THE SPECTRUM OF DIFFERENCE OPERATORS AND ALGEBRAIC CURVES

BY PIERRE VAN MOERBEKE AND DAVID MUMFORD

Reprint from *Acta mathematica*, Vol. 143, 1979

THE SPECTRUM OF DIFFERENCE OPERATORS AND ALGEBRAIC CURVES

BY

PIERRE VAN MOERBEKE(1) and DAVID MUMFORD

Brandeis University, Waltham, Mass., U.S.A. *Cambridge, Mass., U.S.A.*

The explicit linearization of the Korteweg–de Vries equation [10, 18] and the Toda lattice equations [10, 12, 22] led to a theory relating periodic second order (differential and difference) operators to hyperelliptic curves with branch points given by the periodic and antiperiodic spectrum of the original operator. As a result the periodic second order operators with a given spectrum form a torus (except for a lower dimensional submanifold) which is the Jacobi variety of the defining curve. Krichever [15, 16, 17], motivated by further examples in the work of Zaharov-Shabat [30], showed how curves with certain properties lead to commuting differential operators reconfirming forgotten work by Burchnell and Chaundy [6]. Inspired by Krichever's ideas, Mumford [24] establishes then a dictionary between commutative rings of (differential and difference) operators and algebraic curves using purely algebraic methods. As an example, the Hill's operator whose spectrum consists of a finite number of non-degenerate bands leads to a finite number of independent differential operators commuting with the original Hill's operator and this commutative ring defines a curve of finite genus. However, the generic Hill's operator has an infinite number of bands and must be analyzed in terms of a hyperelliptic curve of infinite genus; see McKean and Trubowitz [21]. These analytical techniques have not yet been extended to higher order differential operators so that the correspondence between differential operators and curves, generically of infinite genus, is far from being understood. In view of this, it is important to discuss in detail the correspondence between periodic *difference* operators and algebraic curves (of finite genus). In the second order case, the periodic difference operators are good approximations of the periodic differential operators and

(1) Research for this paper was partially supported by NSF Grant No. MCS-75-05576 A01.

the corresponding curves are also hyperelliptic (see McKean and van Moerbeke [20]). Hopefully periodic differential operators will lead to infinite genus versions of the curves suggested by the difference operators.

In this work, we show that every so-called regular periodic difference operator of any order and not necessarily symmetric leads to a spectral curve $\mathcal{R}$ of a given type and a "regular" point on its Jacobi variety Jac ($\mathcal{R}$) and vice-versa. The regularity is a condition on the "symbol" of the difference operator, which in turn provides information about the infinite points of the spectral curve. Except for a finite number of translates of the theta-divisor, every point of Jac ($\mathcal{R}$) is regular. As a consequence, the isospectral class of regular difference operators C of a given order with a given h-spectrum for all Floquet multipliers h parametrizes the regular points of Jac ($\mathcal{R}$). This is the content of § 2.

How does a linear flow on Jac ($\mathcal{R}$) translate in terms of an isospectral deformation of the difference operators C? It translates into a system of ordinary differential equations, given by Lax-type commutation relations on the original difference operator: $\dot{C}=[C, A^+]$, where A^+ is the upper-triangular part of some operator A, constructed as follows: the linear flow above picks out a specific meromorphic function on $\mathcal{R}$, which is holomorphic on the affine part; this function then maps into the difference operator A. However there is more to it: these flows all derive from Hamiltonians and a symplectic structure reminiscent of the Kostant-Kirillov method of orbits for the group of upper-triangular matrices; for this method, see Kostant [14] and Abraham-Marsden [1]. It can be summarized as follows: the usual Bruhat decomposition of $SL(n, \mathbf{R})$ leads to a natural symplectic structure on the orbits in $\mathcal{N}^*$ in $sl(n, \mathbf{R})^*$ under the action of the triangular subgroup N; this fact does not apply as such because of the periodic and not necessarily symmetric nature of the difference operators, but it is nevertheless suggestive. The result is that the coefficients of the algebraic expressions for $\mathcal{R}$ can be regarded as Hamiltonians (depending on the difference operator C) in involution for the symplectic structure above; they lead to Hamiltonian flows, each of which is linearizable on Jac ($\mathcal{R}$); moreover all linear flows on Jac ($\mathcal{R}$) derive from such Hamiltonians. Its proper group-theoretiinterpretcal ation relates to the Kac-Moody extension of $sl(n, \mathbf{R})$, which will be developed, also for other classical groups, in a forthcoming paper by Adler and van Moerbeke [3]. For relations of this symplectic structure with the Gelfand–Dikii [8] symplectic structure and its group theoretical content, consult Adler [2]. All these considerations specialized to hyperelliptic curves leads then to the explicit linearization of the periodic Toda lattice equations.

In § 3, we deal with a number of interesting special cases. Whenever the curve $\mathcal{R}$ comes from a symmetric difference operator, it carries a natural involution, which, in turn, defines a linear subvariety of Jac ($\mathcal{R}$), called the Prym variety of $\mathcal{R}$. Then the manifold of isospectral *symmetric* operators coincides with Prym ($\mathcal{R}$) and all linear isospectral deformations are generated by meromorphic functions on $\mathcal{R}$, holomorphic on the affine part, as above, and moreover invariant under the involution. A similar statement holds for curves $\mathcal{R}$ defined by self-adjoint operators.

The entries of the difference operators can be regarded as Abelian functions on the Jacobi variety of the corresponding curve; then, using classical formulas (see Fay [11]), the entries can be expressed as quotients of theta functions (§ 5). Most of the results sketched above relate to *periodic* difference operators; its periodic nature is responsible for the division properties of the curve, as will be explained in § 2; in the Toda hyperelliptic case the latter amounts to the existence of two points P and Q on the curve, such that some integer multiple of $Q-P$ vanishes on Jac $(\mathcal{R})$. When the division properties do not hold and for a somewhat more restricted class of divisors, the associated difference operators are merely almost periodic. It remains an interesting open question to characterize those almost periodic difference operators which lead to finite genus curves; this is unknown even for the second order difference (and differential) operator case. Results close to those in paragraphs 2, 3 and 5 have been obtained by Krichever [16].

The relation between difference operators and curves (special curves) have been extended by Mumford [26] to a connection between two-dimensional difference operators and algebraic surfaces (spectral surface). As pointed out for one-dimensional operators, its "symbol" is a zero-dimensional difference operator and it defines the non-affine part of the curve; in the same way, in two dimensions, the "symbol" will be one-dimensional and the "symbol of the symbol" zero-dimensional; they lead to the non-affine behaviour of the spectral surface, which is crucial in the study of the Picard variety for the spectral surface (analogous to the Jacobi variety for curves). In fact, unlike for spectral curves, the spectral surface has trivial Picard variety, so that generic periodic two-dimensional difference operators do not admit isospectral deformations; for a fairly elementary exposition of Mumford's result, see P. van Moerbeke [23].

The first author thanks Professor P. Deligne for many helpful conversations, especially with regard to § 4.

Table of contents:

§ 1. Introduction

Let f be an infinite column vector $f=(\dots f_{-1}, f_0, f_1, \dots)^T$. Let D operate on f as the shift $Df_k=f_{k+1}$. Consider the difference operator C defined by

$$(Cf)_n=\sum_{k=-M'}^{M} c_{n,n+k} f_{n+k}=\left(\sum_{k=-M'}^{M} c_{n,n+k} D^k\right) f_n, \quad c_{i,j}\in \mathbf{C};$$

C acts on f as an infinite band matrix (c_{ij}) acting on f, zero outside the band $-M'\leqslant i-j\leqslant M$; C is said to have support $[-M', M]$. Assume C to be periodic of period N, i.e. $c_{i+N,j+N}=c_{ij}$; this amounts to the commutation relation $CS=SC$, where $S=D^N$. Let $(M,N)=n$ and $(M',N)=n'$; let $M_1 n=M$, $M_1' n'=M'$ and $N_1 n=N$, $N_1' n'=N$.

A difference operator C will be called *regular,* if the n quantities

$$\sigma_i=c_{i,i+M} c_{i+M,i+2M} \cdots c_{i+(N_1-1)M,i+N_1 M}, \quad 1\leqslant i\leqslant n$$

are all different from zero and different from each other and the same for the n' quantities

$$\sigma_i'=c_{i,i-M'} c_{i-M',i-2M'} \cdots c_{i-(N_1'-1)M',i-N_1' M'}, \quad 1\leqslant i\leqslant n'.$$

They involve only boundary elements, i.e., elements on the outer diagonals. Note that $\sigma_{i+n}=\sigma_i$ and $\sigma_{i+n'}'=\sigma_i'$.

A square matrix C_h of order N will be used throughout this paper. It is constructed as follows: if $N>M+M'$ consider the square matrix of order N taken from C, having c_{11} for upper left corner and c_{NN} for lower right corner, put the upper-left and lower-right triangular corners (see Figure 1) respectively in the upper-right and lower-left corner of the square block after multiplication by h^{-1} and h. In general, we write:

$$(C_h)_{i,j}=\sum_{k=-\infty}^{+\infty} h^k\cdot c_{i,j+kN}$$

In fact C_h contains all the information contained in C. Also observe that $C_h D_h=(CD)_h$ for any two difference operators C and D. The determinant of C_h-zI is readily seen to be a polynomial expression in z, h and h^{-1}, which has the form

$$\begin{aligned} F(h,h^{-1},z) &\equiv \det(C_h-zI)\\ &=A_0 h^M+A_1(z)h^{M-1}+\dots+A_M(z)+A_{M+1}(z)h^{-1}+\dots+A_{M+M'}h^{-M'}=0, \end{aligned}$$

where

$$A_0=(-1)^{M(N-M)}\prod_{k=1}^{N} c_{k,k+M}=(-1)^{M(N-M)}\prod_{i=1}^{n}\sigma_i\neq 0,$$

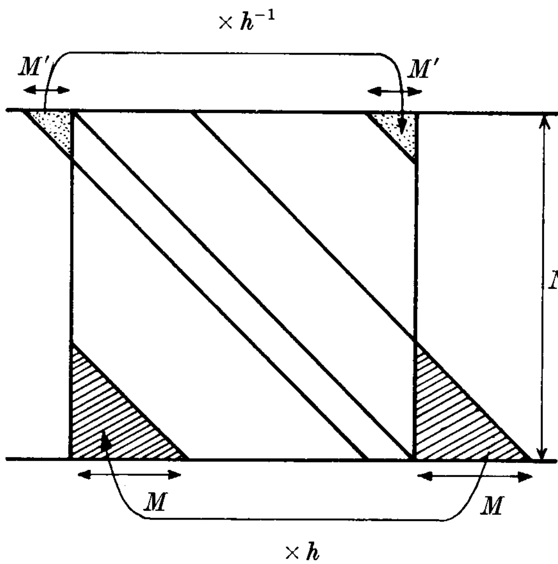

Figure 1

$$A_{M+M'} = (-1)^{M'(N-M')} \prod_{k=1}^{N'} c_{k,k-M'} = (-1)^{M'(N-M')} \prod_{i=1}^{n'} \sigma_i' \neq 0$$

and

$$A_M(z) = (-1)^N z^N + \dots .$$

Further information about the polynomials $A_j(z)$ is contained in Lemma 1, § 2.

For later use, we introduce some geometrical notations. Let $\mathcal{R}$ be an algebraic curve of genus g. We will allow $\mathcal{R}$ to be singular and even reducible, but we will always require $\mathcal{R}$ to be connected and reduced, i.e., no nilpotents in its structure sheaf. We also require that its singular points will be locally isomorphic to singular points of plane curves. In the singular case, the genus g will be the "arithmetic genus" of $\mathcal{R}$, i.e., $\dim H^1(\mathcal{O}_R)$ or $\dim H^0(\omega_R)$ where ω_R are the 1-forms η on $\mathcal{R}$ with poles only at singular points P of $\mathcal{R}$ and at those points

$$\sum_{\substack{\text{Branches } \gamma \\ \text{of } R \text{ at } P}} \operatorname{res}_\gamma (f\eta) = 0 \quad \text{all} \quad f \in \mathcal{O}_{P,R}. \tag{0}$$

At each singular point P, there is a 1-form η with "highest poles at P", i.e., every other 1-form η' satisfying (0) equals $f\eta$, for some $f \in \mathcal{O}_{p,R}$ (cf. Serre [28]). Let Jac ($\mathcal{R}$) be the Jacobian variety of $\mathcal{R}$ (the generalized Jacobian [27, 28], if $\mathcal{R}$ is singular). We will be interested in positive divisors $\mathcal{D}$ on $\mathcal{R}$ of degree g which are sufficiently generic. If $\mathcal{R}$ is smooth, a positive divisor $\mathcal{D}$ is just $\sum_{i=1}^{d} \nu_i$, $\nu_i \in \mathcal{R}$. In the singular case, $\mathcal{D}$ is given by such an expression *and* if k of the ν_i's equal a singular point P, then in addition to P occurring in $\mathcal{D}$ with multiplicities k, we must also give a k-dimensional space of "allowable" poles at P, i.e., a module $M_P(\mathcal{D})$ over $O_{P,\mathcal{R}}$ such that

$$O_{P,\mathcal{R}} \subset M_P(\mathcal{D}) \subset \mathbf{C}(\mathcal{R})$$

with $\mathbf{C}(\mathcal{R})$ being the field of meromorphic functions on $\mathcal{R}$ and such that

$$\dim M_P(\mathcal{D})/O_{P,\mathcal{R}} = k.$$

A general divisor $\mathcal{D}$ is an expression $\Sigma \pm \nu_i$, plus for all singular points, a finitely generated $O_{P,\mathcal{R}}$-module $M_P(\mathcal{D}) \subset \mathbf{C}(\mathcal{R})$ such that if P occurs with multiplicity k in $\mathcal{D}$, then

$$k = \dim (M_P + O_P/O_P) - \dim (M_P + O_P/M_P).$$

For every such $\mathcal{D}$, we define the space of functions with poles at $\mathcal{D}$ as:

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathbf{C}(\mathcal{R}) \,|\, (f) + \mathcal{D} \geqslant 0\}.$$

Here if $\mathcal{R}$ is singular, then at every singular point P, $(f) + \mathcal{D} \geqslant 0$ at P means $f \in M_P(\mathcal{D})$. We define the space of differentials with zeroes at $\mathcal{D}$ as

$$\Omega(-\mathcal{D}) = \{\text{meromorphic differentials } \eta \text{ on } \mathcal{R} \,|\, (\eta) \geqslant \mathcal{D}\}.$$

Here at singular points P, $(\eta) \geqslant \mathcal{D}$ means that for all $f \in M_P(\mathcal{D})$

$$\sum_{\substack{\text{Branches } \gamma \\ \text{of } \mathcal{R} \text{ at } P}} \operatorname{res}_\gamma (f\eta) = 0;$$

the Riemann–Roch theorem tells us as usual that

$$\dim \mathcal{L}(\mathcal{D}) - \dim \Omega(-\mathcal{D}) = \deg \mathcal{D} - g + 1.$$

Now let $\mathcal{D}$ be a positive divisor of degree g. $\mathcal{D}$ is *general* if $\dim \mathcal{L}(\mathcal{D}) = 1$, i.e., $\dim \Omega(-\mathcal{D}) = 0$. $\mathcal{D}$ will be called *regular* with regard to two infinite sequences of smooth points $\{P_i\}_{i \in \mathbf{Z}}$ and $\{Q_i\}_{i \in \mathbf{Z}}$ if[1]

$$\dim \mathcal{L}\left(\mathcal{D} + \sum_{i=1}^{k} P_i - \sum_{i=0}^{k} Q_i\right) = 0$$

---

[1] $\sum_{i=1}^{k} P_i = \sum_{1}^{k} P_i, \quad k \geqslant 1$

$= 0, \qquad k = 0$

$= -P_0 - P_{-1} - \dots - P_{k+1} \quad \text{for} \quad k \leqslant -1$

moreover $\sum_{i=0}^{k} Q_i = \sum_{0}^{k} Q_i, \quad k \geqslant 0$

$= 0 \qquad k = -1$

$= -Q_{-1} - Q_{-2} - \dots - Q_{k+1} \quad \text{for} \quad k \leqslant -2.$

§ 2. The correspondence between difference operators and curves

THEOREM 1. *There is a one-to-one correspondence between the two sets of data:*

(a) *a regular difference operator C of support $[-M', M]$ and period N, modulo conjugation by diagonal periodic operators.*

(b) *a curve $\mathcal{R}$, $(n+n')$ points on $\mathcal{R}$, a divisor $\mathcal{D}$ on $\mathcal{R}$ and two functions h, z on $\mathcal{R}$ subject to several conditions. $\mathcal{R}$ may be singular, but always has genus:*

$$g=\frac{(N-1)(M+M')-(n+n')+2}{2}.$$

The $(n+n')$ points $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_{n'}$ are smooth and have a definite ordering. We define P_i(resp Q_i) for all $i \in \mathbf{Z}$ by $P_{i+n}=P_i$ (resp $Q_{i+n'}=Q_i$). $\mathcal{D}$ has degree g and is regular for these sequences. The functions h and z have zeroes and poles as follows:

$$(h)=-N_1\sum_{i=1}^{n} P_i+N_1'\sum_{i=1}^{n'} Q_i$$

and

$$(z)=-M_1\sum_{i=1}^{n} P_i-M_1'\sum_{i=1}^{n'} Q_i+\textit{a positive divisor not containing the } P_i\textit{'s and } Q_i\textit{'s.}$$

Finally, $z^{N_1}h^{-M_1}$ (resp. $z^{N_1'}h^{M_1'}$) should take on distinct values at the P_i's (resp. the Q_i's).

Remark. The condition that $\mathcal{D}$ be regular reduces in this instance to the vanishing of a finite number of determinants involving differentials.

Proof. We first give the entire proof of this theorem, assuming for (a)$\Rightarrow$(b) that $F(h, h^{-1}, z)$ defines a non-singular curve in $\mathbf{C}^*\times\mathbf{C}$ and for (b)$\Rightarrow$(a) that $\mathcal{R}$ is non-singular. After we add a few words on the modifications necessary to deal with the singular case. First we show that (a) implies (b). The eigenvalues z and h such that

$$Cf=zf \quad \text{and} \quad Sf=hf \tag{1}$$

satisfy

$$\sum_{k=-M'}^{M} c_{n,n+k} f_{n+k}=zf_n, \quad 1\leqslant n\leqslant N \tag{2}$$

with

$$f_{n+k}=h^{\alpha} f_{n+k-\alpha N} \tag{3}$$

where α is the integer such that $\alpha N<n+k<(\alpha+1)N$. So, (2) can be rewritten

$$\sum_{k=-M'}^{M} c_{n,n+k} h^{\alpha} f_{n+k-\alpha N}=zf_n, \quad 1\leqslant n\leqslant N$$

or, what is the same

$$C_h \bar{f} = z\bar{f}$$

where $\bar{f} = (f_1, \ldots, f_N)^T$. Therefore the eigenvalues (z, h) of (1) satisfy

$$\det (C_h - zI) = F(h, h^{-1}, z) = 0, \tag{4}$$

which determines an algebraic curve over $\mathbf{C}$; vice versa, any couple (z, h) satisfying this algebraic relation provides a couple of eigenvalues for (1). Since A_0 and $A_{M+M'}$ in (4) are nonzero, the function h has its poles or zeros only at $z = \infty$. Therefore $F(h, h^{-1}, z)$ with z, h and $h^{-1} \in \mathbf{C}$, defines the affine part $\mathcal{R}_0$ of an algebraic curve $\mathcal{R}$. The equation

$$G(h, z) \equiv h^{M'} f(h, h^{-1}, z)$$

shows that $\mathcal{R}$ is an $M+M'$-sheeted covering of $\mathbf{C}$.

We now turn to the behaviour of the curve at the boundary. For this, we need to analyze the coefficients $A_j(z)$ of h^{M-j} in F very closely:

LEMMA 1. *The functions $A_j(z)$ are polynomials in z of degree k_j satisfying*

$$k_j \leqslant \frac{Nj}{M}, \quad 0 \leqslant j \leqslant M$$

and

$$k_{M+M'-j} \leqslant \frac{Nj}{M'}, \quad 0 \leqslant j \leqslant M'$$

with equality if and only if the right hand side is an integer, i.e., when $j = 0, M_1, 2M_1, \ldots, nM_1 = M$ in the first case and when $j = 0, M'_1, 2M'_1, \ldots, n'M'_1 = M'$ in the second case. For $j = \alpha M_1$, $0 \leqslant \alpha \leqslant n$, the coefficient of

$$z^{k_j} h^{M-j} = z^{\alpha N_1} h^{(n-\alpha) M_1}$$

in $F(h, h^{-1}, z) = 0$ is the symmetric polynomial of degree $n - \alpha$ in σ_i defined as

$$\tau_\alpha = \sum_{1 \leqslant i_1, \ldots, i_\alpha \leqslant n} \prod_{i \neq i_1, \ldots, i_\alpha} \sigma_i, \quad 0 \leqslant \alpha \leqslant n.$$

Likewise, for $j = \alpha M'_1$, $0 \leqslant \alpha \leqslant n'$, the coefficient of

$$z^{k_{M+M'-j}} h^{-M'+j} = z^{\alpha N'_1} h^{-(n'-\alpha) M'_1}$$

is the symmetric polynomial of degree $n' - \alpha$ in σ'_i, defined in a similar way as above.

