p)$, $\ell = k_1 - k_2$, we see that $\mathscr{F}_1'$, $\mathscr{F}_2'$ and ℓ satisfy the conditions of part (b) so that $[C^{(0)}] \in \overline{D}_k^*$.

For the second part, we start with $C = C_1 \cup C_2$, $\mathscr{F}_1$, $\mathscr{F}_2$, and ℓ satisfying (b). Take 2 sections in $\mathscr{F}_1$, one a section of $\mathscr{F}_1(-\ell p)$, spanning a subsheaf $\mathscr{F}_1'$, and 2 sections in $\mathscr{F}_2$, one a section of $\mathscr{F}_2(-\ell p)$, spanning $\mathscr{F}_2'$ and let their ratios define morphisms of the desingularizations $C_i^{(0)\prime}$ of $C_i^{(0)}$:

$$\pi_1\colon\ C_1^{(0)\prime} \to \mathbf{P}^1 \text{ of degree } k_1, \text{ ramified to order } \ell_1 \text{ at } p$$
$$\pi_2\colon\ C_2^{(0)\prime} \to \mathbf{P}^1 \text{ of degree } k_2, \text{ ramified to order } \ell_2 \text{ at } p.$$

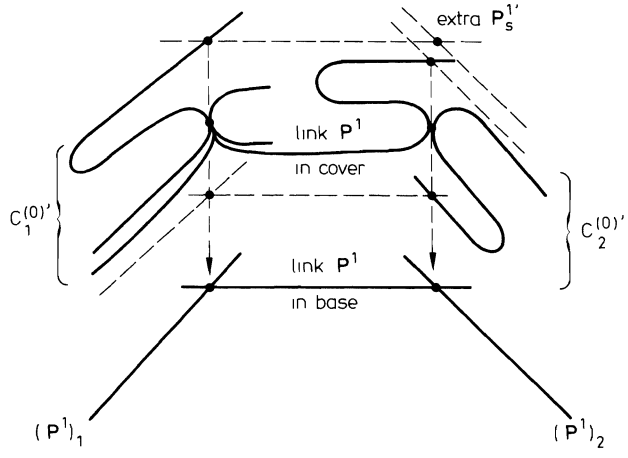
If it happens that $\ell_1 = \ell_2$, we can join them into an admissible cover by identifying p in $C_1^{(0)\prime}$, $C_2^{(0)\prime}$:

Example: $\ell_1 = \ell_2 = 2$, $n = 4$



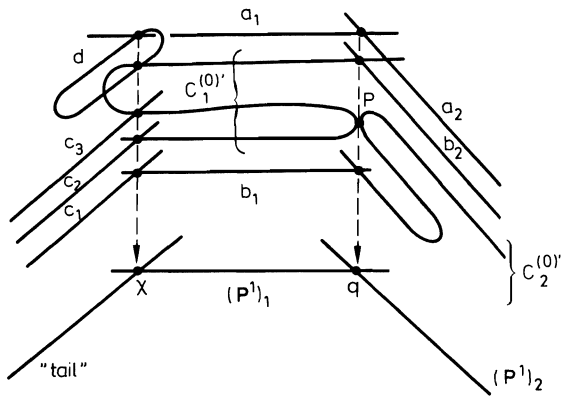
Otherwise, we need an extra $\mathbf{P}^1$ to join them:

Example. $\ell_1 = 3$, $\ell_2 = 2$, $n = 5$



If $\ell_1 > \ell_2$, add a $\mathbf{P}^1$ to the cover of degree ℓ_1 over the link $\mathbf{P}^1$ in the base ramified to order ℓ_1 at the point P in $C_1^{(0),'}$ and to order ℓ_2 at the point $p \in C_2^{(0),'}$. At this point, the covering is "filled in", exactly as in the case of an irreducible C. To make it admissible, extra $\mathbf{P}^1$'s are needed at multiple branch points of π_1 and π_2. To join the two points of $C_1^{(0),'}$ (resp. $C_2^{(0),'}$) over double points of $C_1^{(0)}$ (resp. $C_2^{(0)}$), linking $\mathbf{P}^1$'s are needed. Finally the degree of the whole cover must be augmented to k. Thus for example, in the case $\ell_1 = \ell_2 = 2$, $k_1 = k_2 = 3$, $k = 5$, one would add $\mathbf{P}^1$'s as follows:

Example:



All a_i, b_i, c_i, d are $\mathbf{P}^1$'s. In general, $k_1-\ell_1$ $\mathbf{P}^1$'s (like b_2) are added to $C_1^{(0)}$ to extend its other points over q across $(\mathbf{P}^1)_2$, and $k_2-\ell_2$ $\mathbf{P}^1$'s (like b_1) are added to $C_2^{(0)}$ to extend its other points over q across $(\mathbf{P}^1)_1$.

The main point to be checked is that k is large enough to accommodate all these curves. Thus if

A_i = set of double points of $C_i^{(0)}$ where $\mathscr{F}_i'$ is *not* invertible,

to carry out our constructions, we need

$$(k_1-\ell_1)+(k_2-\ell_2)+\max(\ell_1,\ell_2)+\#A_1+\#A_2$$

sheets, so this number must be $\leqq k$. But, in fact, as above,

$$k_i+\#A_i=\chi(\mathscr{F}_i')+p_a(C_i^{(0)})-1.$$

Moreover, since one of the sections of $\mathscr{F}_i$ used to define π_i vanishes to order at least ℓ, it follows that

$$\begin{aligned}\ell &\leqq \ell_i+\ell(\mathscr{F}_{i,p}/\mathscr{F}_{i,p}')\\ &\leqq \ell_i+\chi(\mathscr{F}_i)-\chi(\mathscr{F}_i').\end{aligned}$$

Thus if $\ell_1\geqq\ell_2$:

$$\begin{aligned}(k_1-\ell_1)+(k_2-\ell_2)+\max(\ell_1,\ell_2)+\#A_1+\#A_2 &= \chi(\mathscr{F}_1')+\chi(\mathscr{F}_2')-\ell_2+p_a(C^{(0)})-2\\ &\leqq \chi(\mathscr{F}_1)+\chi(\mathscr{F}_2)-\ell+2k-3\\ &=(3-k+\ell)-\ell+2k-3\\ &=k. \quad \text{Q.E.D.}\end{aligned}$$

Working this out more explicitly, we can draw a series of Corollaries from this theorem:

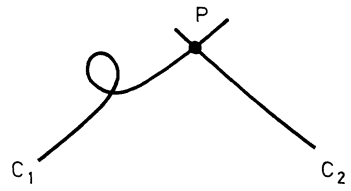
Corollary 1. *Let* $\operatorname{Int}\Delta_0$ *be the locus of irreducible curves* C *with one double point* p. *Writing these curves as the quotient of a curve* C' *of genus* $g-1$ *by identifying* 2 *distinct points* p_1, p_2, *we have:*

$$(\operatorname{Int}\Delta_0)\cap\bar{D}_k=\left\{(C',p_1,p_2)\,\middle|\,\begin{array}{l}\exists \text{ a line bundle } L \text{ on } C' \text{ of degree } k\\ \text{s.t. } h^0(L)\geqq 2,\ h^0(L(-p_1-p_2))\geqq 1\end{array}\right\}.$$

The map $(C', p_1, p_2) \mapsto C'$ *carries all components of this intersection onto* $\mathcal{M}_{g-1}$.

Proof. This comes from part (a) of the theorem. The condition above comes from the case where $\mathcal{F}$ is invertible. The other possibility is that $\mathcal{F}_p \cong \mathfrak{m}_p$. Such an $\mathcal{F}$ is the image of a line bundle M on C' of degree $k-1$ such that $h^0(M) \geqq 2$. Let $L = M(p_1)$ and the condition above is satisfied.