Proof. It proceeds by induction. Let T_i be a typical term of $\det(C_h - zI) = 0$, containing h^i and consider how many times it appears in the determinant of $C_h - zI$. Clearly $T_0 = (-z)^N$. A term T_1 of maximal degree in g is formed by picking one of the entries $(N-k_1+1, k_2)$, i.e., $C_{N-k_1+1, N+k_2}h$ with $1 \leq k_i \leq M$ and $k_1 + k_2 \leq M+1$ and keeping the largest possible number of entries on the diagonal of $C_h - zI$. This choice excludes the entries (k_1, k_2) and $(N-k_1+1, N-k_1+1)$ of the diagonal and forces one to take the elements of the upper outer-diagonal. Since every column must have a representation in T_1, take the entry $(N-k_1+1-M, N-k_1+1)$ of the upper outer-diagonal, which excludes the entry $(N-k_1+1-M, N-k_1+1-M)$ of the diagonal. More generally, if T_1 contains the entry $(N-k_1+1-iM, N-k_1+1-(i-1)M)$, it does not contain the entry $(N-k_1+1-iM, N-k_1+1-iM)$ of the diagonal, as long as $1 \leq i \leq i_0$, where i_0 is the largest i such that $N-k_1+1-iM \geq k_2$. Two cases must now be distinguished: (a) if $N-k_1+1-i_0M = k_2$ the process is terminated and at least i_0+1 number of elements of the diagonal have been excluded and the degree of z in T_1 is bounded above by

$$N - i_0 - 1 = N - 1 - \left[\frac{N-k_1-k_2+1}{M}\right];$$

(b) if $N-k_1+1-i_0M > k_2$, the elements (k_2, k_2) and $(N-k_1+1-i_0M, N-k_1+1-i_0M)$ of the diagonal must be excluded, so that the degree of z in T_1 is bounded above by

$$N - i_0 - 2 = N - 2 - \left[\frac{N-k_1-k_2+1}{M}\right].$$

In either case, these estimates will be maximal provided $k_1 + k_2$ assumes its largest possible value $M+1$, so that the degree k_{M-1} of z in T_1 is bounded by $N - N/M$.

The rest of the argument goes by induction: if $k_{M-j} \leq N(M-j)/M$, then $k_{M-j-i} \leq N(M-j-1)/M$; this is done using the same method as above; pick the entry $(N-k_1+1, k_2)$ in $\det(C_h - zI)$ containing h which does not appear yet in T_j; this excludes a number of diagonal entries, bounded below by N/M, so that the degree k_{M-j-1} of z in T_{j+1} is bounded above by

$$\frac{Nj}{M} - \frac{N}{M}.$$

It remains to establish the second part of Lemma 1. Whenever Nj/M is an integer, this estimate is exact: $k_j = Nj/M$. It is done by exhibiting the term of exact degree Nj/M in z and $M-j$ in h. Consider the expression denoted by σ_i in C; let $\tilde{\sigma}_i$ be the expression obtained in the same fashion in C_h. A factor $C_{i+\alpha M,\, i+(\alpha+1)M}$ yields an h in $\tilde{\sigma}_i$ as soon as $i + \alpha M \leq kN < i + (\alpha+1)M$ for some integer k. Therefore $\tilde{\sigma}_i$ will be of degree

$$\#\{\alpha\in\mathbf{Z};\ 0\leqslant\alpha\leqslant N_1-1 \text{ such that } i+\alpha M\leqslant kN<1+(\alpha+1)M \text{ for some integer } k\}=M_1$$

in h. But $\tilde{\sigma}_i$ can be completed to a term in $\det(C_h-zI)$ by multiplying σ_i by the maximal possible elements of the diagonal. Every factor $C_{i+\alpha M,\, i+(\alpha+1)M}$ in $\tilde{\sigma}_i$ excludes exactly one diagonal element, because the integers $\{i+\alpha M\,|\,0\leqslant\alpha\leqslant N_1-1\}$ are all different modulo N. Therefore $\det(C_h-zI)$ contains a term of degree $N-N_1=(n-1)N_1$ in z and M_1 in h. All possible such terms are obtained by making a sum over the index i from 1 to n.

Moreover every term in $\det(C_h-zI)$ of degree βM_1 $(1\leqslant\beta\leqslant n)$ in h and $(n-\beta)N_1$ in z is obtained in a similar way from considering $\sigma_{i_1}\ldots\sigma_{i_\beta}$ and the corresponding combination $\tilde{\sigma}_{i_1}\ldots\tilde{\sigma}_{i_\beta}$ in C_h and to complete it with diagonal elements of C_h-zI to form a term in $\det(C_h-zI)$. This finishes the proof of Lemma 1.

We now turn to the behaviour of the curve at the boundary: the lemma implies that there are n distinct points $P_1,\ldots,P_n$ covering $z=\infty$, where $h=\infty$ and n' other points $Q_1,\ldots,Q_{n'}$ covering $z=\infty$, where $h=0$. To check this fact, define a local parameter t near each point P_i as follows

$$z=t^{-M_1}\quad\text{and}\quad h=C_it^{-N_1}+\ldots,\quad\text{where}\quad C_i^{M_1}=\frac{1}{\sigma_i}. \tag{5}$$

Near each point Q_i, define another local parameter t such that

$$z=t^{-M_1'}\quad\text{and}\quad h=C_i't^{N_1'}+\ldots,\quad\text{where}\quad C_i'^{\,M_1'}=\sigma_i'.$$

A typical term of $F=0$ containing h^{M-j} $(0\leqslant j\leqslant M)$ looks like

$$z^ih^{M-j}\quad\text{with}\quad 0\leqslant i\leqslant k_j\leqslant\frac{Nj}{M};$$

expressed in the local parameter t it appears as $t^{-M_1i-N_1(M-j)}$; the exponent satisfies $M_1i+N_1(n-j)\leqslant N_1M$ with equality if and only if $i=k_j$ with $j=\alpha M_1$, $\alpha=0,1,\ldots,n$. Therefore F can be expressed near $(z,h)=(\infty,\infty)$ as

$$\sum_{\alpha=0}^{n}(-1)^\alpha\tau_\alpha z^{\alpha N_1}h^{(n-\alpha)M_1}+\text{lower order terms.}$$

Then

$$\begin{aligned}z^{-N}F(h,h^{-1},z)&=\sum_{\alpha=0}^{n}(-1)^\alpha\tau_\alpha\left(\frac{h^{M_1}}{z^{N_1}}\right)^{n-\alpha}+\text{lower order terms}\\&=\prod_{i=1}^{n}\left(\sigma_i\frac{h^{M_1}}{z^{N_1}}-1\right)+\text{lower order terms,}\end{aligned}$$

which implies that $h^{M_1}z^{-N_1}$ assumes n distinct values $1/\sigma_i\neq0$. Therefore the point at $(z,h)=(\infty,\infty)$ separates into n distinct points $P_1,\ldots,P_n$.

The same analysis can now be applied to the point $(z, h) = (\infty, 0)$; there the upshot is that $h^{M_1'} z^{N_1'}$ assumes n' distinct values $\sigma_i' \neq 0$ and therefore the point at $(z, h) = (\infty, 0)$ separates into n' distinct points $Q_1, \ldots, Q_{n'}$. Let Q_i correspond to σ_i'. It follows that

$$(h) = -N_1 \sum_{i=1}^{n} P_i + N_1' \sum_{i=1}^{n'} Q_i$$

and

$$(z) = -M_1 \sum_{i=1}^{n} P_i - M_1' \sum_{i=1}^{n'} P_i + (\text{a positive divisor on } \mathcal{R}_0). \tag{6}$$

We are now in a position to compute the genus of $\mathcal{R}$ from Hurwicz's formula. Relation (6) implies at once that the ramification index of $\mathcal{R}_\infty = \mathcal{R} \setminus \mathcal{R}_0$ equals

$$V_\infty = n(M_1 - 1) + n'(M_1' - 1);$$

whereas the ramification index V_0 of $\mathcal{R}_0$ is given by the number of zeros of the different $\Delta = G_h'(h, z)$ or what is the same, by the number of poles of Δ. Near the point P_i, Δ behaves as

$$\begin{aligned}
\Delta &= \text{constant } (\neq 0) \times h^{M'} \frac{d}{dh} \prod (\sigma_i h^{M_1} - z^{N_1}) + \text{lower order terms} \\
&= \text{constant } (\neq 0) \times h^{M'} \sum_{k=1}^{n} \sigma_k M_1 \prod_{i \neq k} \left(\frac{\sigma_i}{\sigma_k} z^{N_1} + z^{N_1} \right) h^{M_1 - 1} + \text{lower order terms} \\
&= \text{constant } (\neq 0) \times h^{M' + M_1 - 1} z^{(n-1) N_1} + \text{lower order terms} \\
&= \text{constant } (\neq 0) \times t^{-N_1(M_1 + M' - 1) - M_1 N_1 (n-1)} + \text{lower order terms,}
\end{aligned}$$

and, using a similar argument, near Q_i

$$\Delta = \text{constant} (\neq 0) \times t^{-N_1'} + \text{lower order terms.}$$

Therefore

$$V_0 = N(M + M')$$

and

$$g = \frac{V_0 + V_\infty}{2} - (M + M') + 1 = \frac{(N-1)(M + M') - (n + n') + 2}{2}$$

The eigenvectors f common to C and S can be regarded as column vectors of meromorphic functions; using the normalization $f_0 = 1$, $f = (f_1, f_2, f_3, \ldots, f_{N-1}, h)^{\mathrm{T}}$. Since f satisfies

$$(C_h - zI) f = 0,$$

f_k can be expressed as follows

$$f_k = \frac{\Delta_{1,k}}{\Delta_{1,i}} f_i = \frac{\Delta_{2,k}}{\Delta_{2,i}} f_i = \ldots = \frac{\Delta_{N,k}}{\Delta_{N,i}} f_i, \quad 1 \leqslant i, k \leqslant N$$

where $\Delta_{i,j} = (-1)^{i+j} \times (i, j)$th minor of $C_h - zI$. In particular

$$f_k = \frac{\Delta_{N,k}}{\Delta_{N,N}} h = \frac{\Delta_{k,k}}{\Delta_{k,N}} h$$

which expresses each f_k as a rational function in z and h. In order to find the divisor $\mathcal{D}$ on $\mathcal{R}$, it is important to investigate the nature of the poles and zeros of these functions on $\mathcal{R}_\infty$.

LEMMA 2. *The meromorphic functions f_k satisfy the following conditions at infinity, If* $0 \leqslant i \leqslant n-1$, $\alpha \in \mathbf{Z}$

(i) *at* P_k, order $\left(\dfrac{f_{k+n\alpha+i}}{f_k}\right) \geqslant -\alpha$, *with equality if* $i = 0$

(ii) *at* Q_k, order $\left(\dfrac{f_{k+n\alpha+i}}{f_k}\right) \geqslant \alpha$, *with equality if* $i = 0$.

Proof. In order to investigate the poles at $P = \bigcup P_i$, consider the new set of coordinates

$$\alpha = z^{N_1} h^{-M_1} \quad \text{and} \quad \beta = z^{\delta} h^{\gamma}$$

with $\gamma N_1 + \delta M_1 = -1$ with $N_1 > \delta \geqslant 0$. (It is always possible to find two such integers γ and δ, since N_1 and M_1 are relatively prime.) These coordinates are most convenient, because near a point P_i, α and β behave as follows.

$$\begin{aligned} \alpha &= \sigma_i + O(t) \\ \beta &= \text{constant}(\neq 0)t + O(t^2). \end{aligned} \tag{7}$$

The meromorphic functions α and β can also be regarded as the eigenvalues of the commuting operators $A = C^{N_1} S^{-M_1}$ and $B = C^{\delta} S^{\gamma}$ with entries a_{ij} and b_{ij} respectively. In fact A is a lower triangular difference operator (i.e., $a_{ij} = 0$, for $i < j$), whereas B is a strictly lower triangular difference operator, whose first non-zero subdiagonal is at $-n$ (i.e., $b_{ij} = 0$ for $i - n < j$). Moreover $a_{kk} = \sigma_k$, because a_{kk} is obtained from C and S as

$$c_{k,k+M} D^M c_{k,k+M} D^M \dots c_{k,k+M} D^M S^{-M_1} = \sigma_k D^0;$$

therefore A has an n-periodic diagonal with entries $\sigma_k \neq 0$ different from one another. The fact that A and B commute induces relations between the a_{ij}'s and b_{ij}'s. The first one expresses that $a_{kk} = a_{k+n,k+n}$ which is the same as $\sigma_k = \sigma_{k+n}$. From (7), β itself can be used as a local parameter near P_m; let α admit the following Taylor expansion in β near P_m

$$\alpha = \sum_{i=0}^{\infty} \alpha_i \beta^i \quad \text{with} \quad \alpha_0 = \sigma_m.$$

Express the fact that $Bf=\beta f$ and $Af=\alpha f$ starting with f_0:

$$\begin{aligned}
&\dots+b_{0,-n-2}f_{-n-2}+b_{0,-n-1}f_{-n-1}+b_{0,-n}f_{-n}=\beta f_0\\
&\dots+b_{-1,-n-2}f_{-n-2}+b_{-1,-n-1}f_{-n-1}\qquad\qquad=\beta f_{-1}\\
&\dots+b_{-2,-n-2}f_{-n-2}\qquad\qquad\qquad\qquad\qquad=\beta f_{-2}\\
&\text{etc.}
\end{aligned}$$

and

$$\begin{array}{lll}
\dots+a_{0,-n}f_{-n} & +a_{0,-n+1}f_{-n+1}+\dots+a_{0,-2}f_{-2}+a_{0,-1}f_{-1}+\sigma_nf_0 & =(\alpha_0+\alpha_1\beta+\alpha_2\beta^2+\dots)f_0\\
\dots a_{-1,-n}f_{-n} & +a_{-1,-n+1}f_{-n+1}+\dots+a_{-1,-2}f_{-2}+\sigma_{n-1}f_{-1} & =(\alpha_0+\alpha_1\beta+\alpha_2\beta^2+\dots)f_{-1}\\
\dots a_{-2,-n}f_{-n} & +a_{-2,-n+1}f_{-n+1}+\dots+\sigma_{n-2}f_{-2} & =(\alpha_0+\alpha_1\beta+\alpha_2\beta^2+\dots)f_{-2}\\
\dots & \dots & \dots\\
\dots+a_{-n+1,-n}f_{-n}+\sigma_1f_{-n+1} & & =(\alpha_0+\alpha_1\beta+\alpha_2\beta^2+\dots)f_{-n+1}\\
\dots+\sigma_nf_{-n} & & =(\alpha_0+\alpha_1\beta+\alpha_2\beta^2+\dots)f_{-n}
\end{array}$$

This is in fact a finite system of equations, because multiplication by h^{-1} shifts all the indices by $-N$. First consider the point P_n, where $\alpha_0=\sigma_n$. The result of this lemma will be established in this case; the extension to the other points P_i will then be straightforward.

Step 1

In the proof of this lemma, the following statement will be used at several occasions: fix $k\in\mathbf{Z}$, $k\leqslant 1$; if at P_n

$$\text{order}\left(\frac{f_{-(\alpha n+i)}}{f_{-\alpha n}}\right)\geqslant k\quad\forall i\geqslant n$$

then the same inequality holds for $i\geqslant 1$.

Suppose the contrary; then the second system of equations leads to an homogeneous triangular system of $n-1$ equations in $n-1$ unknowns $f_{-i}^{(k_1)}$, $1\leqslant i\leqslant n-1$, where

$$\frac{f_{-(\alpha n+i)}}{f_{-\alpha n}}=f_{-i}^{(k_1)}\beta^{k_1}+\text{lower order terms}\quad\text{with}\quad k_1<k,$$

to wit,

$$\sum_{j=n-1}^{i-1}a_{-i,-j}f_{-j}^{(k_1)}+(\sigma_{n-i}-\sigma_n)f_{-i}^{(k_1)}=0,\quad 1\leqslant i\leqslant n-1.$$

Its only solution is given by $f_{-j}^{(k_1)}=0$ $(1\leqslant j\leqslant n-1)$, since its determinant equals

$$\prod_{i=1}^{n-1}(\sigma_{n-i}-\sigma_n)\neq 0.$$

Step 2

Next, we show that at P_n

$$\text{order}\left(\frac{f_{-(\alpha n+i)}}{f_{-\alpha n}}\right) \geqslant 1 \quad \text{for} \quad i \geqslant 1,$$

with equality if $i=n$.

In view of Step 1, it suffices to show the statement for $i \geqslant n$. Suppose the contrary; then for some $\gamma \geqslant n$

$$\frac{f_{-(\alpha n+\gamma)}}{f_{-\alpha n}} = f_\gamma^{(-k)}\beta^{-k} + \text{lower order terms},$$

with $k \geqslant 0$ and $\gamma \geqslant n$ and with $f_\gamma^{(-k)} \neq 0$. Then $\beta f_{-(\alpha n+\gamma)} f_{-\alpha n}^{-1}$ is of order $-k+1$ and the γth equation of the first system tells you that for some $\gamma_1 \geqslant 2n$,

$$\text{order}\left(\frac{f_{-(\alpha n+\gamma_1)}}{f_{-\alpha n}}\right) = -k+1, \quad \gamma_1 \geqslant 2n$$

and by induction

$$\text{order}\left(\frac{f_{-(\alpha n+\gamma_i)}}{f_{-\alpha n}}\right) = -k+i \quad \text{for some} \quad \gamma_i \geqslant (i+1)n \quad \text{with} \quad i \geqslant 1.$$

In particular for $i=N_1-1$, we find some $\gamma_i \geqslant N$ so that

$$\text{order}\left(\frac{f_{-(\alpha n+\gamma_i)}}{f_{-\alpha n}}\right) = -k+N_1-1.$$

This is a contradiction, because for $k \geqslant N$,

$$\text{order}\left(\frac{f_{-(\alpha n+k)}}{f_{-\alpha n}}\right) = \text{order}\left(\frac{f_{-(\alpha n+k-N)}}{f_{-\alpha n}} h^{-1}\right) \geqslant -k+N_1.$$

To show the equality in Step 2, when $i=n$, notice that for $k \geqslant 1$

$$\text{order}\left(\frac{f_{-(\alpha+1)n-k}}{f_{-\alpha n}}\right) = \text{order}\left(\frac{f_{-(\alpha+1)n-k}}{f_{-(\alpha+1)n}}\right) + \text{order}\left(\frac{f_{-(\alpha+1)n}}{f_{-\alpha n}}\right) \geqslant 2.$$

But in order to satisfy the αnth equation of the first system (which is analogous to the first one of the same system), you must have

$$\text{order}\left(\frac{f_{-(\alpha+1)n}}{f_{-\alpha n}}\right) = 1.$$

Step 3

It is now straightforward to extend Steps 1 and 2 to the other points P_j. More specifically, at the point P_{n-i} (where $\sigma_0=\sigma_{n-1}$), divide both systems by the function f_{-i}. Then the systems so obtained are the same as the ones above, in which f_{-k} is replaced by

$$g_{-k}=\frac{f_{-k-i}}{f_{-i}}$$

so that at P_{n-i}

$$\text{order}\left(\frac{f_{-(\alpha n+i+j)}}{f_{-(\alpha n+i)}}\right)\geqslant 1 \quad \text{for} \quad j\geqslant 1 \quad \text{with equality if} \quad j=n.$$

By multiplication of the numerator and denominator with the same power of h, one observes that the result holds for any $\alpha\in\mathbf{Z}$. Therefore at P_k, letting $k=n-i$, $0\leqslant j\leqslant n-1$

$$\begin{aligned}\text{order}\,\frac{f_{k+\beta n+j}}{f_k}&=\text{order}\left(\frac{f_{-[-(\beta+2)n+i+(n-j)]}}{f_{-(-n+i)}}\right)\\&=\text{order}\left(\frac{f_{-[-(\beta+2)n+i+(n-j)]}}{f_{-[-(\beta+2)+i]}}\right)+\sum_{\alpha=1}^{\beta+1}\text{order}\left(\frac{f_{-[-(\alpha+1)n+i]}}{f_{-[-\alpha n+i]}}\right)\\&\geqslant 1-(\beta+1)=-\beta\end{aligned}$$

with equality if $j=0$.

A similar analysis can be made near the points $Q_1, \ldots, Q_{n'}$, by considering the new coordinates

$$\alpha'=z^{N_1'}h^{M_1'} \quad \text{and} \quad \beta=z^{\delta'}h^{\gamma'}$$

where $\gamma' N_1'-\delta' M_1'=1$. Near a point Q_i, α and β behave as follows

$$\begin{aligned}\alpha'&=\sigma_i'+O(t)\\ \beta&=\text{constant}\;(\neq 0)\,t+O(t^2);\end{aligned}$$

the operators $C^{N_1}S^{M_1}$ and $C^{\delta'}S^{\gamma'}$ are both upper triangular with eigenvalues α' and β' and with a nonzero diagonal in the first case and with a nonzero n'th subdiagonal (above the main diagonal) in the second case. This establishes the result of Lemma 2.

Define now $\mathcal{D}$ to be the *minimal positive divisor* on $\mathcal{R}$ such that[1]

$$(f_k)+\mathcal{D}\geqslant -\sum_{i=1}^{k}P_i+\sum_{i=0}^{k-1}Q_i \quad \text{for all} \quad k\in\mathbf{Z}.$$

---

[1] Recall the convention in the footnote of § 1.

It is finite, since it suffices to consider the functions $f_1, \ldots, f_{N-1}$ only. Note that, by Lemma 2, in the kth inequality above, the 2 divisors have *equal orders* at P_k and at Q_k.

LEMMA 3. *$\mathcal{D}$ is a divisor of order g.*

Proof. In a first step, one shows that every function f in

$L = \{f$ meromorphic with $(f) + \mathcal{D} \geqslant$ any linear combination of P_i and Q_i with coefficients in $\mathbf{Z}\}$

can be expressed as a linear combination of functions f_k. Let L_1 be the linear span of the functions f_k and define the ring

$$R = \mathbf{C}[h, h^{-1}, z]/F(h, h^{-1}, z);$$

R contains z and h. The space L_1 is an R-module, because $h^{\pm 1} f_k = f_{k \pm N}$ and $zf_k = (Cf)_k$. Moreover $L_1 \supset R$, because $f_{\alpha N} = h^\alpha$ and $z = zf_0 = (Cf)_0$. If $L_1 \subsetneqq L$, then there is a maximal ideal m in R such that

$$L_1 \subset mL;$$

a maximal ideal m in R is naturally associated with a point $p \in \mathcal{R}_0$ (i.e., $\mathcal{R} \setminus \bigcup P_i \setminus \bigcup Q_i$) such that

$$m = \{g \in R \mid g(p) = 0\}.$$

In fact $\mathcal{D} - p \geqslant 0$; because if not, all functions in L_1 would vanish at p; but this is a contradiction, since the functions h and h^{-1} have no common zeros on $\mathcal{R}$ and they both belong to L_1. Therefore every function f_k would be such that

$$(f_k) + (\mathcal{D} - p) \geqslant -\sum_{i=1}^{k} P_i + \sum_{i=0}^{k-1} Q_i \quad \forall k \in \mathbf{Z}.$$

This contradicts the fact that $\mathcal{D}$ is minimal. Therefore $L = L_1$. Choose integers k_1 and k_2 such that $k_1 + k_2 + \text{order}\ (\mathcal{D}) > 2g - 2$. In this next step, the dimension of

$$\mathcal{L}(\mathcal{E}) \equiv \mathcal{L}\left(\sum_{i=1}^{k_1} P_i + \sum_{i=0}^{k_2 - 1} Q_i + \mathcal{D}\right)$$

will be counted in two different ways: on the one hand using the Rieman-Roch theorem [1]

$$\dim \mathcal{L}(\mathcal{E}) = k_1 + k_2 + \text{order}\ (\mathcal{D}) - g + 1 \tag{11}$$

---

[1] Since order $(\mathcal{E}) > 2g - 2$, $\dim \Omega(-\mathcal{E}) = 0$.

and on the other hand in a direct way to be explained below. Any function φ in $\mathcal{L}(\mathcal{E})$ can be expressed as a linear combination $\varphi = \sum_{i=-k_2}^{k_1} a_i f_i$. To prove this, let φ have a pole of maximal order[1], say β, among the points P_i and let P_j be the one with maximal index. Then subtract from φ an appropriate multiple of $f_{n(\beta-1)+j}$; the latter belongs to the space $\mathcal{L}(\mathcal{E})$, since $n(\beta-1)+j \leqslant k_1$. The new function obtained in this way belongs to $\mathcal{L}(\mathcal{E}-P_{k_1})$. The same procedure can now be repeated over and over again until you get a function

$$\psi = [\varphi - \text{some linear combination of } f_k \ (-k_2 \leqslant k \leqslant k_1)] \in \mathcal{L}(\mathcal{D}) \subset L.$$

Since $L = L_1$, ψ is a linear combination of the functions $f_k \in L_1$; but no $k \neq 0$ can occur; indeed considering the form f_k with $|k|$ maximal which occurs and considering the pole at P_k (if $k>0$) or at Q_k (if $k<0$), we find $\psi \notin \mathcal{L}(\mathcal{D})$. Therefore $\psi = \text{constant} = \text{constant} \times f_0$. This shows that $\varphi = \sum a_i f_i$ where the summation ranges from $-k_2$ to k_1. But every f_j $(-k_2 \leqslant j \leqslant k_1)$ is in $\mathcal{L}(\mathcal{E})$ and they are independent. Therefore, this second count yields

$$\dim \mathcal{L}(\mathcal{E}) = k_1 + k_2 + 1. \tag{12}$$

Comparing (11) and (12) leads to the conclusion that

$$\text{order } (\mathcal{D}) = g.$$

LEMMA 4. *$\mathcal{D}$ is a regular divisor.*

Proof. Firstly, one shows that $\mathcal{D}$ is general. Consider an integer k_1 such that $k_1 > g-2$; then

$$\dim \mathcal{L}\left(\mathcal{D} + \sum_{i=1}^{k_1} P_i\right) = k_1 + 1$$

because

$$\text{order}\left(\mathcal{D} + \sum_{i=1}^{k_1} P_i\right) = k_1 + g > 2g-2.$$

Then

$$\dim \mathcal{L}\left(\mathcal{D} + \sum_{i=1}^{j} P_i\right) \leqslant j+1 \quad \text{for} \quad 1 \leqslant j \leqslant k_1$$

because $\mathcal{L}(\mathcal{D} + \sum_{i=1}^{j+1} P_i)$ is strictly larger than $\mathcal{L}(\mathcal{D} + \sum_{i=1}^{j} P_i)$, because f_{j+1} belongs to the first space and not to the second. Therefore letting the index j go down by one lowers the dimension by at least one unit. It follows that

$$1 \leqslant \dim \mathcal{L}(\mathcal{D}) \leqslant 1.$$

This is to say that $\mathcal{D}$ is general.

---

(1) Here this statement must be understood as follows: φ has a pole of order β at P_k if the actual order of pole at P_k is $\beta + \gamma$ where γ is the number of times P_k occurs in $\mathcal{D}$.

It remains to be shown that $\mathcal{D}$ is regular. To do this, it suffices to show that

$$\dim \mathcal{L}\left(\mathcal{D}+\sum_{i=1}^{k} P_i - \sum_{i=0}^{k} Q_i\right)=0 \quad \text{for} \quad k\geqslant 0.$$

For $k=0$, we have that

$$\dim \mathcal{L}(\mathcal{D})=1$$

and

$$\mathcal{L}(\mathcal{D}-Q_1)\subsetneqq \mathcal{L}(\mathcal{D})$$

since the function $f_0=1$ belongs to the second space, but not the first. For the induction step note that

$$\dim \mathcal{L}\left(\mathcal{D}+\sum_{i=1}^{k+1} P_i - \sum_{i=0}^{k} Q_i\right)\leqslant \dim \mathcal{L}\left(\mathcal{D}+\sum_{i=1}^{k} P_i - \sum_{i=0}^{k} Q_i\right)+1=1$$

(because $\dim \mathcal{L}(\mathcal{E})$ increases by at most one when you allow one further pole). Since f_{k+1} belongs to $\mathcal{L}(\mathcal{D}+\sum_{i=1}^{k+1} P_i-\sum_{i=0}^{k} Q_i)$ but not to $\mathcal{L}(\mathcal{D}+\sum_{i=1}^{k+1} P_i-\sum_{i=0}^{k+1} Q_i)$, we also have that

$$\dim \mathcal{L}\left(\mathcal{D}+\sum_{i=1}^{k+1} P_i - \sum_{i=0}^{k+1} Q_i\right)=0.$$

This ends the proof that (a) implies (b). The converse statement (that (b) implies (a)) derives from the following observation.

From the Riemann–Roch theorem, from the fact that allowing one extra pole increases $\dim \mathcal{L}$ by at most one, and from the regularity of $\mathcal{D}$ (in that order) one has

$$1\leqslant \dim \mathcal{L}\left(\mathcal{D}+\sum_{i=1}^{k} P_i - \sum_{i=0}^{k-1} Q_i\right)\leqslant \dim \mathcal{L}\left(\mathcal{D}+\sum_{i=1}^{k-1} P_i - \sum_{i=0}^{k-1} Q_i\right)+1=1.$$

Let f_k be the unique element of $\mathcal{L}(\mathcal{D}+\sum_{i=1}^{k} P_i-\sum_{i=0}^{k-1} Q_i)$ up to scalars. It is clear that h is the unique function in $\mathcal{L}(\sum_{i=1}^{N} P_i-\sum_{i=0}^{N-1} Q_i)$. Normalize f_k such that $hf_k=f_{k+N}$ for every $k\in\mathbf{Z}$.

LEMMA 5. *If*

$$f\in\mathcal{L}\left(\mathcal{D}+\sum_{i=1}^{r} P_i - \sum_{i=0}^{s-1} Q_i\right) \quad \textit{with} \quad r\geqslant s, \quad r,s\in\mathbf{Z}$$

then f is a linear combination of f_k $(s\leqslant k\leqslant r)$.

Proof. If $r=s$, the result is trivial. Suppose $r>s$; then, since

$$(f_r)\geqslant -\mathcal{D}-\sum_{1}^{r} P_i+\sum_{0}^{s-1} Q_i=\mathcal{E}$$

and since no meromorphic function has a divisor $\geqslant \mathcal{E}+P_r$, the function f_r has a pole at P_r of exact order equal to

$$\left(\# \text{ of } P_r \text{ appearing in } \sum_1^r P_i\right) + (\# \text{ of } P_r \text{ appearing in } \mathcal{D}).$$

The function f has at P_r a pole of order, at worst, the integer given above. Therefore for some constant c_r

$$f - c_r f_r \in \mathcal{L}\left(\mathcal{D} + \sum_{i=1}^{r-1} P_i - \sum_{i=0}^{s-1} Q_i\right).$$

The same argument can now be applied over and over again so as to find constants c_k such that

$$f - \sum_{k=s}^{r} c_k f_k \in \mathcal{L}\left(\mathcal{D} + \sum_{i=1}^{r-1} P_i - \sum_{i=0}^{s-1} Q_i\right).$$

This implies that

$$f - \sum_{k=s}^{r} c_k f_k = 0$$

which establishes Lemma 5.

The rest of the proof is now straightforward. Consider any meromorphic function u, holomorphic on $\mathcal{R}_0$. Then, for K and K' large enough

$$u f_k \in \mathcal{L}\left(\mathcal{D} + \sum_{i=1}^{k+K} P_i - \sum_{i=0}^{k-K'} Q_i\right)$$

admits an expansion as above. In particular,

$$(z)_\infty = -\sum_{i=1}^{M} P_i - \sum_{i=0}^{M'-1} Q_i$$

so that

$$z f_k = \sum_{i=-M'}^{M} c_{k,k+i} f_i$$

with $c_{k,k+M}$ and $c_{k,k-M'} \neq 0$. Moreover, the difference operator $C=(c_{ij})$ is regular by the last hypothesis of (b). C is periodic as a result of the normalization $hf_k = f_{k+N}$. The functions f_k $(0 \leqslant k \leqslant n-1)$ are defined up to some nonzero multiplicative constant. Such a change of basis, due to multiplying f_k with some nonzero constant, amounts to conjugating C with a diagonal operator of period N.

As promised at the start of the proof, we want to add a few words on the modifications necessary to deal with the singular case. In general, we can always define $\mathcal{R}_0$ to be the affine scheme given by $F(h, h^{-1}, z) = 0$. Since, near $h = \infty$, F can be re-written:

$$z^{-N} F = \prod \left(\sigma_i \frac{h^{M_1}}{z^{N_1}} - 1 \right) + \text{lower order terms},$$

it follows that even if F is reducible, F can have no multiple factors as long as the σ_i remain distinct. Then as above we get a reduced algebraic curve $\mathcal{R}$ with n smooth points over $h = \infty$, n' smooth points over $h = 0$. Next, this $\mathcal{R}$, although possibly reducible, is at least connected. To see this, note by the constructions already given that any polynomial F subject to the restrictions of Lemma 1 arises from an operator C. Thus for generic choices of C, $\mathcal{R}$ is certainly irreducible. If you approximate an arbitrary C by a sequence C_i whose curves $\mathcal{R}_i$ are irreducible, $\mathcal{R}$ appears as the limit of the $\mathcal{R}_i$'s. Thus $\mathcal{R}$ must be connected. In the singular case, the genus g of $\mathcal{R}$ is to be interpreted as the arithmetic genus, i.e., $g = \dim H^1(O_R) = \dim \Omega$, and $2g - 2$ is the degree of the divisor of any differential (η). Then the calculation of g by Hurwicz's formula can be interpreted as calculating the degree of the divisor of (dz), and as such works in all cases. Next, in the definition of the divisor $\mathcal{D}$, we must be careful what we mean at singular points P: at each such P, the "multiplicity" of $\mathcal{D}$ is given more precisely by a module $M_P(\mathcal{D})$. In our case, we define $M_P(\mathcal{D})$ to be the O_p-module generated by the functions f_k. Thus the space L in the proof of Lemma 3 is by definition

$$L = \bigcup_{P \in R_0} M_P(\mathcal{D}),$$

and the rest of the argument goes through without change. For instance, the Riemann–Roch theorem is valid over any such singular $\mathcal{R}$ so the dimension counts all work as before. Q.E.D.

Note that in the correspondence of the theorem, the spectrum of the operators C_h determines the curve $\mathcal{R}$, together with the points $P_1, \ldots, P_n, Q_1, \ldots, Q_{n'}$, and the functions z and h on it. The divisor $\mathcal{D}$ plays the role of the auxiliary parameters which must be given in addition to the spectrum in order to fully recover the operator C. If $\mathcal{R}$ is non-singular, the set of all regular $\mathcal{D}$'s is given by a *Zariski*-open subset of the Jacobian Jac $(\mathcal{R})$ of $\mathcal{R}$. When $\mathcal{R}$ is singular, however, one must distinguish between those operators C which correspond to divisiors $\mathcal{D}$ which are *principal* (i.e., for all singular points P, the module $M_P(\mathcal{D})$ has one generator) and those C corresponding to non-principal $\mathcal{D}$. The first set is again parametrized by a *Zariski*-open subset of the so-called generalized Jacobian, Jac $(\mathcal{R})$, an algebraic group which is an extension of an abelian variety part, and a part isomorphic to $\mathbf{C}^n \times (\mathbf{C}^*)^m$ (cf. Serre [28]). The second set is harder to parametrize: it corresponds to a *Zariski*-open set in the boundary of the *compactified* Jacobian $\overline{\operatorname{Jac}(\mathcal{R})}$ (cf. DeSouza [7], Altman–Kleiman [4, 5]). It can be shown that the operator-theoretic meaning of this distinction is that $\mathcal{D}$ is principal if and only if for those h for which C_h has a multiple eigenvalue, the minimal polynomial of C_h is still its characteristic polynomial (see Mumford [25]). For use later, we will say that C is *of principal type* if the divisor $\mathcal{D}$ is principal. (Note that this always holds if $\mathcal{R}$ is non-singular.)

Besides the divisor $\mathcal{D}$ and the points P_i, Q_i, the curve $\mathcal{R}$ has various other elements of structure on it which are important for later analysis. One of these is the holomorphic differential form

$$\zeta = \frac{dz}{h\dfrac{\partial F}{\partial h}} = -\frac{dh}{h\dfrac{\partial F}{\partial z}}.$$

Clearly, when $\mathcal{R}$ is non-singular, $\partial F/\partial h$ and $\partial F/\partial z$ have no common zeroes on $\mathcal{R}_0$, so ζ *has neither zeroes nor poles in* $\mathcal{R}_0$. When $\mathcal{R}$ is singular, the same thing holds if we interpret zeroes and poles in the sense described in the introduction. Otherwise put, ζ is a generator on $\mathcal{R}_0$ of the sheaf of 1-forms η satisfying, for all $P \in \mathcal{R}_0$:

$$\sum_{\substack{\text{Branches}\\ \text{of } \mathcal{R} \text{ at } P}} \operatorname{res}_\gamma (f \cdot \eta) = 0, \quad \text{all} \quad f \in O_p.$$

LEMMA 7. *If* $\alpha = NM_1 - M_1 - 1$, $\alpha' = NM_1' - M_1' - 1$, *then*

$$(\zeta) = \alpha \sum_{i=1}^{n} P_i + \alpha' \sum_{i=1}^{n'} Q_i.$$

Proof. To study the order of zero or pole of ζ at P_i, we use the expansion

$$z^{-N} F(h, h^{-1}, z) = \prod_{i=1}^{n} \left(\sigma_i \frac{h^{M_1}}{z^{N_1}} - 1 \right) + \text{lower order terms}$$

described above. Since at P_i, in terms of a local parameter t,

$$z = t^{-M_1} + \ldots, \quad h = C_i t^{-N_1} + \ldots,$$

we calculate from this formula that:

$$\begin{aligned} \frac{\partial F}{\partial h} &= (\text{constant}) \cdot z^{N-N_1} \cdot h^{M_1 - 1} + \text{lower order poles} \\ &= (\text{constant}) \cdot t^{-M_1 N + N_1} + \text{lower order poles}. \end{aligned}$$

Substituting into the formula for ζ, one checks that ζ has an α-fold zero at P_i as required. The proof for the Q_i's is similar.

LEMMA 8. *The following inequality holds:*

$$(\Delta_{11}\zeta)_\infty \geqslant -P_1 - Q_1.$$

Proof. Consider the minor Δ_{11} obtained by removing the first row and the first column from $C_h - zI$. The computation of Δ_{11} follows the same argument as in Lemma 1, § 1; the only difference is that all the terms involving σ_1 in $\det(C_h - zI)$ are absent. Therefore, the leading terms in Δ_{11} will be

$$\sum_{\alpha=0}^{n-1} (-1)^\alpha \bar{\tau}_\alpha z^{\alpha N_1 - 1} h^{(n-1-\alpha)M_1} = z^{-1} \prod_{i=2}^{n} (\sigma_i h^{M_1} - z^{N_1})$$

where $\bar{\tau}_\alpha$ denotes the symmetric polynomial of degree α in $(\sigma_2, ..., \sigma_n)$ (instead of $\sigma_1, ..., \sigma_n$ as in Lemma 1). From the expression for the leading term one reads off that Δ_{11} will have a pole of order $M_1(N-1)$ at worst if $h^{-M_1} z^{N_1} = \sigma_1$ (i.e., at P_1) and $M_1(N-1)-1$ at worst if $h^{-M_1} z^{N_1} = \sigma_i$, $i \neq 1$ (i.e., at P_i with $i \neq 1$). An analogous statement can be made about the points Q_i.