Corollary 2. *Let* $\operatorname{Int} \Delta_{0,1}$ *be the locus of curves of the type*



where C_2 *is smooth of genus* $g-1$, *and* C_1 *is* $\mathbf{P}^1$ *with* $0, \infty$ *identified. Describing these curves by the pair* (C_2, p), *we have:*

$$(\operatorname{Int} \Delta_{0,1}) \cap \bar{D}_k = \left\{ (C_2, p) \,\middle|\, \begin{array}{c} \exists \text{ a line bundle } L \text{ on } C_2 \text{ of degree } k \\ \text{s.t. } h^0(L) \geqq 2, h^0(L(-2p)) \geqq 1 \end{array} \right\}.$$

Proof. We use part (b) of the theorem. For $\mathcal{F}_1$ to exist, it is necessary and sufficient that $\chi(\mathcal{F}_1)$, ℓ satisfy:

$$\ell = 1, \chi(\mathcal{F}_1) \geqq 2 \quad \text{or} \quad \ell \geqq 2, \chi(\mathcal{F}_1) \geqq \ell.$$

But this means that $\mathcal{F}_2$ is a line bundle on C_2 such that either

$$\deg \mathcal{F}_2 \leqq k, h^0(\mathcal{F}_2) \geqq 2, h^0(\mathcal{F}_2(-\ell p)) \geqq 1, \ell \geqq 2$$

or

$$\deg \mathcal{F}_2 \leqq k-1, h^0(\mathcal{F}_2) \geqq 2, h^0(\mathcal{F}_2(-p)) \geqq 1.$$

All of these imply the existence of $\mathcal{F}_2$ such that

$$\deg \mathcal{F}_2 = k, h^0(\mathcal{F}_2) \geqq 2, h^0(\mathcal{F}_2(-2p)) \geqq 1.$$

For the next Corollary, we define:

$\mathcal{M}_{g,1}$ = moduli space of pairs (C, p), C smooth of genus g, $p \in C$
$S_{k,\ell,g}$ = subset of $\mathcal{M}_{g,1}$ of pairs C, p such that there exists a line bundle L on C of degree k with $h^0(L) \geqq 2$, $h^0(L(-\ell p)) \geqq 1$.

Corollary 3. *If* $1 \leqq g_1 \leqq k-1$ *and* $g_2 = g - g_1$, *let* $\operatorname{Int} \Delta_{g_1}$ *be the locus of curves* $C = C_1 \cup C_2$ *where* C_1, C_2 *are smooth with genus* g_1, g_2 *and* $C_1 \cap C_2 = \{p\}$. *Note that:*

$$\operatorname{Int} \Delta_{g_1} \cong \mathcal{M}_{g_1,1} \times \mathcal{M}_{g_2,1}.$$

Then

$$(\operatorname{Int} \Delta_{g_1}) \cap \bar{D}_k = \bigcup_{\frac{g+1}{2} \leqq k_1 \leqq \min(k, g_1)} (S_{k_1, 2k_1 - g_1, g_1} \times \mathcal{M}_{g_2,1})$$
$$\cup \bigcup_{\frac{g_2+1}{2} \leqq k_2 \leqq \min(k, g_2)} (\mathcal{M}_{g_1,1} \times S_{k_2, 2k_2 - g_2, g_2}).$$

Proof. We apply (b). Note that as the C_i are smooth, the $\mathscr{F}_i$ are line bundles. Let $k_i = \deg \mathscr{F}_i$. Now k_1, k_2 and ℓ satisfy:

$$(k_1+1-g_1)+(k_2+1-g_2)=\chi(\mathscr{F}_1)+\chi(\mathscr{F}_2)=3-k+\ell$$

hence

$$\begin{aligned} k_1+k_2 &= 1-k+\ell+(g_1+g_2) \\ &= k+\ell \end{aligned}$$

Now part (b) of Theorem 5 asserts that

$$(*) \qquad (\operatorname{Int} \Delta_{g_1}) \cap \bar{D}_k = \bigcup_{\substack{k_1, k_2, \ell \\ \text{with } k_1+k_2=k+\ell}} (S_{k_1,\ell,g_1} \times S_{k_2,\ell,g_2}).$$

But note that:

i) if $2k-\ell-1 \geqq g$, $S_{k,\ell,g} = \mathscr{M}_{g,1}$

ii) $S_{k,\ell,g} \subset S_{k+1,\ell+1,g}$.

For (i), see [7]; for (ii), if L has degree k and puts (C,p) in $S_{k,\ell,g}$, then $L(p)$ has degree $k+1$ and puts (C,p) in $S_{k+1,\ell+1,g}$. Now if k_1, k_2, ℓ satisfy

$$k_1+k_2=k+\ell,$$

note that:

$$\text{either } 2k_1-\ell-1 \geqq g_1, \quad \text{or} \quad 2k_2-\ell-1 \geqq g_2.$$

If not,

$$g_i \geqq 2k_i-\ell, \qquad i=1,2$$

hence

$$2k-1=g_1+g_2 \geqq 2k_1+2k_2-2\ell=2k$$

a contradiction.

Therefore, in the union in $(*)$, one of the 2 factors is always $\mathscr{M}_{g_1,1}$ or $\mathscr{M}_{g_2,1}$. But if

$$2k_1-\ell-1>g_1,$$

then if we replace k_2 by k_2+1 and ℓ by $\ell+1$, then

$$\begin{aligned} S_{k_1,\ell,g_1} \times S_{k_2,\ell,g_2} &= \mathscr{M}_{g_1,1} \times S_{k_2,\ell,g_2} \\ &\subset \mathscr{M}_{g_1,1} \times S_{k_2+1,\ell,g_2+1} \\ &= S_{k_1,\ell+1,g_1} \times S_{k_2+1,\ell,g_2+1}. \end{aligned}$$

Thus in the union in $(*)$, we need only consider the terms with

$$2k_1-\ell-1=g_1, 2k_2-\ell=g_2$$

or

$$2k_1-\ell=g_1, 2k_2-\ell-1=g_2.$$

This is exactly the Corollary.

Note that in the special cases $g_1=1$ or 2, the Corollary reduces to:

$$\begin{aligned} (\operatorname{Int} \Delta_1) \cap \bar{D}_k &= \mathscr{M}_{1,1} \times S_{k,2,g-1}, \\ (\operatorname{Int} \Delta_2) \cap \bar{D}_k &= (S_{2,2,2} \times \mathscr{M}_{g-2,1}) \cup (\mathscr{M}_{2,1} \times S_{k,3,g-2}) \\ &\quad \cup (\mathscr{M}_{2,1} \times S_{k-1,1,g-2}). \end{aligned}$$

Only in these two cases do we have components of $\operatorname{Int}\Delta_{g_1}\cap\bar{D}_k$ consisting entirely of curves with automorphisms: viz. $\mathscr{M}_{1,1}\times S_{k,2,g-1}$ and $S_{2,2,2}\times\mathscr{M}_{g-2,1}$ (the latter means a curve C_1 of genus 2 meeting a curve C_2 of genus $g-2$ at a point p, where $p\in C_1$ is a Weierstrass point).