PROPOSITION 1. *Every regular difference operator leads to* $2N$ *regular divisors* $\mathcal{D}_1^{(i)}$ and $\mathcal{D}_2^{(i)}$ $(1 \leqslant i \leqslant N)$ *of degree* g *(where* $\mathcal{D} = \mathcal{D}_1^{(N)}$*) having the property that for* $1 \leqslant i, j \leqslant N$:

$$\mathcal{D}_1^{(i)} + \mathcal{D}_2^{(j)} - \sum_{k=j}^{i} P_k + \sum_{k=j+i}^{i-1} Q_k \quad \text{if} \quad i > j$$

$$\mathcal{D}_1^{(i)} + \mathcal{D}_2^{(j)} + \sum_{k=i+1}^{j-1} P_k - \sum_{k=i}^{j} Q_k \quad \text{if} \quad i < j$$

$$\mathcal{D}_1^{(i)} + \mathcal{D}_2^{(i)} - P_i - Q_i \quad \text{if} \quad i = j$$

is the divisor of some meromorphic differential ω_{ij}. *Then*

$$\omega_{ij} = \Delta_{ji}\zeta.$$

Moreover,

$$(f_k) = \mathcal{D}_1^{(k)} - \mathcal{D} - \sum_{i=1}^{k} P_i + \sum_{i=0}^{k-1} Q_i.$$

Proof. Since $\mathcal{D}_1^{(N)} = \mathcal{D}$ is general

$$0 \leqslant \dim \mathcal{L}(\mathcal{D} - P_N - Q_N) \leqslant \dim \mathcal{L}(\mathcal{D}) - 1 \leqslant 0.$$

Therefore by the Riemann–Roch theorem

$$\dim \Omega(-\mathcal{D}+P_N+Q_N)=1.$$

Let ω_{NN} be the unique differential, up to some multiplicative constant such that

$$(\omega_{NN}) \geqslant \mathcal{D}-P_N-Q_N.$$

This differential has poles at P_N and Q_N, otherwise $(\omega_{NN}) \geqslant \mathcal{D}-P_N$ or $\geqslant \mathcal{D}-Q_N$ which would contradict the fact that $\mathcal{D}$ is general. Define the positive divisor $\mathcal{D}_2^{(N)}$ such that

$$(\omega_{NN})=\mathcal{D}_1^{(N)}-P_N-Q_N+\mathcal{D}_2^{(N)} \tag{13}$$

and the meromorphic differential

$$\omega_{kN}=f_k\omega_{NN}.$$

These differentials enjoy the property that

$$(\omega_{kN}) \geqslant \mathcal{D}_2^{(N)}-\sum_{i=0}^{k} P_i+\sum_{i=1}^{k-1} Q_i \tag{14}$$

and they are the only ones with this property. Define the positive divisor $\mathcal{D}_1^{(k)}$ such that

$$(\omega_{kN})=\mathcal{D}_1^{(k)}-\sum_{i=0}^{k} P_i+\sum_{i=1}^{k-1} Q_i+\mathcal{D}_2^{(N)}.$$

From (13), it follows also $\mathcal{D}_2^{(N)}$ is general and since ω_{kN} satisfying (14) is unique, $\mathcal{D}_1^{(k)}$ is also general. Therefore we may define ω_{kk} to be the only differential such that

$$(\omega_{kk}) \geqslant \mathcal{D}_1^{(k)}-P_k-Q_k$$

and $\mathcal{D}_2^{(k)}$ such that

$$(\omega_{kk})=\mathcal{D}_1^{(k)}-P_k-Q_k+\mathcal{D}_2^{(k)}.$$

So far we have defined the last column and the diagonal of $\Omega=(\omega_{ij})$. The remaining differentials ω_{ij} are defined such that

$$\frac{\omega_{ij}}{\omega_{Nj}}=\frac{\omega_{iN}}{\omega_{NN}}.$$

The next step is to show that $\omega_{ii}=\Delta_{ii}\zeta$ up to some multiplicative constant, to begin with for $i=N$. By the uniqueness it suffices to prove that

$$(\Delta_{NN}\zeta) \geqslant \mathcal{D}_1^{(N)}-P_N-Q_N.$$

Any regular divisor $\mathcal{D}_1^{(N)}$ can be approximated by regular divisors in $\mathcal{R}_0$. The corresponding operators also approach the original one. Therefore *it is legitimate to assume* $\mathcal{D}_1^{(N)}$ *in* $\mathcal{R}_0$. By Lemma 8,

$$(\Delta_{NN}\zeta)_\infty \geqslant -P_N - Q_N. \tag{15}$$

the zeros of $\Delta_{NN}\zeta$ on $\mathcal{R}_0$ come from zeros of Δ_{NN}, since ζ never vanishes on $\mathcal{R}_0$ (by Lemma 7). Since

$$f_i = \frac{\Delta_{Ni}}{\Delta_{NN}} h = \frac{\Delta_{ii}}{\Delta_{iN}} h, \quad 1 \leqslant i \leqslant N,$$

since $\Delta_{Ni} h$ never has any poles on $\mathcal{R}_0$ and since every point of $\mathcal{D}_1^{(N)}$ appears as a pole of some f_i, the minor Δ_{NN} vanishes at each of the points of $\mathcal{D}_1^{(N)}$;

$$(\Delta_{NN})_0 \geqslant \mathcal{D}_1^{(N)},$$

combined with (15), leads to

$$(\Delta_{NN}\zeta) \geqslant \mathcal{D}_1^{(N)} - P_N - Q_N.$$

To show this statement for $1 \leqslant i \leqslant N-1$, shift the matrix $C_h - zI$ up i levels and to the left i steps(1) and call $(f_1^{(i)}, ..., f_{N-1}^{(i)}, h)$ its eigenvector. Then

$$f_k^{(i)} = f_{k+i} f_i^{-1} = \frac{\Delta_{i,k+i}}{\Delta_{i,i}} \quad \text{and} \quad (f_k^{(i)}) = \mathcal{D}_1^{(k+i)} - \mathcal{D}_1^{(i)} - \sum_{j=i+1}^{k+i} P_j + \sum_{j=i}^{k+i-1} Q_j.$$

Clearly the shifted operator is also regular, leads to the same curve and defines the points P_i and Q_i in the shifted order $P_{i+1}, ..., P_n, \mathrm{P}_1, ..., P_i$ and $Q_{i+1}, ..., Q_n, Q_1, ..., Q_i$. The functions $f_k^{(i)}$ define a regular divisor of order g, which from the relation above must be $\mathcal{D}_1^{(i)}$. Therefore assuming again that by some small deformation $\mathcal{D}_1^{(i)}$ is in $\mathcal{R}_0$,

$$(\Delta_{ii})_0 \geqslant \mathcal{D}_1^{(i)}.$$

But, since

$$(\Delta_{ii}\zeta)_\infty \geqslant -P_i - Q_i$$

it follows that

$$(\Delta_{ii}\zeta) \geqslant \mathcal{D}_1^{(i)} - P_i - Q_i.$$

Therefore $\Delta_{ii}\zeta = \omega_{ii}$. This establishes the fact that $\Delta^{\mathrm{T}}\zeta = \Omega$.

To show that every divisor $\mathcal{D}_2^{(N)}$ is regular, consider the transposed difference operator C^{T}. It leads to the same curve and the eigenvector is given by

$$f_k' = \frac{\Delta_{kN}}{\Delta_{NN}} h^{-1} = \frac{\Delta_{kk}}{\Delta_{Nk}} h^{-1} = \frac{\omega_{Nk}}{\omega_{NN}} h^{-1}$$

(1) The diagona l entries of the new matrix are then $(c_{i+1,i+1} - z, ..., c_{NN} - z, c_{11} - z, ..., c_{ii} - z)$.

with divisor

$$(f'_k) = \mathcal{D}_2^{(k)} - \mathcal{D}_2^{(N)} - \sum_1^k Q_j + \sum_0^{k-1} P_j;$$

hence $\mathcal{D}_2^{(N)}$ and therefore every $\mathcal{D}_2^{(i)}$ $(1 \leqslant i \leqslant N)$ is regular.

This last part of the above proof may be rephrased as asserting that if in the correspondence of Theorem 1, the 2 sets of data:

$$\{\mathcal{R}, \{P_i\}, \{Q_i\}, z, h, \mathcal{D}_1^{(N)}\} \quad \text{and} \quad C$$

correspond to each other, then the modified data

$$\{\mathcal{R}, \{Q_i\}, \{P_i\}, z, h^{-1}, \mathcal{D}_2^{(N)}\} \quad \text{and} \quad C^{\mathrm{T}}$$

also correspond to each other.

Also note that the definition of $\mathcal{D}$ as the least positive divisor such that

$$(f_k) + \mathcal{D} \geqslant -\sum_{i=1}^{k} P_i + \sum_{i=0}^{k-1} Q_i$$

shows immediately that the set of divisors $\mathcal{D}_1^{(k)}$ $(k \in \mathbf{Z})$ have no common points; the same holds for the divisors $\mathcal{D}_2^{(k)}$ $(k \in \mathbf{Z})$.

Finally, in case $\mathcal{R}$ is singular, we must make the definition of $\mathcal{D}_2^{(N)}$ more precise: (1)

$$\mathcal{D}_2^{(N)} = \text{set of points where } (\omega_{NN}) \gneqq \mathcal{D}_1^{(N)} - P_N - Q_N,$$

and if P is a singular point in this set, then

$M_P(\mathcal{D}_2^{(N)})$ = module of meromorphic functions f such that for all

$$g \in M_P(\mathcal{D}_1^{(N)}),\ (f \cdot g \cdot \omega_{NN}) \geqslant 0 \text{ at } P.$$

It follows that:

$M_P(\mathcal{D}_1^{(N)})$ = module of g such that for all $f \in M_P(\mathcal{D}_2^{(N)})$, $(f \cdot g \cdot \omega_{NN}) \geqslant 0$ at P.

So the relation between $\mathcal{D}_1^{(N)}$, $\mathcal{D}_2^{(N)}$ is symmetric and we still write this

$$(\omega_{NN}) = \mathcal{D}_1^{(N)} + \mathcal{D}_2^{(N)} - P_N - Q_N.$$

---

(1) In the language of coherent sheaves,

$$\mathcal{O}(\mathcal{D}_2^{(N)}) = \underline{\mathrm{Hom}}\,(\mathcal{O}(\mathcal{D}_1^{(N)}), \Omega(P_N + Q_N)).$$

In checking the above proof for the singular case, one must use Serre duality for $\mathcal{R}$. For torsion free sheaves $\mathcal{J}$, it says

$$\underline{\mathrm{Ext}}^i\,(\mathcal{J}, \Omega) = 0, \quad i > 0$$

$$H^i(\mathcal{J}) \text{ dual to } H^{1-i}\,(\mathrm{Hom}\,(\mathcal{J}, \Omega)).$$

§ 3. Symmetric and self-adjoint difference operators with examples

There are two interesting special cases of difference operators: the symmetric and self-adjoint difference operators. They both lead to curves with involutions and to divisors with special properties. This is the topic of this chapter. At the end of it some examples will be discussed with applications to inverse spectral problems.

THEOREM 2. *There is a one-to-one correspondence between the following sets of data:*

(1) *a regular, symmetric difference operator C of support $[-M, M]$ of period N, modulo conjugation by periodic diagonal operators, with entries ± 1.*

(2) *a curve $\mathcal{R}$ possibly singular of genus*

$$g = M(N-1) - n + 1$$

with ordered smooth points $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$, two meromorphic functions h and z with the properties given in theorem 1 with $N_1 = N_1'$ and $M_1 = M_1'$ and a regular divisor $\mathcal{D}$. Moreover, $\mathcal{R}$ has an involution τ such that $h^\tau = h^{-1}$, $z^\tau = z$, $\tau(P_k) = Q_k$ and $\mathcal{D}$ has the property that

$$\mathcal{D} + \mathcal{D}^\tau - P_n - Q_n$$

is the divisor of some differential ω on $\mathcal{R}$.

Proof. This theorem results from combining Theorem 1 with the last remark of the previous section. We are simply dealing in (1) with a regular difference operator C mod conjugation by periodic diagonal Λ, such that $C^{\mathrm{T}} = \Lambda \cdot C \cdot \Lambda^{-1}$. In fact, any such C is conjugate to a symmetric C' and this C' is unique up to conjugation by a Λ with entries ± 1. In (2), we are dealing with data $\{\mathcal{R}, \{P_i\}, \{Q_i\}, z, h, \mathcal{D}\}$ such that $\mathcal{R}$ has an automorphism τ carrying this data to $\{\mathcal{R}, \{Q_i\}, \{P_i\}, z, h^{-1}, \mathcal{D}'\}$.

It is useful to see more explicitly how τ arises. In fact, since C is symmetric, the algebraic equation

$$F(h, h^{-1}, z) = \det (C_h - zI)$$

is symmetric with regard to h and h^{-1}, i.e., F is now a function of $h + h^{-1}$ and z. Hence, the map

$$(z, h)^\tau \equiv (z, h^{-1})$$

maps $\mathcal{R}$ into $\mathcal{R}$ and is an involution since $\tau^2 = \text{identity}$. The formula for the genus with $M = M'$ and $n = n'$ simplifies to the one above. Note that $\sigma_i = \sigma'$ and that, because $z^{N_1} h^{-M_1}$ has value σ_i at P_i (resp. $z^{N_1} h^{M_1}$ has value σ_1' at Q_i), therefore $\tau(P_i) = Q_i$.

Finally, because C is symmetric it follows that

$$\Delta_{ij}(z, h) = \Delta_{ji}(z, h^{-1})$$

i.e., as a function on $\mathcal{R}$, $\Delta_{ij} \circ \tau = \Delta_{ji}$. Therefore by Proposition 1, § 1, for all i and j, τ acting on the divisor of ω_{ij} is the divisor of ω_{ji}, i.e.

$$(\mathcal{D}_1^{(i)} + \mathcal{D}_2^{(j)})^\tau = \mathcal{D}_2^{(i)} + \mathcal{D}_1^{(j)}.$$

Since the divisors $\mathcal{D}_1^{(j)}$ have no common points, this implies that $(\mathcal{D}_1^{(i)})^\tau = \mathcal{D}_2^{(i)}$ and in particular

$$\mathcal{D} + \mathcal{D}^\tau - P_n - Q_n = (\omega_{NN}).$$

Remark. Let $\mathcal{R}$ have an involution τ and let O be some origin in Jac $(\mathcal{R})$. Then the *Prym variety* Prym $(\mathcal{R})$ of $\mathcal{R}$ over $\mathcal{S}$ (quotient of $\mathcal{R}$ by τ) may be defined as the set of principal $\mathcal{D} \in \mathrm{Jac}\,(\mathcal{R})$, $\mathcal{D}$ a divisor $\sum_{i=1}^{g} \nu_i$ considered modulo linear equivalence, such that

$$\sum_{i=1}^{g} \int_{0_i}^{\nu_i} \omega + \sum_{i=1}^{g} \int_{0_i^\tau}^{\nu_i^\tau} \omega = 0, \quad \text{(mod periods)}.$$

It is a linear subvariety of Jac $(\mathcal{R})$. Moreover every holomorphic differential on $\mathcal{S}$ can be lifted to a holomorphic differential on $\mathcal{R}$ which is invariant under τ. Therefore it is possible to find a basis $\omega_1, \dots, \omega_g$ such that $\omega_i^\tau = \omega_i$ for $1 \leqslant i \leqslant g_0$ and $\omega_i^\tau = -\omega_i$ for $g_0 < i \leqslant g$. Since the relation above is trivially satisfied for ω's such that $\omega^\tau = -\omega$, it, in fact, reduces to g_0 conditions

$$\sum_{i=1}^{g} \int_{0_i}^{\nu_i} \omega_k = 0, \quad \text{(mod periods)}, \quad 1 \leqslant k \leqslant g_0.$$

We show that the regular part of Prym $(\mathcal{R})$ can be parametrized by symmetric regular difference operators of principal type (p. 123).

The symmetric regular difference operators of principal type lead to curves with an involution and to a principal divisor $\mathcal{D} \in \mathrm{Jac}\,(\mathcal{R})$ such that

$$\mathcal{D} + \mathcal{D}^\tau - P_n - Q_n$$

is the divisor of some differential $\omega_0 = \Delta_{NN} \zeta$; Δ_{NN} is a meromorphic function having for divisor

$$(\Delta_{NN}) = (\omega_0) - (\zeta) = \mathcal{D} + \mathcal{D}^\tau - \alpha \sum_1^n P_i - P_n - \alpha \sum_1^n Q_i - Q_n.$$

Choose the origin to be $\alpha\sum_1^n P_i+P_n$; then Abel's theorem tells you that $\mathcal{D}\in \text{Prym}(\mathcal{R})$; also $\mathcal{D}$ is regular. Conversely, consider a regular principal divisor $\mathcal{D}$ in Prym $(\mathcal{R})$, defined with regard to the origin $\alpha\sum_1^n P_i+P_n$. This implies the existence of a meromorphic function Δ having for divisor

$$\mathcal{D}+\mathcal{D}^\tau-\alpha\sum_1^n P_i-P_n-\alpha\sum_1^n Q_i-Q_n.$$

Then, $\omega_0=\Delta\zeta$ satisfies

$$(\omega_0)=\mathcal{D}+\mathcal{D}^\tau-P_n-Q_n$$

and therefore $\mathcal{D}$ leads to a symmetric principal difference operator.

The next theorem deals with self-adjoint difference operators:

THEOREM 3. *There is a one-to-one correspondence between the following set of data*

(a) *a regular self-adjoint difference operator C of support $[-M, M]$ of period N*

(b) *a curve, possible singular, of genus*

$$g=M(N-1)-n+1$$

with ordered smooth points $P_1, ..., P_n$ and $Q_1, ..., Q_{n'}$, two meromorphic functions h and z with the properties given in Theorem 1 with $N_1'=N_1$ and $M_1=M_1'$ and a divisor $\mathcal{D}$ of degree g. Moreover $\mathcal{R}$ is endowed with an antiholomorphic involution $\tilde{}$ for which $\mathcal{R}\backslash\mathcal{R}_{\mathbf{R}}=\mathcal{R}_+\cup\mathcal{R}_-$ (disconnected) (define $\mathcal{R}_{\mathbf{R}}=\{p\in\mathcal{R}\,|\,\tilde{p}=p\}$), such that $\tilde{P}_i=Q_i$ with $P_i\in\mathcal{R}_+$ and $Q_i\in\mathcal{R}_-$, and such that if $\varphi^*(P)=\overline{\varphi(\tilde{P})}$, then $hh^*=1$ and $z=z^*$. The divisor $\mathcal{D}$ has the property that

$$\mathcal{D}+\tilde{\mathcal{D}}-P_n-Q_n$$

is the divisor of some differential on $\mathcal{R}$, which is real positive(1) *on $\mathcal{R}_{\mathbf{R}}$.*

Remark. Note that the *regularity* of $\mathcal{D}$ is not assumed. In this case, we will prove it using the relation

$$\mathcal{D}+\tilde{\mathcal{D}}-P_n-Q_n=(\omega)$$

where $\omega\geqslant 0$ on $\mathcal{R}_{\mathbf{R}}$.

The proof of this statement goes in two steps: first $\mathcal{D}$ is shown to be general. Indeed, since $\omega\geqslant 0$ on $\mathcal{R}_{\mathbf{R}}$,

$$2\pi i\,\text{Res}_{P_n}\,\omega=-2\pi i\,\text{Res}_{Q_n}\,\omega=\int_{\mathcal{R}_{\mathbf{R}}}\omega>0;$$

(1) This makes sense because $\mathcal{R}_{\mathbf{R}}$ inherits a natural orientation as the boundary of the oriented surface $\mathcal{R}_+$.

P_n and Q_n are poles of ω and therefore neither P_n nor Q_n appears in $\mathcal{D}$ or $\tilde{\mathcal{D}}$. To show that $\mathcal{D}$ is general, it suffices to show that $\mathcal{L}(\mathcal{D}-Q_n)=\{0\}$. Suppose $\mathcal{L}(\mathcal{D}-Q_n)\neq\{0\}$ and let φ satisfy $(\varphi)\geqslant -\mathcal{D}+Q_n$; then $(\varphi)=-\mathcal{D}+Q_n+\mathcal{E}$ for some positive divisor $\mathcal{E}$ and

$$\begin{aligned}(\varphi\varphi^*\omega) &= -\mathcal{D}+Q_n+\mathcal{E}-\mathcal{D}+P_n+\tilde{\mathcal{E}}+\mathcal{D}+\mathcal{D}-P_n-Q_n\\ &=\mathcal{E}+\tilde{\mathcal{E}},\end{aligned}$$

contradicting

$$\int_{\mathcal{R}_{\mathbf{R}}}\varphi\varphi^*\omega=\int_{\mathcal{R}_{\mathbf{R}}}|\varphi|^2\omega>0.$$

In the second step we show that

$$\mathcal{L}\left(\mathcal{D}+\sum_1^{k-1}P_i-\sum_0^{k-1}Q_i\right)=\{0\}$$

by induction. According to Step 1, this holds for $k=1$ and N. So, assume that

$$\mathcal{L}\left(\mathcal{D}+\sum_1^{k}P_i-\sum_0^{k}Q_i\right)=\{0\}.$$

This fact and the Riemann–Roch theorem imply that

$$\dim\mathcal{L}\left(\mathcal{D}+\sum_1^{k}P_i-\sum_0^{k-1}Q_i\right)=1.$$

Let f_k be the unique function in this space. Define

$$\mathcal{D}^{(k)}=(f_k)+\mathcal{D}+\sum_1^{k}P_i-\sum_0^{k-1}Q_i.$$

Then in Jac $(\mathcal{R})$,

$$\mathcal{D}^{(k)}-P_k\equiv\mathcal{D}+\sum_1^{k-1}P_i-\sum_0^{k-1}Q_i.$$

This implies that

$$\dim\mathcal{L}(\mathcal{D}^{(k)})=1$$

and, hence

$$\dim\mathcal{L}(\mathcal{D}^{(k)}-P_k)=0,$$

since P_k does not appear in $\mathcal{D}^{(k)}$; indeed $f_kf_k^*\omega\geqslant 0$ on $\mathcal{R}_{\mathbf{R}}$ and

$$(f_kf_k^*\omega)=\tilde{\mathcal{D}}^{(k)}+\mathcal{D}^{(k)}-P_k-Q_k,$$

so that the integral of $f_kf_k^*\omega$ over $\mathcal{R}_{\mathbf{R}}$ is strictly positive and therefore $f_kf_k^*\omega$ must have at least one pole, the only possible ones being P_k and Q_k.

This also implies that neither P_k, nor Q_k ever appear in $\mathcal{D}^{(k)}$ and, in particular, $\mathcal{D}$ does not contain P_n or Q_n.

Proof of Theorem 3. Given a self-adjoint operator C, we construct a curve $\mathcal{R}$ with the given properties. Since C is self-adjoint,

$$\det(C_h - zI) = F(h, h^{-1}, z) = 0$$

has the following form:

$$A_0 h^M + \bar{A}_0 h^{-M} + \sum_{i=1}^{M} (A_i(z) h^{M-i} + (\overline{A_i(\bar{z})})\, h^{-(M-i)}) = 0. \tag{16}$$

Therefore, the map

$$\tilde{}: (z, h) \to (\bar{z}, \bar{h}^{-1})$$

defines an anti-holomorphic involution from $\mathcal{R}$ into $\mathcal{R}$. But if $|h| = 1$ the finite matrix C_h is self-adjoint and therefore has a real spectrum. Therefore the fixed points $\mathcal{R}_{\mathbf{R}}$ for this map are given by

$$\begin{aligned}\mathcal{R}_{\mathbf{R}} &= \{(z, h) \mid h = \bar{h}^{-1}, z = \bar{z}\}\\ &= \{(z, h) \mid |h| = 1\}\end{aligned}$$

and

$$\mathcal{R} \backslash \mathcal{R}_{\mathbf{R}} = \{|h| > 1\} \cup \{|h| < 1\} = \mathcal{R}_+ \cup \mathcal{R}_-$$

with $\mathcal{R}_+ \ni P_i$ and $\mathcal{R}_- \ni Q_i$; this defines two distinct regions $\mathcal{R}_+$ and $\mathcal{R}_-$, whose boundary is given by $\mathcal{R}_{\mathbf{R}}$ and therefore $\mathcal{R}_{\mathbf{R}}$ is homologous to zero. $\mathcal{R}_{\mathbf{R}}$ will consist of possibly several circles σ_i with a definite orientation as noted in the footnote above. Since $\sigma_i = \bar{\sigma}_i$, we have $\tilde{P}_i = Q_i$; moreover this involution extends to the field of meromorphic functions as follows

$$\varphi^*(p) = (\overline{\varphi(\tilde{p})})$$

and to meromorphic differentials as follows:

$$(\varphi d\psi)^* = \varphi^* d\psi^*.$$

With this definition

$$h^* = h^{-1} \quad \text{and} \quad z^* = z.$$

By Theorem 1, the difference operator C maps into a regular divisor $\mathcal{D}$ such that

$$(\omega) = (\omega_{NN}) = (\zeta \Delta_{NN}) = \mathcal{D} + \mathcal{D}' - P_n - Q_n. \tag{17}$$

Next, we show that $\mathcal{D}' = \tilde{\mathcal{D}}$. But, because C is self-adjoint, it follows that

$$\Delta_{ij}^* = \Delta_{ji}.$$

Therefore, by Proposition 1, for all i and j, $\tilde{}$ acting on the divisor of ω_{ij} is the divisor of ω_{ji}, i.e.,

$$\tilde{\mathcal{D}}_1^{(i)}+\tilde{\mathcal{D}}_2^{(j)}=\tilde{\mathcal{D}}_2^{(i)}+\tilde{\mathcal{D}}_1^{(j)}.$$

Since the divisors $\mathcal{D}_1^{(j)}$ have no common points this implies that $\mathcal{D}_1^{(i)}=\mathcal{D}_2^{(i)}$ and in particular $\mathcal{D}=\tilde{\mathcal{D}}$.

Finally, we show that $\omega\geqslant 0$ on $\mathcal{R}_{\mathbf{R}}$ for some appropriate normalization of ζ. To do this take the differential of F: since z only appears on the diagonal C_h-zI,

$$-\sum_{i=1}^{N}\Delta_{ii}\,dz+h\frac{\partial F}{\partial h}\frac{dh}{h}=0.$$

Using this relation, and choosing $\zeta=-i\,dz(h\,\partial F/\partial h)^{-1}$ and using Proposition 1,

$$\begin{aligned}\omega=\zeta\Delta_{NN}&=-i\Delta_{NN}\frac{dz}{h\dfrac{\partial F}{\partial h}}=\frac{-i\,dh/h}{\sum_{i=1}^{N}\dfrac{\Delta_{ii}}{\Delta_{NN}}}=\frac{-i\,dh/h}{\sum_{i=1}^{N}\dfrac{\Delta_{ii}}{\Delta_{iN}}\,\dfrac{\Delta_{iN}}{\Delta_{NN}}}=\frac{-i\,dh/h}{\sum_{i=1}^{N}\dfrac{\Delta_{Ni}}{\Delta_{NN}}\cdot\dfrac{\Delta_{iN}}{\Delta_{NN}}}\\&=\frac{-i\,dh/h}{\sum_{i=1}^{N}\dfrac{\Delta_{Ni}}{\Delta_{NN}}\cdot\left(\dfrac{\Delta_{Ni}}{\Delta_{NN}}\right)^*}\qquad(\text{since } \Delta_{Ni}^*=\Delta_{iN},\quad 1\leqslant i\leqslant N)\\&=\frac{-i\,dh/h}{\sum_{i=1}^{N}f_if_i^*}\end{aligned}$$

Note that this formula shows that $\omega^*=\omega$. We now show that $\omega\geqslant 0$ on $\mathcal{R}_{\mathbf{R}}$. Indeed on $\mathcal{R}_{\mathbf{R}}$

$$\sum_{i=1}^{N}f_if_i^*=\sum|f_i|^2\geqslant 0.$$

To show that $-i\,dh/h\geqslant 0$ on $\mathcal{R}_{\mathbf{R}}$, let $h=\varrho e^{i\theta}$; at all but a finite number of points, h is a local parameter on $\mathcal{R}$, and θ is a local coordinate on $\mathcal{R}_{\mathbf{R}}$. Since $-i\,dh/h=d\theta$, $\omega\geqslant 0$ at these points, hence by continuity, at all points.

Consider now the converse. The curve $\mathcal{R}$ has the properties listed in (b), in particular it has an antiholomorphic involution $p\to\tilde{p}$ such that $\tilde{P}_i=Q_i$. The curve of fixed points $\mathcal{R}_{\mathbf{R}}=\{p\,|\,p=\tilde{p}\}$ divides $\mathcal{R}$ into two distinct regions $\mathcal{R}_+$ and $\mathcal{R}_-$, the first containing the points P_i and the second the points Q_i. The curve $\mathcal{R}_{\mathbf{R}}$ can thus be regarded as the boundary of $\mathcal{R}_+$ or $\mathcal{R}_-$, (thus $\mathcal{R}_{\mathbf{R}}$ is homologous to zero).

Choose any regular $\mathcal{D}$ such that

$$\mathcal{D}+\tilde{\mathcal{D}}-P_n-Q_n=(\omega)$$

where $\omega = \omega^*$ and it is real positive on $\mathcal{R}_{\mathbf{R}}$. This means that if $z_0 \in \mathcal{R}_{\mathbf{R}}$ and t is a local parameter at z_0 where t is real on $\mathcal{R}_{\mathbf{R}}$, $\operatorname{Im} t > 0$ on $\mathcal{R}_+$, $\operatorname{Im} t < 0$ on $\mathcal{R}_-$, then

$$\omega = a(t)\,dt, \quad a(\bar{t}) = \overline{a(t)}, \quad a(t) \geqslant 0 \quad \text{if } t \in \mathbf{R}.$$

Let f_k be the usual meromorphic functions associated to $\mathcal{D}$, i.e., such that

$$(f_k) \geqslant -\sum_1^k P_i + \sum_0^{k-1} Q_i - \mathcal{D}.$$

Let

$$\mathcal{D}_k \equiv (f_k) + \mathcal{D} + \sum_1^k P_i - \sum_0^{k-1} Q_{k-1}.$$

Normalize f_k as before such that $f_{k+N} = hf_k$. We now define a scalar product between the f_k's, i.e.,

$$(f_k, f_l) = \int_{\mathcal{R}_{\mathbf{R}}} f_k f_l^* \,\omega = \int_{\mathcal{R}_{\mathbf{R}}} f_k \bar{f}_l \,\omega.$$

When $k \neq l$, $(f_k, f_l) = 0$; indeed, for $k > l$

$$\begin{aligned}(f_k f_l^* \omega) &= \mathcal{D}_k - \mathcal{D} - \sum_1^k P_i + \sum_0^{k-1} Q_i + \tilde{\mathcal{D}}_l - \tilde{\mathcal{D}} - \sum_1^l \tilde{P}_i + \sum_0^{l-1} \tilde{Q}_i + \mathcal{D} + \tilde{\mathcal{D}} - P_n - Q_n \\ &= \mathcal{D}_k + \tilde{\mathcal{D}}_l - \sum_l^k P_i + \sum_{l+1}^{k-1} Q_i;\end{aligned}$$

it tells you that $f_k f_l^* \omega$ has no other poles but at some of the points P_i, i.e., in the region $\mathcal{R}_+$ only. Since $\mathcal{R}_{\mathbf{R}}$ is the boundary of that region and homologous to zero and since

$$\sum_{P_i} \operatorname{Res} f_k f_l^* \,\omega = 0,$$

the conclusion above follows. A similar argument proves the assertion when $k < l$. For $k = l$,

$$(f_k, f_k) = \int_{\mathcal{R}_{\mathbf{R}}} f_k f_k^* \,\omega = \int_{\mathcal{R}_{\mathbf{R}}} |f_k|^2 \,\omega > 0$$

again because on $\mathcal{R}_{\mathbf{R}}$, ω is non-negative.

Since

$$2\pi i \operatorname{Res}_{P_k} (f_k f_k^* \,\omega) = \int_{\mathcal{R}_{\mathbf{R}}} |f_k|^2 \,\omega > 0$$

the f_k's can be multiplied with positive real constants such that

$$\operatorname{Res}_{P_k} (f_k f_k^* \,\omega) = -i. \tag{20}$$

This is compatible with the normalization $hh^*=1$; it suffices to multiply ω with a real multiplicative constant (which can still be done) such that

$$\mathrm{Res}_{P_n}(\omega) = -i.$$

Consider now φ such that $\varphi^*=\varphi$. The matrix of the operator $C[\varphi]$ associated to φ is defined by:

$$\varphi f_k = \sum_l a_{kl} f_l. \tag{21}$$

Now

$$\sum_{i=1}^{n} \mathrm{Res}_{P_i}(\varphi f_k f_j^* \omega) = \sum_l a_{kl} \sum_{i=1}^{n} \mathrm{Res}_{P_i}(f_l f_j^* \omega) = -i a_{kj}. \tag{22}$$

Also

$$\begin{aligned} \overline{\sum_{i=1}^{n} \mathrm{Res}_{P_i}(\varphi f_k f_j^* \omega)} &= \sum_{i=1}^{n} \mathrm{Res}_{Q_i}(\varphi f_k f_j^* \omega)^* \\ &= \sum_{i=1}^{n} \mathrm{Res}_{Q_i}(\varphi f_k^* f_j \omega) \\ &= -\sum \mathrm{Res}_{P_i}(\varphi f_j f_k^* \omega) \\ &= i a_{jk}. \end{aligned}$$

Therefore $\bar{a}_{kj}=a_{jk}$, i.e., the operator $C[\varphi]$ associated to φ is self-adjoint. In particular, $z^*=z$; therefore the operator C is self-adjoint. The rest of this chapter will be devoted to the application of these theorems to a few examples.

1. Consider a second order symmetric difference operator[1] of period N, i.e.,

$$(Cf)_k = b_{k-1} f_{k-1} + a_k f_k + b_k f_{k+1} \quad \text{with} \quad a_{k+N}=a_k,\ b_{k+N}=b_k \in \mathbf{C}.$$

Here $M=M'=1$ and $n=1$. It is regular as soon as $b_k \neq 0$ $(1 \leqslant k \leqslant N)$. Then

$$C_h - zI = \begin{bmatrix} a_1 - z & b_1 & 0 & & \cdots & & b_N h^{-1} \\ b_1 & a_2 - z & b_2 & & & & \vdots \\ 0 & b_2 & a_3 - z & & & & \\ \vdots & & & \ddots & & & \\ & & & & a_{N-2} - z & b_{N-2} & 0 \\ & & & & b_{N-2} & a_{N-1} - z & b_{N-1} \\ b_N h & \cdots & & & 0 & b_{N-1} & a_N - z \end{bmatrix}$$

(1) Observe that any second order difference operator can be symmetrized by conjugation with a diagonal matrix.

is a tridiagonal period matrix, with determinant

$$F(h, h^{-1}, z) = (-1)^{N+1}\left(\prod_1^N b_i(h+h^{-1}) - P(z)\right) = 0,$$

where $P(z)$ is a polynomial of degree N with leading coefficient $=1$. Setting $\prod_1^N b_i = A \neq 0$

$$h(z) = \frac{1}{2A}\left(P(z) \pm \sqrt{P(z)^2 - 4A^2}\right) = \frac{1}{\frac{1}{2A}\left(P(z) \mp \sqrt{P(z)^2 - 4A^2}\right)}$$

Therefore the curve is hyperelliptic of genus $g = N-1$ with two points P and Q at infinity; besides $(h) = -NP + NQ$. Moreover, switching the sign of the radical in the formula above amounts to changing h into h^{-1}; therefore the involution τ coincides with the hyperelliptic involution. The fixed points for this involution are given by the 2N points where $h = \pm 1$, i.e., the branch points. Let α_i and β_i be homology cycles $(1 \leq i \leq g)$; then the fact that $(h) = -NP + NQ$ implies the existence of a closed loop $\sum n_i\alpha_i + m_i\beta_i$ such that

$$\sum_1^g n_i \int_{\alpha_i} \omega + \sum_1^g m_i \int_{\beta_i} \omega = N \int_P^Q \omega$$

for every holomorphic differential. This amounts to g relations between the branch points. So, any hyperelliptic curve coming from such a tridiagonal matrix satisfies these relations and vice-versa.

The inverse problem, as discussed in [12, 22], is an immediate consequence of Theorem 2. Let the spectrum of C_1 (i.e., C_h for $h=1$) and $\prod_1^N b_i$ be known. Let also the spectrum of the matrix C^0 be known; C^0 is formed from C_1 after removal of the last row and column. Then, the matrix C is completely known up to at most 2^{N-1} ambiguities.

Indeed, the knowledge of $A = \prod_1^N b_i$ and the spectrum of C_1 determines the equation

$$A(h+h^{-1}) - P(z) = 0$$

and therefore the hyperelliptic curve $\mathcal{R}$. The matrix C^0 has $N-1=g$ spectral points in $\mathbf{C}$. They can be lifted up to $\mathcal{R}$ in 2^g different ways, if no one of them coincides with the branch points and if no two of them coincide. Each of these ways leads to a regular divisor $\mathcal{D}$ of order g such that

$$\mathcal{D} + \mathcal{D}^\tau - P - Q = \left(\frac{\Delta_{NN}(z)}{\sqrt{P(z)^2 - 4A^2}}\, dz\right).$$

Each one of the regular divisions $\mathcal{D}$ determines in a unique way a periodic tridiagonal matrix C modulo conjugation by diagonal matrices with entries ± 1, such that the spectrum of C^0 is the one given above. Observe also that Prym $(\mathcal{R}) = \text{Jac}(\mathcal{R})$.

2. Let C be a self-adjoint periodic difference operator(1)

$$(Cf)_h = \bar{b}_{k-1} f_{k-1} + a_k f_k + b_k f_{k+1} \quad \text{with} \quad a_{k+N} = a_k \in \mathbf{R},\ b_{k+N} = b_k \neq 0.$$

Then

$$F(h, h^{-1}, z) = (-1)^{N+1} (Ah + \bar{A} h^{-1} - P(z)) = 0$$

where $P(z) = z^N + \ldots$ is a real polynomial of degree N. This defines a hyperelliptic curve whose branch points are located at the values of z where $h(z) = \pm |A|/A$; for each of these values of h, the matrix $C_h - zI$ is self-adjoint. Therefore the branch points are real. The involution $\tilde{}$ transforms a point of $\mathcal{R}$ as follows: take the complex conjugate in $\mathbf{C}$ and flip sheets.

3. Next consider a symmetric fourth-order difference operator

$$(Cf)_k = c_{k-2} f_{k-2} + b_{k-1} f_{k-1} + a_k f_k + b_k f_{k+1} + c_k f_{k+2}$$

with $a_{k+N} = a_k$, $b_{k+N} = b_k$ and $c_{k+N} = c_k$. Here $M = M' = 2$, so that $(N, M) = n = 1$ or 2; so, a distinction must be made between N odd and N even. When N is odd, the regularity reduces to the condition that $c_k \neq 0$ $(1 \leqslant k \leqslant N)$ and when N is even, it reduces to $c_k \neq 0$ $(1 \leqslant k \leqslant N)$ and

$$c_1 c_3 c_5 \ldots c_{N-1} \neq c_2 c_4 c_6 \ldots c_N.$$

In either case

$$F(h, h^{-1}, z) = \det (C_h - zI) = A(h + h^{-1})^2 + P_1(z)(h + h^{-1}) + P_2(z)$$

with $P_2(z) = (-z)^N + \ldots$ and $\deg P_1(z) \leqslant [N/2]$ with equality if N is even. This implies that $\mathcal{R}$ is a double covering of the hyperelliptic curve $\mathcal{S}$, defined by

$$Ag^2 + P_1(z) g + P_2(z) = 0$$

ramified at the $2N$ points on $\mathcal{S}$ where $g = h + h^{-1} = \pm 2$, i.e., where $h = \pm 1$. When N is odd, $\mathcal{S}$ is ramified at infinity, so that $\mathcal{R}$ has two points P and Q covering $z = \infty$. If N is even, $\mathcal{S}$ is not ramified at infinity, so that $\mathcal{R}$ has 4 points P_1, P_2, Q_1 and Q_2 covering infinity. Then

$$\begin{aligned}(h) &= -NP + NQ && N \text{ odd}\\ &= -\frac{N}{2}(P_1 + P_2) + \frac{N}{2}(Q_1 + Q_2) && N \text{ even.}\end{aligned}$$

$\mathcal{R}$ has genus $g = 2N - 2$ or $2N - 3$ according to whether N is odd or even.

(1) Unlike in the symmetric case, a second order difference operator cannot necessarily be made self-adjoint by conjugation with a diagonal matrix.

The inverse problem can now be formulated as follows. From the knowledge of the spectrum, the antiperiodic spectrum of C and $A=\prod_1^N c_i$, you can reconstruct the curve $\mathcal{R}$. A generic set of g points in $\mathbf{C}$ leads to 4^g regular fourth order difference operators.

Indeed from A, from the periodic and antiperiodic spectrum of C you know the polynomials

$$4A+2P_1(z)+P_2(z)=(-z)^N+\dots$$

and

$$4A-2P_1(z)+P_2(z)=(-z)^N+\dots$$

Therefore P_1 and P_2 are known; this defines $\mathcal{R}$ completely. The generic set of g points in $\mathbf{C}$ can be lifted up to $\mathcal{R}$ in 4^g ways, defining 4^g different divisors $\mathcal{D}$. Each one of those leads to a fourth order periodic difference operator which is not necessarily symmetric. Only, when $\mathcal{D}$ is in Prym ($\mathcal{R}$), C can be made symmetric by conjugation with a diagonal operator. This imposes $g_0=[(N-1)/2]$ conditions on the choice of the g points in $\mathbf{C}$, expressing the fact that they must be roots of the function $\Delta_{NN}(z)$; the latter is symmetric in h and h^{-1} if it is to come from a symmetric operator.

§ 4. Flows on the Jacobi variety and symplectic structures

As a result of § 2, the Jacobi variety (except for a lower dimensional manifold) can be parametrized by difference operators of a given order with the same h-spectrum. Therefore the linear flows on Jac ($\mathcal{R}$) (with regard to the group structure) can be regarded as isospectral flows on the space of difference operators. This section shows that these flows can be expressed in terms of Lax-type commutation relations. Let $\mathcal{A}$ denote the ring of meromorphic functions on $\mathcal{R}$, holomorphic on $\mathcal{R}_0$. Since z, h, h^{-1} are affine coordinates on $\mathcal{R}_0$, $\mathcal{A}$ is the ring of polynomials in z, h, h^{-1}. Let $\{\omega_k\}$ be a basis for the space of holomorphic differentials.