Corollary 2 plus the case $g_1=1$ combine to say:

Corollary 4.

$$\begin{pmatrix}\textit{locus of curves } C_1\cup C_2,\ C_2 \textit{ smooth of genus } g-1\\ C_1 \textit{ smooth or singular of genus } 1,\ C_1\cap C_2=\{p\}\end{pmatrix}\cap\bar{D}_k=\bar{\mathscr{M}}_{1,1}\times S_{k,2,g-1}.$$

§5. Counting Pencils on the Generic Curve

We need to refine the results in Corollaries 1 and 3 of §4 by determining the intersection multiplicities in the intersections $\bar{D}_k\cap\operatorname{Int}\Delta_i$, and we will need to count the intersections of these cycles with the curves in Δ_i obtained by varying the double points used to get a singular stable curve in Δ_i. In order to do both, we need two fundamental results counting pencils on the generic curve of genus g, which are essentially but not completely contained in [7]. The results are:

Theorem A. *For all $g\geqq 1$ and all d such that*

$$\frac{g}{2}+1\leqq d\leqq g+1,$$

let

$$a(d,g)=(2d-g-1)\frac{g!}{d!(g-d+1)!}.$$

Then for almost all pairs (C,p), C a curve of genus g, $p\in C$, there is a finite number $a(d,g)$ of line bundles L on C of degree d such that

$$(*)\qquad h^0(L)\geqq 2,\ h^0(L(-2d-g-1)p))\geqq 1.$$

Moreover, for each L, $h^0(L)=2$, L is generated by $H^0(L)$, and $H^0(L)$ defines a covering

$$\pi\colon C\to\mathbf{P}^1$$

of degree d with all ordinary branch points except for one $(2d-g-1)$-fold branch point at p, all lying over distinct points of $\mathbf{P}^1$.

Theorem B. *For all $g\geqq 1$ and all d such that*

$$\frac{g}{2}+1\leqq d\leqq g$$

let

$$b(d,g)=(2d-g-1)(2d-g)(2d-g+1)\frac{g!}{d!(g-d)!}.$$

Then for almost all curves C of genus g, there is a finite number $b(d,g)$ of pairs (L,p), L a line bundle on C of degree d and points $p \in C$ such that

$$(**) \qquad h^0(L) \geqq 2,\ h^0(L(-(2d-g)p)) \geqq 1.$$

Moreover, for each L, $h^0(L)=2$, L is generated by $H^0(L)$ and $H^0(L)$ defines a covering

$$\pi\colon C \to \mathbf{P}^1$$

of degree d with all ordinary branch points and, if $2d-g \geqq 3$, one $(2d-g)$-fold branch point, all lying over distinct points of $\mathbf{P}^1$.

Proof. Theorem A follows directly from results of [7]. To see this, note first that if C is a general curve, $p \in C$ a general point, then by a naive dimension count there will exist no map

$$\pi\colon C \to \mathbf{P}^1,$$

ramified at p, whose branch divisor B on $\mathbf{P}^1$ is supported on fewer than $3g+1$ points; or any such map π of degree $d-k$ with a $(2d-g-1-k)$-fold branch point at p for $k>0$; or any such map of degree d with a $(2d-g-1)$-fold branch point at p such that

$$h^0(C, \pi^* \mathcal{O}_{\mathbf{P}^1}(1)) \geqq 3.$$

Thus, if (C,p) is general, any line bundle L of degree d on C satisfying $(*)$ must satisfy as well the rest of the conditions of the theorem. It then remains only to count the number of points of intersection in the dth symmetric product C_d of C of the cycles

$$C_d^1 = \{D \mid h^0(C, \mathcal{O}_C(D)) \geqq 2\}$$

and

$$X_p^{2d-g-1} = \{D \mid D-(2d-g-1)p \geqq 0\}.$$

This is readily done: the class of the cycle C_d^1 is found to be

$$c_d^1 = \frac{\theta^{g-d+1}}{(g-d+1)!} - \frac{x\theta^{g-d}}{(g-d)!}$$

where x is the class of the divisor $X_p = \{D \mid D-p \geqq 0\}$ and θ the pullback from the Jacobian of C of the class of the θ-divisor. By Poincaré's formula, the intersection numbers

$$x^{d-\alpha}\theta^{\alpha} = \frac{g!}{(g-\alpha)!}$$

and hence the intersection number

$$(c_d^1 \cdot x^{2d-g-1}) = a(d,g).$$

Finally, in [7] it is shown that if (C,p) is general, then C_d^1 is reduced, and C_d^1 intersects X_p^{2d-g-1} transversely in exactly $a(d,g)$ points, proving Theorem A.[3]

The proof of Theorem B, by contrast, requires a little more care. As in the first case, naive dimension counts show us that, for general C, no line bundle L of degree d on C can satisfy (**) and violate any of the remaining conditions. The problem is thus reduced to computing the intersection number in C_d of the cycle C_d^1 with the diagonal

$$\Delta^{2d-g}=\{D \mid D-(2d-g)p \geqq 0 \quad \textit{for some } p\in C\}$$

and showing that this intersection is transverse.

The intersection number is readily computed. The class of C_d^1 is as before; and for $p+q=g-d+1$ the degree of the pullback, via the diagonal map

$$\begin{aligned}\Delta\colon C\times C_{g-d} &\to C_d\\ (p,E) &\mapsto E+(2d-g)p\end{aligned}$$

of the class $x^p\cdot\theta^q$ is given by

$$\Delta^*(x^p\theta^q)=\frac{g!}{(g-p)!}(2d-g)^2p+(2d-g)q$$

(cf. [11, 2]). Combining these yields the intersection number

$$(C_d^1\cdot\Delta^{2d-g})=b(d,g).$$

It remains to check that for general C this intersection is transverse. To begin with, we may identify the tangent spaces involved as follows:

i) At any point $D\in C_d$, we have natural identifications

$$T_D(C_d)=\Gamma(\mathcal{O}_C(D)/\mathcal{O}_C)$$

and

$$T_D^*(C_d)=\Gamma(K_C/K_C(-D)),$$

with the pairing of tangent and cotangent spaces given by the residue.

ii) If $D=m\cdot p+q_1+\ldots+q_{d-m}$ with $p,q,\ldots,q_{d-m}$ all distinct, then Δ^m is smooth at D with tangent space equal to the subspace of $\Gamma(\mathcal{O}_C(D)/\mathcal{O}_C)$:

$$T_D(\Delta^m)=\text{Annihilator}[\Gamma(K_C((m-1)p-D)/K_C(-D))]$$

and

iii) The tangent space to C_d^1 at D with $h^0(C,\mathcal{O}_C(D))=2$ is given as the annihilator of the image of the map

$$\tilde{\mu}_0=r\circ\mu_0$$

[3] In point of fact, the statement made in [7] is that C_d^1 intersects the cycle $x_{p_1}\cap\ldots\cap x_{p_{2d-g-1}}$ transversely for $p_1,\ldots,p_{2d-g-1}$ general points of C; but the argument applies equally in this case

where

$$\mu_0: H^0(C, \mathcal{O}(D)) \otimes H^0(C, K_C(-D)) \to H^0(C, K_C)$$

is multiplication, and

$$r: H^0(C, K_C) \to H^0(C, K_C/K_C(-D))$$

is evaluation at D.

Statements i) and iii) are standard (cf. [2]), and ii) is elementary. Combining them, we see that a divisor D on C, whose associated line bundle $L = \mathcal{O}_C(D)$ satisfies the conditions of Theorem B, is a transverse point of intersection of C_d^1 and Δ^{2d-g} if and only if

$$H^0(C, K_C(-2D + (2d-g-1)p)) = 0.$$

To see that this in fact holds, let D be such a divisor, and

$$\pi: C \to \mathbf{P}^1$$

be the corresponding map; let

$$R_\pi = (2d-g-1)p + q_1 + \ldots + q_{3g-1}$$

be the ramification divisor of π. Let $\mathscr{H}$ be the versal deformation space for the map π, and let $\mathscr{H}' \subset \mathscr{H}$ be the subvariety of $\mathscr{H}$ of maps with a $(2d-g)$-fold branch point over $\pi(p)$. Then the tangent space to $\mathscr{H}$ at π is given by

$$T_\pi(\mathscr{H}) = H^0(C, \eta)$$

where η is the normal sheaf of π, defined by the exact sequence

$$0 \longrightarrow \theta_C \longrightarrow \pi^* \theta_{\mathbf{P}^1} \longrightarrow \eta \longrightarrow 0,$$

and the differential of the map

$$\phi: \mathscr{H} \to \mathscr{M}_g$$

is given by the coboundary map

$$H^0(C, \eta) \to H^1(C, \theta_C)$$

in this sequence. Finally, we may identify the tangent space to $\mathscr{H}'$ at π with the subspace of $T_\pi(\mathscr{H}) = H^0(C, \eta)$ of sections of η vanishing in a neighborhood of p, i.e., the sections of η':

$$0 \longrightarrow \theta_C \longrightarrow \pi^* \theta_{\mathbf{P}}(-(m-1)p) \longrightarrow \eta' \longrightarrow 0.$$

Now, we observe that since π is a point in a general fiber of $\phi|_{\mathscr{H}'}$, by Sard's theorem the differential ϕ_*, restricted to $T_\pi(\mathscr{H})$, must be surjective. This means that the map d below is surjective:

$$H^0(\eta') \xrightarrow{d} H^1(\theta_C) \longrightarrow H^1(\pi^* \theta_{\mathbf{P}^1}(-(m-1)P)) \longrightarrow 0.$$

But $\theta_{\mathbf{P}^1} \cong \mathcal{O}_{\mathbf{P}^1}(2)$ and $\pi^* \mathcal{O}_{\mathbf{P}^1}(1) = \mathcal{O}_C(D)$, hence

$$\begin{aligned}(0) &= H^1(\mathcal{O}_C(2D-(m-1)P)) \\ &= H^0(K_C(-2D+(m-1)P))^*,\end{aligned}$$

as required.

Using Theorems A and B, we can work out the intersection multiplicities in $(\operatorname{Int}\Delta_i) \cap \bar{D}_k$:

Theorem 6. a) *In the notation of Corollary 1, Δ_0 and $\bar{D}_k$ intersect generically transversely:*

$$(\operatorname{Int}\Delta_0)\cdot\bar{D}_k = \left\{(C', p_1, p_2) \,\middle|\, \begin{array}{c} \exists\ L \text{ on } C' \text{ of degree } k \\ h^0(L) \geqq 2,\ h^0(L(-p_1-p_2)) \geqq 1 \end{array}\right\} \text{ w. mult. one}$$

b) *In the notation of Corollary 3, if $g_1 \geqq 3$*

$$\begin{aligned}(\operatorname{Int}\Delta_{g_1})\cdot\bar{D}_k = &\sum_{\frac{g_1+1}{2} \leqq k_1 \leqq \min(k, g_1)} a(k_1+g_2-k+1, g_2)\cdot S_{k_1, 2k_1-g_1, g_1} \times \mathcal{M}_{g_2, 1} \\ &+ \sum_{\frac{g_2+1}{2} \leqq k_2 \leqq \min(k, g_2)} a(k_2+g_1-k+1, g_1)\, \mathcal{M}_{g_1, 1} \times S_{k_2, 2k_2-g_2, g_2}.\end{aligned}$$

This is still true for $g_1 = 1$ and 2 if the intersection is taken, not in $\bar{\mathcal{M}}_g$, but in the universal deformation space of a curve in $\operatorname{Int}\Delta_{g_1}$ (n.b. $\operatorname{Int}\Delta_1$ and $\operatorname{Int}\Delta_2$ have divisors in the singular locus of $\bar{\mathcal{M}}_g$, so the intersection product is not well-defined).

Proof of a). In the set-theoretic intersection $\operatorname{Int}\Delta_0 \cap \bar{D}_k$, there is an open dense set consisting of those $C = C'/(p_1 \sim p_2)$ for which

0) C' has no automorphisms

i) there is a *unique* line bundle L on C' of degree k with $h^0(L) \geqq 2$, $h^0(L(-p_1-p_2)) \geqq 1$.

ii) for this L, $h^0(L) = 2$ and $H^0(L)$ generates L

iii) if $\pi\colon C' \to \mathbf{P}^1$ is the covering of degree k defined by $H^0(L)$, then π has only ordinary branch points q_i and $\pi(q_i)$, $\pi(p_i)$ are all distinct points of $\mathbf{P}^1$.

In fact, by Theorem B, for almost all C' of genus $g-1 = 2k-2$, there are $b(k, 2k-2)$ pairs (L, q) of line bundles L such that $\deg L = k$, $h^0(L) \geqq 2$ and points $q \in C'$ such that

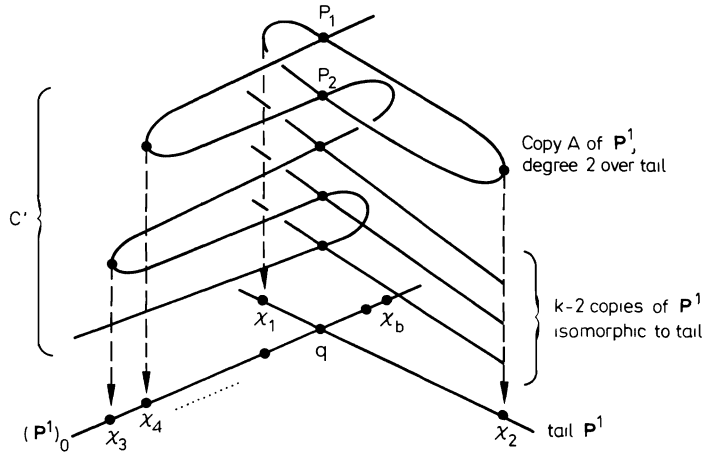
$$h^0(L(-2q)) \geqq 1.$$

Each distinct L_k defines a covering $\pi_k\colon C' \to \mathbf{P}^1$, hence a curve

$$\begin{aligned}&\Gamma_k \subset C' \times C' \\ &\Gamma_k = \{(x, y) \mid \pi_k x = \pi_k y\}.\end{aligned}$$

Since almost all $(p_1, p_2) \in C' \times C'$ are on a unique curve Γ_k if they are on any of them, this proves i), ii) and iii).

Now for such $C = C'/(p_1 \sim p_2)$, $[C]$ is the image of *a unique* point of $\bar{H}_{k,b}$, namely the admissible cover $\pi\colon C'' \to P$ of degree k:



More precisely, if Σ_b is the permutation group on b letters, Σ_b acts on $\bar{H}_{k,b}$ by permuting the labelling of the branch points $\{x_i\}$ and this covering is the unique point of $\bar{H}_{k,b}/\Sigma_b$ over $[C]$. Note that this admissible covering has an automorphism ϕ of order 2, however:

$\phi = \mathrm{id}$, on C'

$\phi|_{\mathrm{tail}\,\mathbf{P}^1} =$ automorphism fixing q, interchanging the two branch points x_1, x_2 on the tail

$\phi|_{\mathrm{copy}\,A} =$ automorphism fixing p_1, p_2, interchanging the two ramified points.

However, C'' has *no* automorphisms over P fixing all the points of ramification. Therefore, if z is the point of $\bar{H}_{k,b}$ representing $\pi\colon C'' \to P$, z is a smooth point but if $\sigma \in \Sigma_b$ is the permutation interchanging x_1, x_2 and fixing the other branch points, it follows that $\sigma z = z$. In fact, σ fixes the smooth divisor $\Delta \subset \bar{H}_{k,b}$ of all admissible coverings like $\pi\colon C'' \to P$ where $P = (\mathbf{P}^1)_0 \cup (\mathrm{tail}\,\mathbf{P}^1)$ but where $x_3, \ldots, x_b$ are allowed to vary in $(\mathbf{P}^1)_0$, and C' varies accordingly. There is a set $t_1, \ldots, t_{b-3}$ of local coordinates on $\bar{H}_{k,b}$ near z such that at all points of $\pi^{-1}(q)$, the universal family

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\pi"] \arrow[dr] & & \mathcal{P} \arrow[dl] \\ & \bar{H}_{k,b} & \end{tikzcd}$$

has local equations

$$xy = t_1,$$

$t_1 = 0$ is the local equation of Δ, and

$$\begin{aligned} \sigma^*(t_1) &= -t_1 \\ \sigma^*(t_i) &= \quad t_i, \qquad 2 \leqq i \leqq b-3. \end{aligned}$$

Then $t_1^2, t_2, \ldots, t_{b-3}$ are local coordinates on $\bar{H}_{k,b}/\{e,\sigma\}$ near the image of z.