THEOREM 4. *Every linear flow on* Jac ($\mathcal{R}$)

$$\sum_{i=1}^{g}\int_{\nu_i(0)}^{\nu_i(t)}\omega_k=a_k t,\quad 1\leqslant k\leqslant g,$$

is associated with a function $u=P(z,h,h^{-1})$ *in* $\mathcal{A}$ *such that*

$$a_k=\sum_{i=1}^{n}\mathrm{Res}_{P_i}(\omega_k u).$$

This flow is equivalent to the system of differential equations, given by (1)

$$\dot{C} = [C[u]^+, C] \quad \text{or} \quad \dot{C} = [C[u]^{[+]}, C]$$

where $C[u] = P(C, S, S^{-1})$.

These two equations give flows in the space of periodic difference operators that differ by conjugation by a periodic diagonal $\Lambda(t)$.

This theorem is equally valid whether $\mathcal{R}$ is singular or not. It is important to realize that by considering integrals of the differentials $\eta \in \Omega$, we get for singular $\mathcal{R}$ also an Abel mapping from principal divisors mod linear equivalence to points of the generalized Jacobian variety. A good reference for this is Serre [28] or Rosenlicht [27].

In the proof of the theorem, to avoid questions of convergence, it is best to approximate the given $\mathcal{D}$ by divisors $\mathcal{D}(s)$ made up of smooth points of $\mathcal{R}$, and correspondingly approximate C by $C(s)$. If the flow through $C(s)$ is given by our formula, then by passing to the limit, so is the flow through C. We first prove 2 simple lemmas:

LEMMA 1. *Let* $\mathcal{D} = \sum \nu_i$ *be a regular point of* Jac $(\mathcal{R})$. *Let* $\mathcal{D}(t) = \sum \nu_i(t)$ *and* $\mathcal{D}'(t) = \sum \nu_i'(t)$ *be in a small enough neighborhood of* $\mathcal{D}$, *such that*

$$\sum_{i=1}^{g} \int_{\nu_i}^{\nu_i(t)} \omega_j = a_j t + O(t^2), \quad 1 \leq j \leq g$$

and

$$\sum_{i=1}^{g} \int_{\nu_i(t)}^{\nu_i'(t)} \omega_j = O(t^2), \quad 1 \leq j \leq g$$

then

$$f_k(t) - f_k = O(t)$$

and

$$f_k(t) - f_k'(t) = O(t^2)$$

uniformly over any open V *such that* $\bar{V} \subset \mathcal{R}_0 \setminus \mathcal{D}$, *where* f_k, $f_k(t)$ *and* $f_k'(t)$ *correspond to* $\mathcal{D}$, $\mathcal{D}(t)$ *and* $\mathcal{D}'(t)$ *respectively.*

The proof follows at once from the fact that the functions f_k depend analytically on their poles. This dependence will be made explicit in § 4, where f_k will be expressed as quotients of theta functions.

---

(1) For any difference operator C, define

$$C_{ij}^+ = \begin{cases} C_{ij} & \text{if } i < j \\ 0 & \text{if } i \geq j \end{cases} \quad \text{and} \quad (C^{[+]})_{ij} = \begin{cases} C_{ij} & \text{if } i < j \\ \tfrac{1}{2} C_{ii} & \text{if } i = j \\ 0 & \text{if } i > j. \end{cases}$$

Define C^- in the same way and $C^{[-]} = C - C^{[+]}$

LEMMA 2. *Consider a point* $P \in \mathbf{C}$, *a holomorphic differential* $\omega = \varphi\, dz$ *in the neighborhood* V *of* P *and an analytic function* u *in* V *with a pole of order* n *at* P. *Consider* $t \in \mathbf{C}$ *small enough, so that the* n *points* $P_i(t)$, *where* $u(P_i(t)) + t^{-1} = 0$, *belong to* V. *Then*

$$\lim_{t\to 0} \frac{d}{dt} \sum_{i=1}^{n} \int_P^{P_i(t)} \omega = -\operatorname{Res}_P \omega u.$$

Proof. Let $\omega = d\psi$ with $\psi(P) = 0$. Then for any path π enclosing the zeros $P_i(t)$ of $u + t^{-1}$,

$$\begin{aligned}
\frac{1}{t}\sum_{i=1}^{n}\int_P^{P_i(t)} \omega &= \frac{1}{t}\sum \psi(P_i(t)) \\
&= \frac{1}{t}\sum \operatorname{Res}_{P_i(t)} \frac{u'}{u+\dfrac{1}{t}}\psi \\
&= \sum \operatorname{Res}_{P_i(t)} \frac{u'}{1+tu}\psi \\
&= \frac{1}{2\pi i}\int_\pi \frac{u'}{1+tu}\psi\, dz.
\end{aligned}$$

When t tends to zero, the right hand side tends to

$$\frac{1}{2\pi i}\int_\pi u'\psi\, dz = -\frac{1}{2\pi i}\int_\pi u\omega = -\operatorname{Res}_P (u\omega)$$

Proof of Theorem 4. Consider $u \in \mathcal{A}$. Assume $\mathcal{D}$ in $\mathcal{R}_0$; the proof extends easily to the case where $\mathcal{D}$ is not in $\mathcal{R}_0$. Then u splits into two functions g_+ and g_- according to

$$u = uf_0 = \sum_{i=-K'}^{K} c_{N,i} f_i = \sum_{i=-K'}^{0} c_{N,i} f_i + \sum_{i=1}^{K} c_{N,i} f_i = g_- + g_+.$$

Then, since

$$(g_+) \geqslant -\sum_1^K P_i + Q_1 - \mathcal{D} \quad \text{and} \quad (g_-) \geqslant -\sum_{-1}^{-K'} Q_i - \mathcal{D} \tag{25}$$

we have that

$$\left(g_+ + \frac{1}{t}\right) = -\sum_1^K P_i - \mathcal{D} + \sum_1^K P_i(t) + \mathcal{D}(t) \tag{26}$$

and

$$\left(g_- - \frac{1}{t}\right) = -\sum_{-1}^{-K'} Q_i - \mathcal{D} + \sum_{-1}^{-K'} Q_i(t) + \mathcal{D}'(t)$$

where $\mathcal{D}(t)$ (resp. $\mathcal{D}'(t)$) is a divisor of order g, near $\mathcal{D}$ and each $P_i(t)$ (resp. $Q_i(t)$) is near P_i (resp. Q_i), as near as you wish by choosing t small enough. Let $\mathcal{D} = \sum_1^g \nu_i(0) = \sum_1^g \nu_i$,

$\mathcal{D}(t)=\sum_1^g \nu_i(t)$ and $\mathcal{D}'(t)=\sum_1^g \nu_i'(t)$. Then $\nu_i(t)$ as a function of t, is holomorphic in t, because near ν_i, the function g_+ behaves as

$$g_+=\frac{b_{-1}}{s}+b_0+b_1 s+\dots$$

in the local parameter s. Putting $g_+ + t^{-1}=0$ leads to an expansion of s as a power series in t. Using Abel's theorem

$$\sum \int_{\nu_i}^{\nu_i(t)} \omega = -\sum_{i=1}^{K} \int_{P}^{P_i(t)} \omega \quad \text{and} \quad \sum \int_{\nu_i}^{\nu_i'(t)} \omega = -\sum_{i=1}^{K'} \int_{Q_i}^{Q_i(t)} \omega \tag{27}$$

for every holomorphic differential ω. From Lemma 4 it follows that

$$\lim_{t\downarrow 0} \frac{1}{t} \sum_{i=1}^{K} \int_{P_i}^{P_i(t)} \omega = -\sum_{i=1}^{n} \operatorname{Res}_{P_i}(\omega g^+). \tag{28}$$

The same argument applies to the function $g_- - t^{-1}$, yielding

$$\lim_{t\to 0} \frac{1}{t} \sum_{i=1}^{K'} \int_{Q_i}^{Q_i(t)} \omega = \sum_{i=1}^{n} \operatorname{Res}_{Q_i}(\omega g^-).$$

Observe the change in sign with regard to (28), as a result of considering the function $g_- - t^{-1}$. In view of (27), (28) and the fact that $\mathcal{D}$ belongs to $\mathcal{R}_0$, we conclude that

$$\sum_{i=1}^{g} \int_{\nu_i}^{\nu_i(t)} \omega = -t \sum_{i=1}^{n} \operatorname{Res}_{P_i}(\omega u) + O(t^2) \tag{29}$$

and

$$\sum_{i=1}^{g} \int_{\nu_i}^{\nu_i'(t)} \omega = t \sum_{i=1}^{n} \operatorname{Res}_{Q_i}(\omega u) + O(t^2).$$

Therefore also

$$\sum \int_{\nu_i(t)}^{\nu_i'(t)} \omega = O(t^2).$$

Since the regular divisors form an open subset in Jac $(\mathcal{R})$, take t small enough, such that $\mathcal{D}(t)$ and $\mathcal{D}'(t)$ are still regular. Denote by $f_k(t)$ (resp. $f_k'(t)$) the unique meromorphic function corresponding to $\mathcal{D}(t)$ (resp. $\mathcal{D}'(t)$). Then

$$((1+tg_+)f_k(t)) \geqslant -\sum_1^k P_i - \sum_1^K P_i - \mathcal{D} + \sum_0^{k-1} Q_i + \sum_1^K P_i(t)$$

and

$$((1-tg_-)f_k'(t)) \geqslant \sum_0^{k-1} Q_i - \sum_{-1}^{-K'} Q_i - \mathcal{D} - \sum_1^k P_i + \sum_{-1}^{-K'} Q_i(t).$$

From Lemma 5 (§ 1), it follows that $(1+tg_+)f_k(t)$ and $(1-tg_-)f_k(t)$ have an expansion in terms of the $f_i(0)$'s

$$(1+tg_+)f_k(t) = \sum_{i \geqslant k} a_{ki}^+(t) f_i(0)$$

and

$$(1-tg_-)f_k'(t) = \sum_{i \leqslant k} a_{ki}^-(t) f_i(0).$$

The difference of these two equations reads

$$\begin{aligned}\sum (a_{ki}^+(t) - a_{ki}^-(t)) f_i(0) &= f_k(t) - f_k'(t) + t(g_+ f_k(t) + g_- f_k'(t)) \\ &= tuf_k(0) + tu(f_k(t) - f_k(0)) + (f_k(t) - f_k'(t))(1 - tg_-) \\ &= t \sum C[u]_{ki} f_i(0) + O(t^2)\end{aligned}$$

uniformly over any open set such that $\bar{V} \subset \mathcal{R}_0 \setminus \mathcal{D}$. Since the functions $f_i(0)$ are independent, we conclude that

$$a_{ki}^+(t) - a_{ki}^-(t) = tC[u]_{ki} + O(t^2).$$

Since $f_k(t)$ is defined up to multiplication with some function of t, it can be determined such that $a_{kk}^+(t) = 1 + O(t^2)$ or $a_{kk}^+(t) = 1 + \frac{1}{2}tC[u]_{kk} + O(t^2)$. Therefore $a_{ki}^+(t) = \delta_{ki} + tC[u]_{ki} + O(t^2)$ or $a_{ki}^+(t) = \delta_{ki} + tC[u]_{ki}^+ + O(t^2)$.

Finally write the column vector $(1+tg_+)zf(t)$ in two different ways, using the results above. On the one hand

$$\begin{aligned}(1+tg_+)zf(t) = (1+tg_+)C(t)f(t) &= C(t)(1+tg_+)f(t) \\ &= C(t)(I + tC[u]^+ + O(t^2))f(0)\end{aligned}$$

and on the other hand

$$\begin{aligned}z(1+tg_+)f(t) &= z(I + tC[u]^+ + O(t^2))f(0) \\ &= (I + tC[u]^+ + O(t^2))C(0)f(0).\end{aligned}$$

Both relations are valid for all $(z, h) \in V$. In each of the cases $C[u]^+$ may be replaced by $C[u]^{[+]}$. Then also

$$(I + tC[u]^+ + O(t^2))^{-1} C(t)(I + tC[u]^+ + O(t^2))f(0) = C(0)f(0). \tag{30}$$

So, the dependence of $\mathcal{D}(t)$ on t given by (26) (at least for small enough t) can be expressed equivalently as (29) or (30). Differentiating both (25) and (30)with regard to t and letting $t\downarrow 0$, we conclude that the flow

$$\dot{C} = [C[u]^+, C] \quad \text{or} \quad \dot{C} = [C[u]^{[+]}, C]$$

is equivalent to the motion

$$\sum_i \dot{\nu}_i(0)\,\omega(\nu_i(0)) = -\sum_j \operatorname{Res}_{P_j}(\omega u) \quad \text{for every holomorphic } \omega.$$

The proof of Theorem 4 is finished, if every flow is shown to occur in this fashion. It suffices to show that the mapping

$$u \to \sum_{i=1}^{n} \operatorname{Res}_{P_i}(\omega_k u)$$

maps $\mathcal{A}$ *onto* $\mathbf{C}^g$. Observe that it is possible to find a function in $\mathcal{A}$ with arbitrarily prescribed polar parts at all of the points P_i, provided arbitrarily large poles are allowed at the points Q_i. But the power series expansions of $\omega_1, \ldots, \omega_g$ at P_1, say, are linearly independent, so their Nth order truncations are independent for $N \gg 0$. Thus a suitable u with Nth order poles at P_1 and regular at the other P_i will give any sequence of g constants

$$\sum_{i=1}^{n} \operatorname{Res}_{P_i}(\omega_k u).$$

It is useful to have an explicit basis for the space of holomorphic differential forms on $\mathcal{R}$. The result is this: the forms

$$\begin{gathered} h^k z^i \eta, \quad i \geqslant 0 \\ kN_1 + iM_1 \leqslant \alpha \equiv NM_1 - M_1 - 1 \\ -kN_1' + iM_1' \leqslant \alpha' \equiv NM_1' - M_1' - 1 \end{gathered}$$

are such a basis. One way to check this is to note that these forms are linearly independent and to prove there are g of them, using a counting argument. Another method comes from the toroidal embedding. We explain both methods. The first one is based on a combinatorial lemma:

LEMMA 3.

$$\#\{i, k \geqslant 1 \quad \text{such that} \quad N_1 k + M_1 i \leqslant N_1 M_1 n - 1\} = \frac{n}{2}(nN_1M_1 - N_1 - M_1 - 1) + 1$$

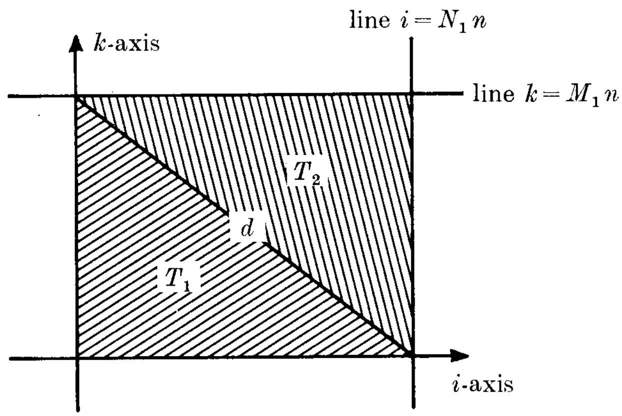

Figure 2

Proof. In Figure 2, we observe that

lattice points on $T_1 \cup T_2 \cup d=(N_1n-1)(M_1n-1)$

lattice points on $T_1=$ # lattice points on T_2, because $(i, k) \to (N_1n-i, M_1n-k)$ interchanges them.

lattice points on $d=n-1$.

The result follows at once from this count.

The rest of the argument goes as follows: the set of (i, k) such that

$$\begin{aligned} i \geqslant 0, \quad kN_1+iM_1 &\leqslant nN_1M_1-M_1-1 \\ -kN_1'+iM_1' &\leqslant n'N_1'M_1'-M_1-1 \end{aligned}$$

can be decomposed into

$$\{i \geqslant 0,\ k \geqslant 1 \quad \text{such that} \quad kN_1+(i+1)M_1 \leqslant nN_1M_1-1\},$$

$$\{i \geqslant 0,\ k \leqslant -1 \quad \text{such that} \quad -kN_1'+(i+1)M_1' \leqslant n'N_1'M_1'-1\}$$

and

$$\{i \geqslant 0, \quad \text{such that} \quad (i+1)M_1 \leqslant nN_1M_1-1\}$$

whose total cardinal—according to Lemma 3—equals

$$\frac{n}{2}(nN_1M_1-N_1-M_1-1)+1+\frac{n'}{2}(n'N_1'M_1'-N_1'-M_1'-1)+1+N-1=g$$

As announced, the second method comes from the theory of toroidal embeddings (Kempf et al. [13]). We merely sketch this. The idea is to embed all the Riemann Surfaces

$\mathcal{R}$ asociated to C with fixed N, M, M' in one rational surface X, and to write holomorphic 1-forms on $\mathcal{R}$ as residues of 2-forms on X with poles along $\mathcal{R}$. More precisely, X is the union of 3-affine pieces:

$$X_1 = \operatorname{Spec} \mathbf{C}[\ldots, z^i h^k, \ldots]_{\substack{i \geq 0 \\ -M_1' i + N_1' k \geq 0}}$$

$$X_2 = \operatorname{Spec} \mathbf{C}[\ldots, z^i h^k, \ldots]_{\substack{i \geq 0 \\ -M_1 i - N_1 k \geq 0}}$$

$$X_3 = \operatorname{Spec} \mathbf{C}[\ldots, z^i h^k, \ldots]_{\substack{-M_1' i + N_1' k \geq 0 \\ -M_1 i - N_1 k \geq 0}}.$$

X contains the affine surface Spec $\mathbf{C}[z, h, h^{-1}]$ which contains the affine curve $\mathcal{R}_0$. It is not hard to check that the closure of $\mathcal{R}_0$ in X is precisely $\mathcal{R}$ and that $\mathcal{R}$ misses the singular points of X. The maps $z \mapsto \lambda z$, $h \mapsto \mu h$ extend to automorphisms of X, so X is a "torus embedding" in the sense of [13]; in fact in the notation of the book, it is the one associated to the simplicial subdivision of the plane into the 3 sectors:

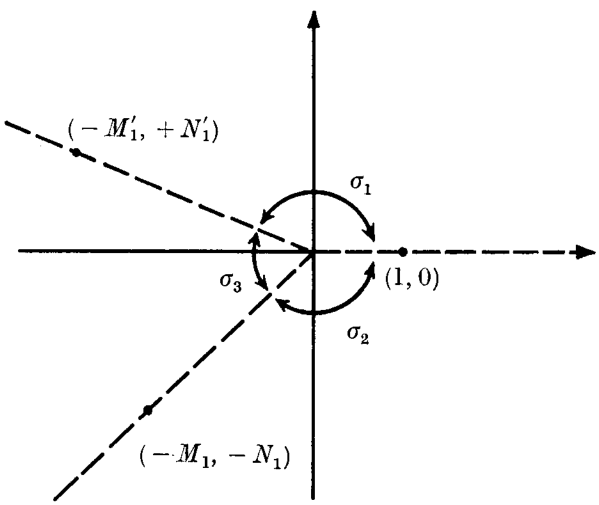

Figure 3

If $\omega(\mathcal{R})$ is the sheaf of meromorphic 2-forms on X, holomorphic outside the singular points of X except for simple poles at $\mathcal{R}$, then residue sets up an exact sequence:

$$H^0(\omega) \to \Gamma(\omega(\mathcal{R})) \to \Gamma(\Omega^1_{\mathcal{R}}) \to H^1(\omega)$$

and as $H^i(\omega)$, $H^{2-i}(\mathcal{O}_x)$ are dual and $H^1(\mathcal{O}_x) = H^2(\mathcal{O}_x) = 0$ (cf. [13], p. 44), it follows that

$$\text{res:} \quad \Gamma(\omega(\mathcal{R})) \xrightarrow{\approx} \Gamma(\Omega_{\mathcal{R}}).$$

Explicitly this means that every holomorphic 1-form on $\mathcal{R}$ is uniquely expressible as

$$\operatorname{Res}_{\mathcal{R}}\left(\frac{g\cdot dz\wedge dh}{F(h,h^{-1},z)\cdot z\cdot h}\right)$$

where $g=g(z,h,h^{-1})$ is chosen so that the 2-form in parenthesis has no poles other than $F=0$. There are 3 possible curves: (1) $h\in\mathbf{C}^*$, $z=0$, (2) $h^{M_1'}\cdot z^{N_1'}\in\mathbf{C}^*$, $h=0$, (3) $h^{M_1}z^{-N_1}\in\mathbf{C}^*$, $h^{-1}=0$. Checking the order of pole at each of these, we find that if

$$g=\sum a_{ik}h^k z^i$$

then $a_{ik}\neq 0$ only if

$$\begin{aligned} i&\geq 1\\ -iM_1'+kN_1'&\geq 1-NM_1'\\ -iM_1-kN_1&\geq 1-NM_1. \end{aligned}$$

Theorem 4 shows that the isospectral flows for difference operators (written in the Lax form) can actually be linearized: they are linear flows on the Jacobi variety of the corresponding curve. In the second part of this section, we show that these flows derive from Hamiltonians according to a co-symplectic structure on the space of all periodic difference operators suggested by group-theoretical considerations. Consider the group G of lower triangular invertible matrices (including the diagonal) of order N. Let g be the Lie algebra of lower triangular matrices and g' be its dual, namely the space of upper triangular matrices. g and g' are paired by the trace of the product. Let $\sigma\in G$, $X\in g$ and $f\in g'$; the adjoint action amounts to conjugation $\sigma\cdot X=\sigma^{-1}X\sigma$ and its coadjoint action amounts to conjugation and projection on g', i.e. $\sigma\cdot f=(\sigma^{-1}f\sigma)-(\sigma^{-1}f\sigma)^{-}$. Fix an element $f\in g'$ and consider the orbit $G\cdot f\subset g'$ of f under the action of the group G. According to a theorem by Kirillov and Kostant [14] the orbit $G\cdot f$ is endowed with a natural symplectic form, i.e., an alternating two-form in the tangent plane T_h to the orbit $G\cdot f$ at a given point h. Since G acts on g' by conjugation,

$$T_h=\{\text{locus of points } \xi_h A,\ A\in g,\ \text{where } \xi_h A=[h,A]-[h,A]^{-}\};$$

let $\xi_h A$ and $\xi_h B\in T_h$; then

$$\omega(\xi_h A,\xi_h B)\equiv \operatorname{Tr}\,(h[A,B])$$

is a non-singular alternating 2-form, so that the orbit $G\cdot f$ is even dimensional. Instead of introducing a skew-symmetric form on the tangent planes to each orbit $G\cdot f$, it is completely equivalent to introduce a skew-symmetric form on the co-tangent planes. Such a form is just a skew-symmetric Poisson-bracket on pairs of functions on $G\cdot f$, whose value at each point is bilinear in the differentials of these functions. This is called a co-symplectic structure, and this structure fits together into one co-symplectic structure on g', inducing all the separate ones on the orbits. It is given by

$$\{f, g\}(x) = \mathrm{Tr}\,(x\cdot[df(x), dg(x)])$$

(where $df(x)$ and $dg(x)$, being linear functions on g', can be identified with elements of g, hence can be bracketed).