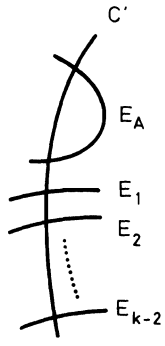
Now consider the local analytic curve γ: $\Delta_s \to \bar{H}_{k,b}/\{e,\sigma\}$ given by $t_1^2 = s$, $t_2 = \ldots = t_{b-3} = 0$. Consider the diagram:

$$\begin{tikzcd}
\bar{H}_{k,b}/\{e,\sigma\} \arrow[r, "\lambda"] & \bar{D}_k \arrow[d, phantom, "\cap"] \\
 & \bar{\mathfrak{m}}_g \arrow[d, phantom, "\cup"] \\
\Delta \arrow[uu, hook, "\gamma"] & \operatorname{Int} \Delta_0
\end{tikzcd}$$

We shall check that the curve $\lambda \circ \gamma$ in $\bar{\mathcal{M}}_g$ is transverse to $\operatorname{Int} \Delta_0$ at the point $[C]$. This implies that the divisor $\lambda^*(\operatorname{Int} \Delta_0)$ on $\bar{H}_{k,n}/\{e,\sigma\}$ is Δ with multiplicity one, hence $\bar{D}_k$ and $\operatorname{Int} \Delta_0$ meet transversely. To see that $\lambda \circ \gamma$: $\Delta \to \bar{\mathcal{M}}_g$ is transverse to Δ_0, consider the restriction of $\mathscr{C}$ to $t_2 = \ldots = t_{b-3} = 0$. It is a family of curves with only ordinary double points:

$$\begin{tikzcd}
\mathscr{C}_1 \arrow[d] \\
\Delta_{t_1}
\end{tikzcd}$$

whose fibres are smooth if $t_1 \neq 0$, and is



over $t_1 = 0$. Moreover, $\mathscr{C}_1$, as a surface, is smooth because its local equation is $xy = t_1$. It follows that on $\mathscr{C}_1$,

$$(E_A^2) = -2, (E_1^2) = \ldots = (E_{k-2}^2) = -1.$$

Blowing down the E's, we get a family of stable curves

$$\begin{tikzcd}
\mathscr{C}_2 \arrow[d] \\
\Delta_{t_1}
\end{tikzcd}$$

with the same curves when $t_1 \neq 0$, and with fibre C over $t_1 = 0$. Since E_A blows down to an ordinary surface double point, $\mathscr{C}_2$ has equation $xy = t_1^2$ at the image of E_A. But $\mathscr{C}_2/\Delta_{t_1}$ is just the pull-back to Δ_{t_1} of the family of curves over Δ_s given by the morphism $\lambda \circ \gamma \colon \Delta_s \to \overline{\mathscr{M}}_g$. Thus it is induced by a family

$$\begin{array}{c} \mathfrak{C}_3 \\ \downarrow \\ \Delta_s \end{array}$$

where $\mathscr{C}_3$ has local equation $x \cdot y = s$ at the image of E_A and is again smooth. This means that $\mathscr{C}_3$ restricts to the universal deformation of the singular point of C, hence is a curve on $\overline{\mathscr{M}}_g$ transverse to the locus Δ_0 of singular curves.

Proof of b). The situation is similar with the two sums, so let's take a sufficiently general point $[C] \in S_{k_1, 2k_1 - g_1, g_1} \times \mathscr{M}_{g_2, 1}$ and compute the intersection multiplicity here. Now $C = C_1 \cup C_2$, C_1 of genus g_1, C_2 of genus g_2, $C_1 \cap C_2 = \{p\}$. Since by Theorem B for almost all C_1 of genus g_1, there are $b(k_1, g_1)$ pairs (L_1, p) such that $\deg L_1 = k_1$, $h^0(L_1) \geqq 2$, $h^0(L_1(-(2k_1 - g_1)p)) \geqq 1$, and since for almost all C_1, the points p are distinct, it follows that for almost all $(C_1, p) \in S_{k_1, 2k_1 - g_1, g_1}$, there is a unique L_1 of degree k_1 with $h^0(L_1) \geqq 2$, $h^0(L_1(-(2k_1 - g_1)p)) \geqq 1$. Moreover, for this L_1, $h^0(L_1) = 2$ and $H^0(L_1)$ defines a covering

$$\pi_1 \colon C_1 \to (\mathbf{P}^1)_1$$

with general branching except for a $(2k_1 - g_1)$-fold branch point at p. As for (C_2, p), this is a general point of $\mathscr{M}_{g_2, 1}$, hence by Theorem A for all k_2 it has exactly $a(k_2, g_2)$ line bundles L_2 of degree k_2 with

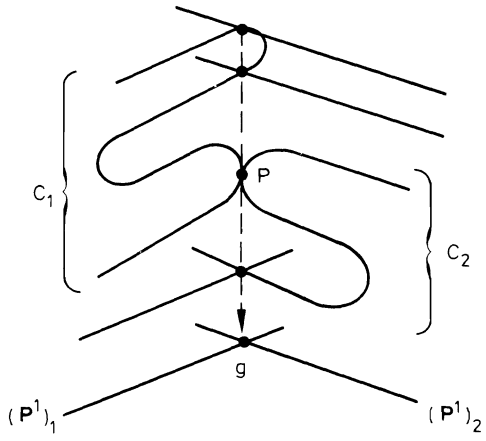
$$h^0(L_2) \geqq 2, \qquad h^0(L_2(-(2k_2 - g_2 - 1)p)) \geqq 1.$$

Moreover, for all of these, $h^0(L_2) = 2$ and $H^0(L_2)$ defines

$$\pi_2 \colon C_2 \to (\mathbf{P}^1)_2$$

with general branching except for a $(2k_2 - g_2)$-fold branch point at p. In particular, there are no line bundles L at all of degree k_2 with $h^0(L_2) \geqq 2$, $h^0(L_2(-(2k - g_2)p)) \geqq 1$. A little reflection shows that the only ways to lift C to a point of $\bar{H}_{k,b}$ are to use the admissible coverings defined by L_1 on C_1 of degree k_1 and L_2 on C_2 of degree k_2 where both have ℓ-fold branching at p and $\ell = 2k_1 - g_1 = 2k_2 - g_2 - 1$:

Example. $\ell = 2$,
$k_1 = 4$,
$k_2 = 3$,
$k = 5$.



Here the degree of π is k and over $(\mathbf{P}^1)_1$ we have C_1 of degree k_1 and $k-k_1$ copies of $\mathbf{P}^1$ mapping isomorphically to $(\mathbf{P}^1)_1$; and over $(\mathbf{P}^1)_2$, we have C_2 of degree k_2 and $k-k_2$ copies of $\mathbf{P}^1$ mapping isomorphically to $(\mathbf{P}^1)_2$.

In order to do this, we must choose k_2 by the equation:

$$2k_2-g_2-1=2k_1-g_1,$$

i.e.,

$$\begin{aligned} k_2 &= k_1+\frac{g_2+1-g_1}{2} \\ &= k_1+g_2+\frac{1-g_1-g_2}{2} \\ &= k_1+g_2-k+1. \end{aligned}$$

Now, whereas L_1 is unique, there are $a(k_2,g_2)$ choices for L_2, so this gives exactly $a(k_2,g_2)$ points of $\bar{H}_{k,b}/\Sigma_b$ over $[C]\in\bar{D}_k$. Thus $\bar{D}_k$ is a divisor with $a(k_2,g_2)$ branches at $[C]$, and what the theorem says is that each of these branches meets $\operatorname{Int}\Delta_{g_1}$ transversely. Assuming to begin with that $g_1\geqq 3$, the argument is similar to case (a) except simpler because the covering $\pi\colon C''\to P=(\mathbf{P}^1)_1\cup(\mathbf{P}^1)_2$ has no automorphisms. Thus $\bar{H}_{k,b}$ is smooth at the point z representing this covering and z is not fixed by any $\sigma\in\Sigma_b$. If we embed $\pi\colon C''\to P$ in a one-dimensional family of admissible coverings, we get

$$\begin{tikzcd} \mathscr{C}_1 \arrow[rr, "\pi"] \arrow[dr] & & \mathscr{P} \arrow[dl] \\ & \Delta_t & \end{tikzcd}$$

where locally near p

$$\begin{aligned} \mathscr{C}_1 &\text{ is } x\cdot y=t, \\ \mathscr{P} &\text{ is } u\cdot v=t^\ell, \\ \pi &\text{ is } u=x^\ell, v=y^\ell. \end{aligned}$$

Thus $\mathscr{C}_1$ is a smooth surface. All the extra rational curves in the fibre of $\mathscr{C}_1$ over $t=0$ are curves E with $(E^2)=-1$. Blowing these down, we get a family of stable curves

$$\mathscr{C}_2 \to \Delta_t$$

where $\mathscr{C}_2$ is still smooth and the fibre over $t=0$ is C. Therefore this family defines a curve in $\overline{\mathscr{M}}_g$ transverse to Δ_{g_1} at $[C]$.

Now in case $g_1=1$ or 2, there is the extra complication that C may have automorphisms. Rather than working on the coarse moduli space, it is more convenient to use the same argument to analyze the intersection of the 2 divisors $\operatorname{Int}\Delta_{g_1}$ and $\bar{D}_k$ in the universal deformation Δ^{3g-3} of C. The universal deformation of $\pi\colon C''\to P$ gives us a family

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "\pi"] \arrow[dr] & & P \arrow[dl] \\ & \Delta^{b-3} & \end{tikzcd}$$

plus a uniformization

$$\Delta^{b-3} \to \bar{D}_k \subset \Delta^{3g-3}$$

of the branch of $\bar{D}_k$ near $[C]$ defined by $\pi\colon C''\to P$. Restricting this deformation to a one-dimensional family as above, we get a curve on Δ^{b-3} mapped to a curve in Δ^{3g-3} transverse to the boundary component. Hence these divisors meet transversely in Δ^{3g-3}.

§ 6. The Class of the Divisor $\bar{D}_k$, II

In this section we finally compute the divisor class of $\bar{D}_k$ in $\operatorname{Pic}(\overline{\mathscr{M}}_{g,\mathrm{reg}})$ as a combination of $\lambda, \delta_0, \ldots, \delta_{k-1}$. We start with the relation

$$\bar{D}_k \equiv a\lambda - b_0\delta_0 - b_1\delta_1 - \ldots - b_{k-1}\delta_{k-1}$$

proved in § 3, but with unknown constants a, b_i, and we determine the constants by restricting this relation to suitable curves in $\overline{\mathscr{M}}_g$.

To describe the general method, suppose D is any divisor on $\overline{\mathscr{M}}_g$, i.e., a combination of codimension 1 subvarieties, but not necessarily a Cartier divisor (= one with a single local equation $f\in\mathbf{C}(\mathscr{M}_g)^*$ everywhere). Assume

$$D \equiv a\lambda - b_0\delta_0 - \ldots - b_{k-1}\delta_{k-1}$$

in $\operatorname{Pic}(\overline{\mathscr{M}}_{g,\mathrm{reg}})$. Let S be a smooth projective curve, let

$$\pi\colon \mathscr{C} \to S$$

be a family of stable curves of genus g, and let

$$\gamma\colon S \to \overline{\mathscr{M}}_g$$

be the induced morphism. Then first of all if $\gamma(S)\not\subset \operatorname{Supp}(D)$ we can define the divisor $\gamma^* D$ in a canonical way (even though $\gamma(S)$ may meet $\operatorname{Sing}(\overline{\mathscr{M}}_g)\cap \operatorname{Supp} D$). From general principles, we can apply the homomorphism:

$$\mathrm{Pic}(\overline{\mathcal{M}}_{g,\mathrm{reg}}) \to \mathrm{Pic}_{\mathrm{fun}}(\overline{\mathcal{M}}_g)$$

to $\mathcal{O}_{\overline{\mathcal{M}}_{g,\mathrm{reg}}}(D)$ and "evaluate" the image on the family $\pi: \mathscr{C} \to S$, to get a line bundle on S. But even more, for every $s \in S$, let

$$\pi_s: \mathscr{C}_s \to \Delta^{3g-3}$$

be the universal deformation of the curve C_s which is the fibre of π over s. Then the morphism γ factors:

$$\begin{pmatrix}\text{neigh of}\\ s\in S\end{pmatrix} \xrightarrow[\gamma_1]{} \Delta^{3g-3} \xrightarrow[\gamma_2]{} \overline{\mathcal{M}}_g.$$

$\gamma_2^*(D)$ is a divisor on Δ^{3g-3}, hence a Cartier divisor (as Δ^{3g-3} is smooth), and we then pull it back by γ_1 as Cartier divisor:

$$\gamma^*(D) \underset{\mathrm{def}}{=} \gamma_1^*(\gamma_2^* D).$$

On the other hand, $\gamma^*(\lambda)$ is defined directly as c_1 of the line bundle $\Lambda^g \pi_* \omega_{\mathscr{C}/S}$, and $\gamma^* \delta_i$ are defined directly too as in Knudsen [10]. That is to say, if $\gamma(S) \not\subset \Delta_i$, $\gamma^* \delta_i$ is represented by the Cartier divisor $\gamma^* \Delta_i$ $(i \neq 1)$ or $\frac{1}{2}\gamma^* \Delta_1$ $(i=1)$. And if $\gamma(S) \subset \Delta_i$ and the curves C_s have exactly one double point x_s of type i, then $\gamma^* \delta_i$ is c_1 of the line boundle:

$$s \mapsto \Lambda^2(\Omega^1_{C_s} \otimes \mathbf{k}(x_s))^*.$$

For our first family, choose general curves C_1, C_2 of genera $\alpha \leqq k-1$ and $2k-1-\alpha$ respectively. Choose $p \in C_2$ a general point and let $S_2 = C_1 \times \{p\} \subset C_1 \times C_2$; denote by S_1 the diagonal $\Delta \subset C_1 \times C_1$; and let

$$\pi: T \to C_1$$

be the curve over C_1 obtained by identifying S_1 and S_2 in $C_1 \times C_1 \cup C_1 \times C_2$ over C_2. This is, the family whose fiber over $q \in C_1$ is the reducible curve obtained by identifying $q \in C_1$ with $p \in C_2$.