The definition of this co-symplectic form can be adapted after some changes to the case of periodic symmetric difference operators and to the case of periodic non-symmetric difference operators. This leads to the Poisson bracket defined below. Before proceeding we need the following definitions.

Let $\mathcal{M}$ be the vector space of N-periodic infinite matrices C such that

$$C_{ij} = 0 \quad \text{if } |i-j| > K \text{ for some } K.$$

Define Tr (C), for $C \in \mathcal{M}$ to be $\sum_{i=1}^{N} C_{ii}$.

Put on $\mathcal{M}$ an inner product

$$\langle C, D\rangle = \mathrm{Tr}\,(CD^{\mathrm{T}}) = \sum_{\substack{(i,j)\in \mathbf{Z}^2 \\ \text{in cosets of} \\ (N,N)\mathbf{Z}}} C_{ij} D_{ij}.$$

A functional F on $\mathcal{M}$ is called differentiable, if there is a matrix $\partial F/\partial C$ in $\mathcal{M}$ such that for all D

$$\lim_{\varepsilon \downarrow 0} \frac{F(C+\varepsilon D) - F(C)}{\varepsilon} = \left\langle \frac{\partial F}{\partial C}, D \right\rangle.$$

Taking D given by

$$D_{ij} = \begin{cases} 1 & \text{if} \quad (i,j) = (i_0 + kN, j_0 + kN), \quad \text{some } k \\ 0 & \text{otherwise} \end{cases}$$

it follows that

$$\left(\frac{\partial F}{\partial C}\right)_{i_0, j_0} = \frac{\partial F}{\partial C_{i_0 j_0}}.$$

A simple identity which will be useful is: $\langle [A, B], C\rangle = \langle [A^{\mathrm{T}}, C], B\rangle$. Define the following bracket between two differentiable functionals F and G on $\mathcal{M}$

$$\{F, G\} = \left\langle \left[\left(\frac{\partial F}{\partial C}\right)^{[+]}, \left(\frac{\partial G}{\partial C}\right)^{[+]}\right] - \left[\left(\frac{\partial F}{\partial C}\right)^{[-]}, \left(\frac{\partial G}{\partial C}\right)^{[-]}\right], C \right\rangle.$$

LEMMA 5. $\{\ ,\ \}$ satisfies the Jacobi identity.

Proof. In general, when we are dealing with a Poisson bracket structure on a vector space, we can make a preliminary reduction in the proof of Jacobi's identity as follows: write

$$\{f, g\}(x) = \sum A_{ij}(x) \cdot \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j}.$$

Then

$$\frac{\partial}{\partial x_k}\{f, g\} = \sum \frac{\partial A_{ij}}{\partial x_k} \cdot \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} + A_{ij} \cdot \frac{\partial^2 f}{\partial x_i \partial x_k} \cdot \frac{\partial g}{\partial x_j} + A_{ij} \cdot \frac{\partial f}{\partial x_i} \cdot \frac{\partial^2 g}{\partial x_j \partial x_k}.$$

We claim that when you evaluate $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}$ the terms involving the 2nd derivatives of f, g and h *always* cancel out. This is easy to check directly, and it also follows because Jacobi's identity is equivalent to $d\omega = 0$, ω the dual 2-form; and $d\omega = 0$ is automatic when the coefficients A_{ij} are constant.

In our case

$$\frac{\partial}{\partial C}\{F, G\} = (\text{2nd derivative terms}) + \left[\left(\frac{\partial F}{\partial C}\right)^{[+]}, \left(\frac{\partial G}{\partial C}\right)^{[+]}\right] - \left[\left(\frac{\partial F}{\partial C}\right)^{[-]}, \left(\frac{\partial G}{\partial C}\right)^{[-]}\right].$$

and

$$\left(\frac{\partial}{\partial C}\{F, G\}\right)^{[+]} = (\text{2nd derivative terms}) + \left[\left(\frac{\partial F}{\partial C}\right)^{[+]}, \left(\frac{\partial G}{\partial C}\right)^{[+]}\right]$$

(since neither of the Lie bracket terms have any diagonal entries). Thus

$$\begin{aligned}\{H, \{F, G\}\} &= (\text{2nd derivative terms}) \\ &\quad + \left\langle \left[\left(\frac{\partial H}{\partial C}\right)^{[+]}, \left[\left(\frac{\partial F}{\partial C}\right)^{[+]}, \left(\frac{\partial G}{\partial C}\right)^{[+]}\right]\right] + \left[\left(\frac{\partial H}{\partial C}\right)^{[-]}, \left[\left(\frac{\partial F}{\partial C}\right)^{[-]}, \left(\frac{\partial G}{\partial C}\right)^{[-]}\right]\right], C \right\rangle.\end{aligned}$$

Writing this out for $\{H, \{F, G\}\}$, $\{F, \{G, H\}\}$, $\{G, \{H, F\}\}$ and summing, the right hand side is zero by the usual Jacobi identity for Lie brackets.

THEOREM 5. *The linear flows on* Jac $(\mathcal{R})$ *are Hamiltonian flows with regard to the Poisson bracket* $\{\ ,\ \}$. *In particular, in Poisson bracket notation, a typical flow*

$$\dot{C} = [C, (S^{-k}C^l)^{[+]}]$$

can be written as

$$\dot{c}_{ij} = \{F, c_{ij}\}$$

where

$$F(C) = \frac{1}{l+1} \operatorname{Tr} (S^{-k} C^{l+1}).$$

Proof. To begin with,

$$\frac{\partial}{\partial C}\operatorname{Tr}(S^{-k}C^{l+1}) = (l+1)(S^{-k}C^l)^{\mathrm{T}}.$$

Indeed, by direct calculation,

$$\frac{\partial}{\partial c_{\alpha\beta}}\operatorname{Tr}(S^{-k}C^{l+1}) = \frac{\partial}{\partial c_{\alpha\beta}}\sum c_{i,i_1}c_{i_1,i_2}\dots c_{i_{l-1},i_l}c_{i_l,i+kN}$$

where the sum extends over $1 \leqslant i \leqslant N$, $|i-i_1| \leqslant K$, $|i_1-i_2| \leqslant K$, ..., and $|i_l-(i+kN)| \leqslant K$,

$$= \sum_{m=0}^{l}\sum c_{i,i_1}c_{i_1,i_2}\dots c_{i_{m-1},\alpha}c_{\beta,i_{m+2}}\dots c_{i,i+kN}$$

the latter sum extending over $1 \leqslant i \leqslant N$, $|i-i_1| \leqslant K$, $|i_1-i_2| \leqslant K$, ..., $|i_{m-1}-\alpha| \leqslant K$, $|\beta-i_{m+2}| \leqslant K$, ..., $|i_l-(i+kN)| \leqslant K$

$$\begin{aligned} &= \sum_{m=0}^{l}\sum c_{\beta,i_{m+2}}\dots c_{i_l,i+kN}c_{i+kN,i_1+kN}\dots c_{i_{m-1}+kN,\alpha+kN} \\ &= (l+1)(S^{-k}\cdot C^l)_{\beta,\alpha}. \end{aligned}$$

Let E_{ij} be the "elementary" matrix

$$(E_{ij})_{k,l} = \begin{cases} 1 & \text{if } (k,l) = (i+lN, j+lN), \quad \text{some } l\in\mathbf{Z} \\ 0 & \text{otherwise.}\end{cases}$$

We check that the derivative of the functional $C \mapsto c_{ij}$ is given by:

$$\frac{\partial}{\partial C}(c_{ij}) = E_{ij}.$$

Thus we can calculate $\{F, c_{ij}\}$:

$$\{F, c_{ij}\} = \langle[((S^{-k}\cdot C^l)^{\mathrm{T}})^{[+]}, E_{ij}^{[+]}] - [((S^{-k}\cdot C^l)^{\mathrm{T}})^{[-]}, E_{ij}^{[-]}], C\rangle.$$

If $i<j$, $E_{ij}^{[-]}=0$, $E_{ij}^{[+]}=E_{ij}$, so

$$\begin{aligned}\{F, c_{ij}\} &= \langle[((S^{-k}\cdot C^l)^{[-]})^{\mathrm{T}}, E_{ij}], C\rangle \\ &= \langle[(S^{-k}\cdot C^l)^{[-]}, C], E_{ij}\rangle \\ &= [(S^{-k}\cdot C^l)^{[-]}, C]_{ij}\end{aligned}$$

If $i>j$, $E_{ij}^{[+]}=0$, $E_{ij}^{[-]}=E_{ij}$, so

$$\begin{aligned}\{F, c_{ij}\} &= -\langle[((S^{-k}\cdot C^l)^{[+]})^{\mathrm{T}}, E_{ij}], C\rangle \\ &= -\langle[(S^{-k}\cdot C^l)^{[+]}, C], E_{ij}\rangle \\ &= -[(S^{-k}\cdot C^l)^{[+]}, C]_{ij}.\end{aligned}$$

If $i=j$, we get the sum of half of each. Using the fact that C commutes with $S^{-k}C^l$, we find

$$[(S^{-k}\cdot C^l)^{[-]}, C] = -[(S^{-k}\cdot C^l)^{[+]}, C],$$

so in all cases, we get

$$\{F, c_{ij}\} = -[(S^k\cdot C^l)^{[+]}, C]_{ij} = [C, (S^k\cdot C^l)^{[+]}]_{ij}$$

as asserted.

THEOREM 6. *Any two functionals* $\mathrm{Tr}\,(S^{-k}C^{l+1})$ *have Poisson bracket zero: i.e., we have a set of Hamiltonians in involution. If to every* $C\in\mathfrak{M}$ *we associate the coefficients of* h^kz^l *in* $\det\,(C_h-zI)$, *these functionals are also in involution and through each* C, *generate the same set of flows.*

Proof. The first step is to show that any two expressions of the form

$$H_i = \mathrm{Tr}\,(S^{-k_i}C^{l_i+1}) \tag{32}$$

commute with one another for the symplectic structure. Consider the Hamiltonian vector field $\mathbf{X}_i$ derived from H_i, acting on differentiable functionals:

$$\mathbf{X}_i(F) = \{H_i, F\}.$$

Theorem 5 tells you that this vector field acts on C as follows.

$$\dot{C} = [C, (S^{-k}C^l)^{[+]}]. \tag{33}$$

This flow preserves the h-periodic spectrum of C (i.e., the spectrum of C_h) for every $h\in\mathbf{C}$. Therefore it preserves $\mathrm{Tr}\,(C_h)^\alpha = \mathrm{Tr}\,(C^\alpha)_h$ for every non-negative integer α, and in particular the coefficient of h^β in $\mathrm{Tr}\,(C_h)^\alpha$, namely $\mathrm{Tr}\,(S^{-\beta}C^\alpha)$. Therefore this flow leaves invariant every H_i, i.e., $\{H_i, H_j\}=0$; also, the Lie bracket $\{\mathbf{X}_i, \mathbf{X}_j\}$ vanishes, because

$$\begin{aligned}\{\mathbf{X}_i, \mathbf{X}_j\}F &= (\mathbf{X}_i\mathbf{X}_j - \mathbf{X}_j\mathbf{X}_i)F\\ &= \{H_i\{H_j, F\}\} - \{H_j, \{H_i, F\}\}\\ &= \{\{H_i, H_j\}, F\}\\ &= 0\end{aligned}$$

using Jacobi's identity. Finally the coefficient of z^{N-l} in $\det\,(C_h-zI)$ is a polynomial in $\mathrm{Tr}\,(C_h)^i$ for $1\leqslant i\leqslant l$ and $\mathrm{Tr}\,(C_h)^l$ is a polynomial in the coefficients of z^{N-i}, $1\leqslant i\leqslant l$. Therefore the coefficient of h^kz^l in $\det\,(C_h-zI)$ will be a polynomial in the coefficients of h^k in $\mathrm{Tr}\,C_h^i$ for $1\leqslant i\leqslant l$; i.e. in the quantities $\mathrm{Tr}\,S^{-k}C^i$. This proves the second assertion of the theorem. Observe that since C_h is $N\times N$, the expressions $\operatorname{Tr}(C_h)^i$ for $i>N$ are linear combinations of $\operatorname{Tr}(C_h)^i$ with $i\leqslant N$. So, no new functionals arise by considering $\operatorname{Tr} S^{-k}C^i$ for $i>N$.

Remark 1. Some of the coefficients appearing in $\det(C_h-zI)=0$ lead to identically zero vector fields. For instance the coefficients of z^{N-1} equal $\operatorname{Tr} C$; but since $\partial(\operatorname{Tr} C)/\partial C=I$, its Hamiltonian vector field vanishes.

Remark 2. Consider the special case of symmetric difference operators. Let $\mathcal{A}_0\subset\mathcal{A}$ be the subring of functions u such that $u^\tau=u$. The functions of $\mathcal{A}_0$ lead to linear flows in Prym $(\mathcal{R})$, because for $1\leqslant k\leqslant g_0$ (for which $\omega_k^\tau=\omega_k$)

$$\sum_{i=1}^{n}\operatorname{Res}_{P_i}(\omega_k u)=\sum_{i=1}^{n}\operatorname{Res}_{Q_i}(\omega_k^\tau u^\tau)=\sum_{i=1}^{n}\operatorname{Res}_{Q_i}(\omega_k u)$$

and, moreover

$$\sum_{i=1}^{n}\operatorname{Res}_{P_i}(\omega_k u)+\sum_{i=1}^{n}\operatorname{Res}_{Q_i}(\omega_k u)=0.$$

Therefore

$$\sum_{i=1}^{g}\int_{O_i}^{v_i}\omega_k=0\quad\text{for}\quad 1\leqslant k\leqslant g_0.$$

Since $\mathcal{A}$ is the polynomial ring in h, h^{-1} and z, the ring $\mathcal{A}_0$ is the polynomial ring in $h+h^{-1}$ and z. Therefore all the flows in Prym $(\mathcal{R})$ translate into flows of the type

$$\begin{aligned}\dot{C}&=[C,((S^k+S^{-k})C^i)^+]\\&=\tfrac{1}{2}[C,((S^k+S^{-k})C^i)^+-((S^k+S^{-k})C^i)^-].\end{aligned}$$

Notice that if C is symmetric, $(S^k+S^{-k})C^i$ is also symmetric and the flow above is generated by an antisymmetric operator, which indeed preserves the symmetry of C.

Examples.

1. Let C be an infinite (generic) tridiagonal matrix of period N. Let $\mathcal{R}$ be the hyperelliptic curve associated with it; let P and Q be the two points at infinity. Moreover, with the notation used in Example 1 (§ 2), a basis of holomorphic differentials is given by

$$\omega_k=\frac{z^{k-1}\,dz}{\sqrt{R(z)}},\quad\text{where } R(z)=P(z)^2-4A^2.$$

Moreover, since the order of zero of ω_k at P or Q equals $g-k$,

$$\begin{aligned}a_k=\operatorname{Res}_P(\omega_k z^j)&=0\quad\text{for } k<g-j+1\quad 1\leqslant j\leqslant g\\&\neq 0\quad\text{for } k=g-j+1\quad 1\leqslant j\leqslant g.\end{aligned}$$

Therefore a complete set of flows is given by the functions $z, z^2, \ldots, z^g$, so that the most general isospectral flow for C (i.e., leaving the spectrum of C and A unchanged) is given by a polynomial $P(z)$ of degree at most g:

$$\dot{C} = \tfrac{1}{2}[C, P(C)^+ - P(C)^-].$$

The Poisson Bracket between two functionals F and G has the simplified form

$$\{F, G\} = \left\langle \begin{pmatrix} \dfrac{\partial F}{\partial a} \\ \dfrac{\partial F}{\partial b} \end{pmatrix}^{\mathrm{T}}, \; J \begin{pmatrix} \dfrac{\partial G}{\partial a} \\ \dfrac{\partial G}{\partial b} \end{pmatrix} \right\rangle$$

where $\partial F/\partial a$ and $\partial F/\partial b$ are the column vectors whose elements are given by $\partial F/\partial a_i$ and $\partial F/\partial b_i$ respectively and J is defined as the $2n \times 2n$ antisymmetric matrix

$$\begin{pmatrix} 0 & B \\ -B^{\mathrm{T}} & 0 \end{pmatrix}$$

where

$$B = 2 \begin{pmatrix} b_1 & 0 & 0 & \cdots & -b_N \\ -b_1 & b_2 & 0 & & \vdots \\ 0 & -b_2 & b_3 & & \vdots \\ \vdots & & & & \vdots \\ 0 & \cdots & & -b_{N-1} & b_N \end{pmatrix}$$

The symplectic structure is given by

$$\omega = \sum_{j=2}^{N} da_j \wedge \sum_{j \leqslant i \leqslant N} \frac{db_i}{b_i}.$$

The g independent quantities in involution, leading to g independent flows, are given by $\beta_2, \ldots, \beta_N$, where

$$\det (C_h - zI)\big|_{h=i} = A_1(z) = (-1)^N z^N + \sum_{i=1}^{N} \beta_i z^{N-i}.$$

An equivalent set is given by $N-1$ points chosen from the spectrum of C_1 or C_{-1} (i.e., $N-1$ branch points of the hyperelliptic curve) or, alternatively, by the quantities

$$\operatorname{Tr} C^k, \quad 2 \leqslant k \leqslant N.$$

2. Consider the symmetric fourth order difference operator

$$(Cf)_k = c_{k-2}f_{k-2} + b_{k-1}f_{k-1} + a_k f_k + b_k f_{k+1} + c_k f_{k+2}$$

and assume N odd. Then $\mathcal{R}$ is a double covering of the hyperelliptic curve

$$F(g, z) \equiv Ag^2 + P_1(z)g + P_2(z) = 0,$$

where $\deg P_1 \leqslant (N-1)/2$ (generically, $=$) and $\deg P_2 = N$. The differentials on $\mathcal{R}_0$ or the symmetric (sheet invariant) differentials on $\mathcal{R}$ are given by linear combinations of

$$\omega_k = z^{k-1}(F'_g)^{-1}dz, \quad 1 \leqslant k \leqslant \frac{N-1}{2} = g_0;$$

this basis can be completed with the antisymmetric (for the involution τ) differentials

$$\omega = \frac{g^i}{h-h^{-1}} z^{k-1}(F'_g)^{-1}dz \begin{cases} i=1, & 1 \leqslant k \leqslant \dfrac{N-1}{2} \quad \text{or} \\ i=0, & 1 \leqslant k \leqslant N-1. \end{cases}$$

Since the dimension of the Prym variety equals $\frac{3}{2}(N-1)$, one expects to find $\frac{3}{2}(N-1)$ functions in $\mathcal{A}_0$ leading to independent flows in Prym ($\mathcal{R}$), namely

$$z, z^2, \ldots, z^{N-1}$$

and

$$(h+h^{-1})z^{(N+1)/2}, \ldots, (h+h^{-1})z^{N-1}.$$

The second sequence starts with the power $(N+1)/2$ in z, because this is the smallest possible power for which $h^{-1}z^{(N+1)/2}$ has an upper triangular part. This set can be completed to a set of flows spanning the whole of Jac ($\mathcal{R}$), by adding $(N-1)/2$ independent flows transversal to Prym ($\mathcal{R}$); they are generated by the functions

$$h^{-1}z^{(N+1)/2}, \ldots, h^{-1}z^{N-1}.$$

Finally, a set of integrals in involution spanning out all the linear flows in Jac ($\mathcal{R}$) is given by the coefficients of $h^k z^i$ in the algebraic expression $F(h, h^{-1}, z) = 0$: the $(N-1)/2$ coefficients of $P_1(z)$ (except for the highest order coefficient, which leads to a zero vector field) counted twice (once as coefficient of hz^k and once of $h^{-1}z^k$) and the $N-1$ coefficients of $P_2(z)$ (except for the coefficient in z^N and z^{N-1}, which again lead to zero vector fields).