The degrees of the divisors λ and δ_i on T are readily calculated: first of all, we see that $\pi_* \omega_{T/C_1}$ is the trivial bundle $(H^0(C_1, \omega_{C_1}) \oplus H^0(C_2, \omega_{C_2})) \otimes \mathcal{O}_{C_1}$, so that $\deg \lambda = 0$. Clearly, $\deg \delta_i = 0$ on T for $i \neq \alpha$; and since the normal space to Δ_α in $\overline{\mathcal{M}}_{2k-1}$ at the point T_q is the tensor product $T_q(C_1) \otimes T_p(C_2)$, we have

$$\begin{aligned}\deg \delta_\alpha &= \deg(N_{\Delta_\alpha/\mathcal{M}} \otimes \mathcal{O}_T)\\ &= \deg(N_{S_1/C_1 \times C_2} \otimes N_{S_2/C_1 \times C_2})\\ &= 2-2\alpha.\end{aligned}$$

It remains to calculate the degree of the divisor $\bar{D}_k$ on T. By § 4, Theorem 5, Corollary 3, set-theoretically

$$\bar{D}_k \cap C_1 = \left\{ q \in C_1 \;\middle|\; \begin{array}{l}\text{For some } i, 0 \leqq i \leqq \dfrac{\alpha-1}{2}\\ \exists \text{ line bundle } L \text{ on } C_1 \text{ of degree } \alpha - i\\ \text{with } h^0(L) \geqq 2, h^0(L(-(\alpha-2i)q)) \geqq 1\end{array} \right\}$$

which by Theorem B consists in $b(\alpha-i,\alpha)$ points. But by Theorem 6, the multiplicity of each point in $a(k-i,2k-1-\alpha)$. (Note that for generic C_1, the divisor $S_{\alpha-i,\alpha-2i,\alpha}$ on $\mathscr{M}_{\alpha,1}$ and the curve $\{(C_1,q)|q\in C_1\}$ on $\mathscr{M}_{\alpha,1}$ must meet transversely.) Therefore

$$\begin{aligned}\deg_{C_1}\bar{D}_k &= \sum_{i=0}^{\frac{\alpha-1}{2}} b(\alpha-i,\alpha)\,a(k-i,2k-1-\alpha)\\ &= \sum_{i=0}^{\alpha/2}\frac{\alpha!(2k-1-\alpha)!}{(\alpha-i)!\,i!(k-i)!\,(k-\alpha+i)!}(\alpha-2i-1)(\alpha-2i)^2(\alpha-2i+1)\\ &= \frac{1}{2}\sum_{i=0}^{\alpha}\frac{\alpha!(2k-1-\alpha)!}{(\alpha-i)!\,i!(k-i)!\,(k-\alpha+i)!}(\alpha-2i-1)(\alpha-2i)^2(\alpha-2i+1).\end{aligned}$$

(The last equality coming from the fact that the sum is unaltered under the substitution $i\rightsquigarrow\alpha-i$.) Now, writing

$$\begin{aligned}&(\alpha-2i-1)(\alpha-2i)^2(\alpha-2i+1)\\ &\quad=[(k-i)-(k-\alpha+i)]\times[(\alpha-i)(\alpha-i-1)(\alpha-i-2)-3(\alpha-i)(\alpha-i-1)\,i\\ &\qquad+3(\alpha-i)\,i(i-1)-i(i-1)(i-2)+3(\alpha-i)(\alpha-i-1)-3i(i-1)],\end{aligned}$$

this sum becomes

$$\begin{aligned}&\frac{\alpha(\alpha-1)(\alpha-2)}{2}\sum_{i=0}^{\alpha}\Bigg[\binom{\alpha-3}{i}\binom{2k-1-\alpha}{k-i-1}-3\binom{\alpha-3}{i-1}\binom{2k-1-\alpha}{k-i-1}\\ &\qquad+3\binom{\alpha-3}{i-2}\binom{2k-1-\alpha}{k-i-1}-\binom{\alpha-3}{i-3}\binom{2k-1-\alpha}{k-i-1}\\ &\qquad-\binom{\alpha-3}{i}\binom{2k-1-\alpha}{k-i}+3\binom{\alpha-3}{i-1}\binom{2k-1-\alpha}{k-i}\\ &\qquad-3\binom{\alpha-3}{i-2}\binom{2k-1-\alpha}{k-i}+\binom{\alpha-3}{i-3}\binom{2k-1-\alpha}{k-i}\Bigg]\\ &+\frac{\alpha(\alpha-1)}{2}\sum_{i=0}^{\alpha}\Bigg[3\binom{\alpha-2}{i}\binom{2k-1-\alpha}{k-i-1}-3\binom{\alpha-2}{i-2}\binom{2k-1-\alpha}{k-i-1}\\ &\qquad-3\binom{\alpha-2}{i}\binom{2k-1-\alpha}{k-i}+3\binom{\alpha-2}{i-2}\binom{2k-1-\alpha}{k-i}\Bigg]\\ &\qquad=\frac{\alpha(\alpha-1)(\alpha-2)}{2}\left[-6\binom{2k-4}{k-2}+8\binom{2k-4}{k-3}-2\binom{2k-4}{k-4}\right]\\ &\qquad\quad+\frac{\alpha(\alpha-1)}{2}\left[3\binom{2k-3}{k-2}-3\binom{2k-3}{k-3}\right]\\ &\qquad=-6\alpha(\alpha-1)(\alpha-2)\cdot\frac{(2k-4)!}{k!(k-2)!}+6\alpha(\alpha-1)\frac{(2k-3)!}{k!(k-2)!}\\ &\qquad=6\alpha(\alpha-1)(2k-1-\alpha)\frac{(2k-4)!}{k!(k-2)!}.\end{aligned}$$

We conclude, then, that

$$b_\alpha = \frac{\deg D_k}{2\alpha - 2} = \frac{3\alpha(2k-1-\alpha)}{k} \cdot \frac{(2k-4)!}{(k-1)!\,(k-2)!} \tag{3}$$

for $\alpha \geqq 2$.

Note that if we choose to vary the point $p \in C_2$ and fix q to be a general point of C_1, the resulting family $\pi\colon T \to C_2$ has intersection numbers

$$\begin{aligned} \deg \lambda &= 0, \\ \deg \delta_i &= 0, \qquad i \neq \alpha, \\ \deg \delta_\alpha &= 2 - 2(2k-1-\alpha), \\ &= -2(2k-2-\alpha) \end{aligned}$$

and, as before,

$$\begin{aligned} \deg D_k &= \sum_{i=0}^{\frac{\alpha+1}{2}} a(\alpha - i + 1, \alpha) \cdot b(k-i, 2k-1-\alpha) \\ &= \sum_{i=0}^{\frac{\alpha+1}{2}} \frac{\alpha!(2k-1-\alpha)!}{(\alpha-i+1)!\, i!(k-i)!\,(k+i-\alpha-1)!} (\alpha - 2i)(\alpha - 2i + 1)^2 (\alpha - 2i + 2). \end{aligned}$$

Setting $\beta = \alpha + 1$, this is

$$\begin{aligned} &\sum_{i=0}^{\beta/2} \frac{(\beta-1)!\,(2k-\beta)!}{(\beta-i)!\, i!(k-i)!\,(k+i-\beta)!} (\beta - 2i - 1)(\beta - 2i)^2 (\beta - 2i + 1) \\ &= \frac{2k-\beta}{\beta} \cdot 6\beta(\beta-1)(2k-\beta-1) \frac{(2k-4)!}{k!(k-2)!} \\ &= 6\alpha(2k-\alpha-1)(2k-\alpha-2) \frac{(2k-4)!}{k!(k-2)!}. \end{aligned}$$

As before, then, we conclude that

$$b_\alpha = \frac{\deg D_k}{2(2k-2-\alpha)} = \frac{3\alpha(2k-\alpha-1)}{k} \frac{(2k-4)!}{(k-1)!\,(k-2)!}$$

The difference here is that this formula is now established as well for $\alpha = 1$, i.e.,

$$b_1 = \frac{6(k-1)}{k} \cdot \frac{(2k-4)!}{(k-1)!\,(k-2)!}. \tag{4}$$

Our second family of curves lies entirely in Δ_1, consisting of a fixed curve C_2 of genus $g-1$, plus a variable elliptic curve E attached at a constant point of C_2. To construct it, let $\pi_1 : X \to B$ be a map from a smooth surface X to a curve B, whose fibers are all stable curves of genus 1; let $S_1 \subset X$ be a section of the map. On the other hand, let C_2 be a general curve of genus $g-1$, $p \in C_2$ a general point, and $S_2 = B \times \{p\} \subset B \times C_2$. Finally, we let

$$\pi\colon U \to B$$

be the curve over B obtained by identifying $S_1 \subset X$ and $S_2 \subset B \times C_2$.

To compute the degrees of the various divisors λ, δ_α, D_k on U, note that if we set

$$d = -\deg N_{S_1/X}$$

then the degree of the j-function associated to π_1 is just $12d$; accordingly

$$\begin{aligned} \deg \delta_0 &= 12d, \\ \deg \lambda &= \deg \pi_* \omega_{U/B} \\ &= \deg(\pi_1)_* \omega_{X/B} \\ &= d, \end{aligned}$$

and

$$\begin{aligned} \deg \delta_1 &= \deg(N_{S_1/X} \otimes N_{S_2/B \times C_2}) \\ &= \deg N_{S_1/X} \\ &= -d. \end{aligned}$$

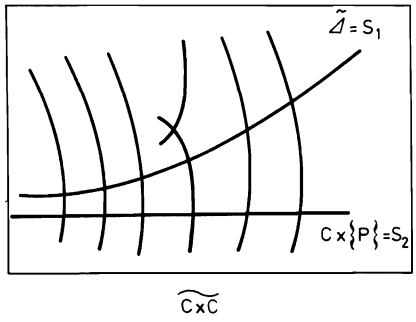
Of course, $\deg \delta_\alpha = 0$ for $\alpha \geqq 2$. On the other hand, we see from Corollary 4, § 4, that U *is disjoint from* $\bar{D}_k$, since C_2, being general, possesses no line bundles L of degree $k-1$ with $h^0(L)=2$ and only finitely many of degree k; and p, being a general point of C_2, will not be a branch point of any of the associated coverings $\pi\colon C_2 \to \mathbf{P}^1$. We conclude, then, immediately

$$a \cdot \deg \lambda - b_0 \deg \delta_0 - b_1 \deg \delta_1 = 0,$$

i.e.,

$$a - 12b_0 + b_1 = 0. \tag{5}$$

For our last family, take C a general curve of genus $2k-2$, $p \in C$ a general point. Let S_1 and S_2 be the proper transforms of the diagonal Δ and the cross-section $C \times \{p\}$ in the blow-up $\widetilde{C \times C}$ of $C \times C$ at the point (p,p). S_1 and S_2 being disjoint, we may identify them in $\widetilde{C \times C}$ over C to obtain a family



$W \xrightarrow{\pi} C$ of stable curves of genus g over C.

For this family, we have clearly

$$\begin{aligned} \deg \delta_1 &= 1, \\ \deg \delta_\alpha &= 0, \qquad \alpha > 1 \end{aligned}$$

and

$$\begin{aligned}\deg \delta_0 &= \deg(N_{S_1/\widetilde{C\times C}} \otimes N_{S_2/\widetilde{C\times C}})\\ &= (2-2(2k-2)-1)-1\\ &= 4-4k.\end{aligned}$$

We also have the sequence

$$0 \to H^0(C, \omega_C) \otimes \mathcal{O}_C \to \pi_* \omega_{W/C} \to \mathcal{O}_C \to 0$$

where the right-hand map is given by taking the residue at S_1, from which we conclude that

$$\deg \lambda = c_1 \pi_* \omega_{W/C} = 0.$$

Finally, since C and $p \in C$ are general, C will possess exactly $\frac{(2k-2)!}{(k-1)!(k)!}$ line bundles L of degree k with $h^0(L)=2$, each of which will have a unique section zero at p; and none of these sections will have a multiple zero. There are thus a total of $\frac{(2k-2)!}{(k-1)!(k)!}(k-1)$ points $q \in C$ such that $h^0(L(-p-q))=1$ for one of these L's. By Theorem 6a, these points occur with multiplicity one in the divisor induced by $\overline{D}_k$ on C, i.e.,

$$\deg_C \overline{D}_k = \frac{(2k-2)!}{k!(k-2)!}$$

and we conclude that

$$4(k-1)b_0 - b_1 = \frac{(2k-2)!}{k!(k-2)!}. \tag{6}$$

Since from (4) we have

$$b_1 = \frac{6(k-1)}{k} \frac{(2k-4)!}{(k-1)!(k-2)!}.$$

this yields

$$\begin{aligned}b_0 &= \frac{1}{4(k-1)}\left(\frac{(2k-2)!}{k!(k-2)!} + \frac{6(k-1)}{k}\frac{(2k-4)!}{(k-1)!(k-2)!}\right)\\ &= \frac{1}{4}\cdot\frac{(2k-4)!}{k!(k-2)!}(2(2k-3)+6)\\ &= \frac{(2k-4)!}{(k-1)!(k-2)!}\end{aligned}$$

and applying the relation (5), we have in turn

$$\begin{aligned}a &= 12b_0 - b_1\\ &= \left(12 - \frac{6(k-1)}{k}\frac{(2k-4)!}{(k-1)!(k-2)!}\right)\\ &= \left(6+\frac{6}{k}\right)\cdot\frac{(2k-4)!}{(k-1)!(k-2)!}.\end{aligned}$$

All in all, then, the coefficients appearing in the expression (1) are

$$a = 6(k+1)c,$$
$$b_0 = kc, \qquad b_\alpha = 3\alpha(2k-\alpha-1)c$$

where

$$c = \frac{(2k-4)!}{k!(k-2)!}.$$

References

1. Anderson, G.: Theta functions and holomorphic differential forms on compact quotients of bounded symmetric domains. Ph.D. Thesis, Princeton, 1980
2. Arbarello, E., Cornalba, M., Griffiths, P., Harris, J.: Topics in the theory of algebraic curves. In Princeton Univ. Press (1982 in press)
3. Brylinski, J.-L.: Propriétés de ramification à l'infini du groupe modulaire de Teichmüller. Ann. Ec. Norm. Sup., **12**, 295 (1979)
4. Freitag, E.: Der Körper der Siegelschen Modulfunktionen. Abh. Math. Sem. Univ. Hamburg,
5. Freitag, E.: Die Kodairadimension von Körpern automorpher Funktionen. Crelle, **296**, 162 (1977)
6. Fulton, W.: Hurwitz schemes and the irreducibility of moduli of algebraic curves, Annals of Math., **90**, 542 (1969)
7. Griffiths, P., Harris, J.: On the variety of special linear systems on a general algebraic curve. Duke Math. J., **47**, 233 (1980)
8. Grothendieck, A., Murre, J.P.: The tame fundamental group of a formal neighborhood of a divisor with normal crossings on a scheme. Lecture Notes Berlin-Heidelberg-New York: Springer Vol. 208. 1971
9. Kleiman, S., Laksov, D.: Another proof of the existence of special divisors. Acta Math., **132**, 163 (1974)
10. Knudsen, F.: The projectivity of the moduli space of stable curves. Math. Scand. (in press 1982)
11. MacDonald, I.G.: Symmetric products of an algebraic curve. Topology, **1**, 319 (1962)
12. Mumford, D.: Hirzebruch's Proportionality Principle in the non-compact case. Inv. Math., **42**, 239 (1977)
13. Mumford, D.: Stability of projective varieties. L'Ens. Math., **23**, 39 (1977)
14. Murre, J.P.: An introduction to Grothendieck's of the fundamental group, Tata Institute Lecture Notes; Bombay (1967)
15. Reid, M.: Canonical 3-folds. Les Journées de Géometrie Algébrique d'Angers, 1979, A. Beauville, ed.
16. Stillman, M.: PhD Thesis, Harvard, 1983
17. Tai, Y.-S.: Pluri-canonical differentials on the Siegel modular variety. Invent. Math. (in press 1982)

Oblatum 3-XI-1981

Added in Proof

Since this article was written, 2 improvements have been made in Theorem 3. John Harer has proven that $\mathrm{Pic}(M_g^0)$ is infinite cyclic, hence *any* divisor D on M_g^0 satisfies

$$[D] \equiv a \cdot \lambda, \qquad a \in \mathbf{Q}.$$

Secondly George Kempf has been able to carry through the calculation of the a_k of Theorem 3 via Porteous' formula, confirming that

$$a_k = 6(2k-4)!\,(k+1)/k!\,(k-2)!.$$