3. A symmetric sixth order difference operator leads to a double covering of the curve

$$F(g, z) = Ag^3 + P_1(z)g^2 + P_2(z)g + P_3(z) = 0.$$

Assume that N is not a multiple of 3. Then all the flows in Prym $(\mathcal{R})$ are given by linear combinations of the functions

$$z, z^2, \ldots, z^{N-1}, \quad (h+h^{-1})z^{i_0}, \ldots, (h+h^{-1})z^{N-1},$$

where i_0 is the smallest integer $> N/3$, and

$$(h+h^{-1})^2 z^{i_1}, \ldots, (h+h^{-1})^2 z^{N-1},$$

where i_1 is the smallest integer $> 2N/3$, and the flows in Jac $(\mathcal{R})$ by these and the transversal flows to Prym $(\mathcal{R})$ generated by

$$h^{-1}z^{i_0}, \ldots, h^{-1}z^{N-1}$$

and

$$h^{-2}z^{i_1}, \ldots, h^{-2}z^{N-1}.$$

§ 5. Theta functions and difference operators

Certain theta-identities allow us to provide explicit formulas for the operator C in terms of the curve $\mathcal{R}$. They are very similar to Cor. 2.19, p. 33 in Fay [11]. To fix notations, we assume a basis $\{\omega_i\}$ of holomorphic 1-forms chosen; we write Abel's mapping from the curve $\mathcal{R}$ to its Jacobian Jac $(\mathcal{R})$ by

$$P \mapsto \int_{P_0}^{P} \boldsymbol{\omega}.$$

We fix an *odd* theta characteristic $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ such that the theta function $\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ does not vanish identically at all points

$$\int_{P_0}^{P} \boldsymbol{\omega}$$

(this exists; cf. Fay [11], p. 16). We write θ for $\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ for short. The theta-identity we want is this:

PROPOSITION. *There is a constant c_1 depending only on $\mathcal{R}$ and $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ such that for all*

$$x_1, \ldots, x_M, \; P_1, \ldots, P_M, \; Q_1, \ldots, Q_{M-1} \in \mathcal{R}, \; \mathbf{e} \in C^g$$

we have

$$\det_{1\leq i,j\leq M}\left\{\theta\left(\mathbf{e}+\sum_{\alpha=1}^{j-1}\int_{P_\alpha}^{Q_\alpha}\boldsymbol{\omega}+\int_{P_j}^{x_i}\boldsymbol{\omega}\right)\cdot\prod_{\alpha=1}^{j-1}\theta\left(\int_{Q_\alpha}^{x_i}\boldsymbol{\omega}\right)\prod_{\alpha=j+1}^{M}\theta\left(\int_{P_\alpha}^{x_i}\boldsymbol{\omega}\right)\right\}$$

$$=c_1\cdot\prod_{j=1}^{M-1}\theta\left(\mathbf{e}+\sum_{\alpha=1}^{j}\int_{P_\alpha}^{Q_\alpha}\boldsymbol{\omega}\right)\cdot\prod_{1\leq i<j\leq M}\theta\left(\int_{x_i}^{x_j}\boldsymbol{\omega}\right)\cdot\prod_{1\leq\beta<\alpha\leq M}\theta\left(\int_{Q_\alpha}^{P_\alpha}\boldsymbol{\omega}\right)\cdot\theta\left(\mathbf{e}+\int_{\Sigma_1^M P_\alpha}^{\Sigma_1^M x_i}\boldsymbol{\omega}\right)$$

The proof follows the standard classical procedure (cf. Fay [11], Prop. 2.16 and the references given there): we check that the right hand side of the equation and all terms in the determinant on the left hand side are in the same line bundle over $\mathcal{R}^{3M-1}\times\mathrm{Jac}\,(\mathcal{R})$ and that they have the same zeroes. To see the first, we assume more generally that we have 3M variable points of Jac $(\mathcal{R})$ and consider both sides as sections of a line bundle on Jac $(\mathcal{R})^{3M}$: these bundles are products of pull-backs by linear maps Jac $(\mathcal{R})^{3M}\to$ Jac $(\mathcal{R})$ of the standard line bundle with section θ, and to check they are equal it suffices to check the corresponding assertion for the Hermitian forms representing the 1st Chern class of these bundles. If B is the Hermitian form of the standard ample bundle on Jac $(\mathcal{R})$, this comes down to checking that all the bilinear forms

$$\sum_{i=1}^{M}\left[B\left(e+x_{\sigma_i}-\sum_{\alpha=1}^{i}P_\alpha+\sum_{\alpha=1}^{i-1}Q_\alpha\right)+\sum_{\alpha=1}^{i-1}B(x_{\sigma_i}-Q_\alpha)+\sum_{\alpha=i+1}^{M}B(x_{\sigma_i}-P_\alpha)\right],$$

σ any permutation of $\{1,\ldots,M\}$

and

$$\sum_{i=1}^{M-1}B\left(e-\sum_{\alpha=1}^{i}P_\alpha+\sum_{\alpha=1}^{i}Q_\alpha\right)+\sum_{1\leq i<j\leq N}B(x_i-x_j)+\sum_{1\leq\beta<\alpha\leq M}B(P_\alpha-Q_\beta)+B\left(e-\sum_{\alpha=1}^{M}P_\alpha+\sum_{i=1}^{M}x_i\right)$$

are all equal. This is elementary. As for the 2nd step, fix P_i, Q_j and $\mathbf{e}$ and consider as functions of $x\in\mathcal{R}$:

$$\psi_k(x)=\theta\left(\mathbf{e}+\sum_{\alpha=1}^{k-1}\int_{P_\alpha}^{Q_\alpha}\boldsymbol{\omega}+\int_{P_k}^{x}\boldsymbol{\omega}\right)\cdot\prod_{\alpha-1}^{k-1}\theta\left(\int_{Q_\alpha}^{x}\boldsymbol{\omega}\right)\cdot\prod_{\alpha=k+1}^{M}\theta\left(\int_{P_\alpha}^{x}\boldsymbol{\omega}\right).$$

Let 0 be some origin on $\mathcal{R}$. Then recall (for instance from Siegel [29]) that for some constant $\mathbf{c}$ (Riemann's constant):

$$\theta(\mathbf{t}-\mathbf{c})=0$$

if and only if

$$\mathbf{t}=\sum_{2}^{g}\int_{\eta_i}^{0}\boldsymbol{\omega}\quad\text{(mod. periods)}$$

for some positive divisor $\sum_2^g \eta_i$ of order $g-1$. Moreover the g roots $q_1, \ldots, q_g$ of

$$\theta\left(\int_0^q \boldsymbol{\omega} - \mathrm{s} - \mathrm{c}\right) = 0$$

satisfy the relation

$$\sum_1^g \int_0^{q_i} \boldsymbol{\omega} = \mathrm{s} \quad \text{(mod. periods)}.$$

The vector $\mathrm{e} \in C^g$ defines a divisor $\sum_1^g q_i - 0$ of degree $g-1$ and hence a line bundle L_{e} if we write

$$\mathrm{e} + \mathrm{c} = \sum_1^g \int_{q_i}^0 \boldsymbol{\omega}.$$

Then the zeros of

$$x \to \theta\left(\mathrm{e} + \sum_1^{k-1} \int_{P_\alpha}^{Q_\alpha} \boldsymbol{\omega} + \int_{P_k}^x \boldsymbol{\omega}\right)$$

satisfy the relation

$$\sum_1^g \int_0^{q_i} \boldsymbol{\omega} = -\mathrm{e} - \sum_1^{k-1} \int_{P_\alpha}^{Q_\alpha} \boldsymbol{\omega} + \int_0^{P_k} \boldsymbol{\omega} - \mathrm{c} \quad \text{(mod. periods)},$$

i.e., this function is a section of $L_{\mathrm{e}}(\sum_1^k P_\alpha - \sum_1^{k-1} Q_\alpha)$. Moreover x will be a zero if and only if

$$\mathrm{e} + \sum_1^{k-1} \int_{P_\alpha}^{Q_\alpha} \boldsymbol{\omega} + \int_{P_k}^x \boldsymbol{\omega} = \sum_2^g \int_{\eta_i}^0 \boldsymbol{\omega},$$

i.e., if and only if (by Abel's theorem),

$$\Gamma\left(L_{\mathrm{e}}\left(\sum_1^k P_\alpha - \sum_1^{k-1} Q_\alpha - x\right)\right) \neq (0).$$

Similarly for any point $R \in \mathcal{R}$,

$$x \to \theta\left(\int_R^x \boldsymbol{\omega}\right)$$

is a section of $L_0(R)$. But as θ is an odd theta function, $\theta(0) = 0$; and this section of $L_0(R)$ is zero at R, i.e., L_0 itself has a section φ with $g-1$ zeros $\mathcal{D}_0$ and the function $\theta(\int_R^x \boldsymbol{\omega})$ has its zeros at $\mathcal{D}_0 + R$. Thus ψ_k is a section of

$$L_{\mathrm{e}}\left(\sum_{\alpha=1}^k P_\alpha - \sum_{\alpha=1}^{k-1} Q_\alpha\right) \otimes O\left(\sum_{\alpha=1}^{k-1} (\mathcal{D}_0 + Q_\alpha)\right) \otimes O\left(\sum_{\alpha=k+1}^{M} (\mathcal{D}_0 + P_\alpha)\right)$$
$$\cong L_{\mathrm{e}}\left(\sum_{\alpha=1}^M P_\alpha\right) \otimes O((M-1)\mathcal{D}_0).$$

In fact, the last $(M-1)$ factors all vanish on $\mathcal{D}_0$, so ψ_k comes just from a section of

$$L_e\left(\sum_{\alpha=1}^{M} P_\alpha\right).$$

This bundle has degree $g-1+M$, so by Riemann–Roch, we may expect $\psi_1, ..., \psi_M$ to be a basis of its sections. Now $\det(\psi_i(x_j)) \neq 0$ if and only if no linear combination of the sections ψ_i is zero at all the points $x_1, ..., x_M$. Consider the various ways the left-hand side can be zero:

1. If $\theta(\mathbf{e}+\sum_{\alpha=1}^{j}\int_{P_\alpha}^{Q_\alpha}\boldsymbol{\omega})=0$, then by formula 45, Fay [11], one sees that

$$\theta\left(\mathbf{e}+\sum_{\alpha=1}^{j}\int_{P_\alpha}^{Q_\alpha}\boldsymbol{\omega}+\int_{P_{j+1}}^{x}\boldsymbol{\omega}\right)\cdot\theta\left(\int_{Q_j}^{x}\boldsymbol{\omega}\right)$$

and

$$\theta\left(\mathbf{e}+\sum_{\alpha=1}^{j-1}\int_{P_\alpha}^{Q_\alpha}\boldsymbol{\omega}+\int_{P_j}^{x}\boldsymbol{\omega}\right)\cdot\theta\left(\int_{P_{j+1}}^{x}\boldsymbol{\omega}\right)$$

are linearly dependent. (Take Fay's e to be our $(\mathbf{e}+\sum_i^j\int_{P_\alpha}^{Q}\boldsymbol{\omega})$, and take his y to be suitably general.) Thus ψ_j and ψ_{j+1} are linearly dependent and the determinant is zero.

2. If $\theta(\int_{x_i}^{x_j}\boldsymbol{\omega})=0$, then either $x_i=x_j$ and the determinant is zero or $x_i\in\mathcal{D}_0$. The left-hand side vanishes to order $M-1$ along the divisor $x_i\in\mathcal{D}_0$, but so does the whole ith row of the determinant.

3. If $\theta(\int_{Q_{\beta_1}}^{P_{\alpha_1}}\boldsymbol{\omega})=0$, then either $P_{\alpha_1}=Q_{\beta_1}$ or $P_{\alpha_1}\in\mathcal{D}_0$ or $Q_{\beta_1}\in\mathcal{D}_0$. If $P_{\alpha_1}=Q_{\beta_1}$, $\beta_1<\alpha_1$, then every section ψ_k vanishes at P_{α_1}. If $\mathbf{e}$ is sufficiently general, this means that all ψ_k are sections of

$$L_{\mathbf{e}}\left(\sum_{\alpha\neq\alpha_1} P_\alpha\right)$$

of degree $g-2+M$, hence with only $M-1$ sections, hence the ψ_k's are linearly dependent and the determinant is zero. For special values of $\mathbf{e}$, but $P_{\alpha_1}=Q_{\beta_1}$, the determinant is still zero by continuity. The left-hand side vanishes to order α_1-1 along the divisor $P_{\alpha_1}\in\mathcal{D}_0$, but (α_1-1)-columns (given by $j+1\leqslant\alpha_1$) of the determinant also vanish here.

4. Finally, if $\theta(\mathbf{e}+\sum_1^M\int_{P_i}^{x_i}\boldsymbol{\omega})=0$, then $L_{\mathbf{e}}(\sum_1^M P_\alpha-\sum_1^M x_i)$ has a section. To show the determinant is zero, we may assume $\theta(\mathbf{e}+\int_{P_1}^{Q_1}\boldsymbol{\omega})\neq 0$, i.e. $\Gamma(L_{\mathbf{e}}(P_1-Q_1))=(0)$. But then

$$\dim\Gamma\left(L_{\mathbf{e}}\left(\sum_1^M P_\alpha\right)\right)\leqslant\dim\Gamma\left(\frac{L_{\mathbf{e}}\left(\sum_1^M P_\alpha\right)}{L_{\mathbf{e}}(P_1-Q_1)}\right)=M.$$

So either the ψ_k are linearly dependent and we are done, or they span $\Gamma(L_{\mathbf{e}}(\sum_1^M P_\alpha))$. In this case, some combination is zero at all the points $x_1, ..., x_M$, so again $\det \psi_k(x_i)=0$.

This proves the proposition because the divisor of the left-hand side is greater than or equal to that of the right, but both divisors come from zeroes of sections of the same line bundle. Renumbering the P_α's, Q_α's from $-M'$ to M instead of 1 to M, and shifting $\mathbf{e}$ by

$$\sum_{\alpha=-M'}^{0} \int_{P_\alpha}^{Q_\alpha},$$

the formula reads:

Corollary.

$$\det_{\substack{-M'\leqslant j\leqslant M\\ 1\leqslant i\leqslant M+M'+1}} \left\{\theta\left(\mathbf{e}+\sum_{\alpha=1}^{j-1}\int_{P_\alpha}^{Q_\alpha}\boldsymbol{\omega}+\int_P^{x_i}\boldsymbol{\omega}\right)\cdot\prod_{\alpha=-M'}^{j-1}\theta\left(\int_{Q_\alpha}^{x_i}\boldsymbol{\omega}\right)\cdot\prod_{\alpha=j+1}^{M}\theta\left(\int_{P_\alpha}^{x_i}\boldsymbol{\omega}\right)\right\}$$

$$=c_1\prod_{j=-M'}^{M-1}\theta\left(\mathbf{e}+\sum_{\alpha=1}^{j}\int_{P_\alpha}^{Q_\alpha}\boldsymbol{\omega}\right)\cdot\prod_{1\leqslant i<j\leqslant M+M'+1}\theta\left(\int_{x_i}^{x_j}\boldsymbol{\omega}\right)\cdot\prod_{-M'\leqslant\beta<\alpha\leqslant M}\theta\left(\int_{Q_\beta}^{P_\alpha}\boldsymbol{\omega}\right)\theta\left(\mathbf{e}+\int_{\sum_{-M}^{0}Q_\alpha+\sum_1^M P_\alpha}^{\sum_1^{M+M'+1}x_i}\boldsymbol{\omega}\right)$$

Now we apply this to our curve $\mathcal{R}$ with given points $P_1, \ldots, P_n, Q_1, \ldots, Q_{n'}$ and function z. By assumption

$$(z) = -M_1\sum_{i=1}^{n} P_i - M_1\sum_{i=1}^{n'} Q_i + \sum_{i=1}^{M+M'} R_i$$

for some set of points $R_1, \ldots, R_{M+M'}$. In this case, the function z can be expanded:

$$z(x)=c_2\cdot\frac{\prod_{i=1}^{M+M'}\theta\left(\int_{R_i}^{x}\boldsymbol{\omega}\right)}{\left(\prod_{\alpha=1}^{n}\theta\left(\int_{P_\alpha}^{x}\boldsymbol{\omega}\right)\right)^{M_1}\cdot\left(\prod_{\alpha=1}^{n'}\theta\left(\int_{Q_\alpha}^{x}\boldsymbol{\omega}\right)\right)^{M_1'}}$$

where c_2 is a suitable constant. Moreover, if $\mathcal{D}$ is a regular divisor of degree g and we define

$$\mathbf{e}=\int_{\mathcal{D}}^{Q_0+\mathcal{D}_0}\boldsymbol{\omega};$$

then we claim that for suitable constants λ_k:

$$f_k(x)=\lambda_k\cdot\frac{\theta\left(\mathbf{e}+\sum_{\alpha=1}^{k-1}\int_{P_\alpha}^{Q_\alpha}\boldsymbol{\omega}+\int_{P_k}^{x}\boldsymbol{\omega}\right)\prod_{\alpha=0}^{k-1}\theta\left(\int_{Q_\alpha}^{x}\boldsymbol{\omega}\right)}{\theta\left(\mathbf{e}+\int_{Q_0}^{x}\boldsymbol{\omega}\right)\prod_{\alpha=1}^{k}\theta\left(\int_{P_\alpha}^{x}\boldsymbol{\omega}\right)}.$$

(Here $k \geqslant 1$; an obvious modification holds if $k \leqslant 0$.) In fact, it is immediate by the functional equation of θ that this is a meromorphic function on $\mathcal{R}$. The factors on the right give it zeroes at $Q_0, \ldots, Q_{k-1}$ and poles at $P_1, \ldots, P_k$. The other factor in the denominator satisfies

$$\theta\left(\mathrm{e} + \int_{Q_0}^{x} \boldsymbol{\omega}\right) = \theta\left(\int_{\mathcal{D}}^{x+\mathcal{D}_0} \boldsymbol{\omega}\right)$$

which is zero if $x \in \mathcal{D}$ because then

$$x + \mathcal{D}_0 - \mathcal{D} \equiv \mathcal{D}_0 - \text{positive divisor of degree } g-1.$$

These properties characterize f_k up to scalars. Now apply the Corollary replacing e by $\mathrm{e} + \sum_{\alpha=1}^{k-1} \int_{P_\alpha}^{Q_\alpha} \boldsymbol{\omega}$, renumbering P_α, Q_α by $P_{\alpha+k}$, $Q_{\alpha+k}$, letting $x_i = R_i$, $1 \leqslant i \leqslant M+M'$ and $x_{M+M'+1} = x$, and expanding the determinant along the 1st row:

$$\sum_{j=-M'+k}^{M+k} (-1)^j \theta\left(\mathrm{e} + \sum_{\alpha=1}^{j-1} \int_{P_\alpha}^{Q_\alpha} \boldsymbol{\omega} + \int_{P_j}^{x} \boldsymbol{\omega}\right) \cdot \prod_{\alpha=-M'+k}^{j-1} \theta\left(\int_{Q_\alpha}^{x} \boldsymbol{\omega}\right) \cdot \prod_{\alpha=j+1}^{M+k} \theta\left(\int_{P_\alpha}^{x} \boldsymbol{\omega}\right)$$

$$\times \det_{\substack{-M'+k \leqslant l \leqslant M+k \\ 1 \leqslant i \leqslant M+M' \\ l \neq j}} \left\{\theta\left(\mathrm{e} + \sum_{\alpha=1}^{l-1} \int_{P_\alpha}^{Q_\alpha} \boldsymbol{\omega} + \int_{P_l}^{R_i} \boldsymbol{\omega}\right) \cdot \prod_{\alpha=-M'+k}^{l-1} \theta\left(\int_{Q_\alpha}^{R_i} \boldsymbol{\omega}\right) \cdot \prod_{\alpha=l+1}^{M+k} \theta\left(\int_{P_\alpha}^{R_i} \boldsymbol{\omega}\right)\right\}$$

$$= \pm c_1 \prod_{j=-M'+k}^{M+k-1} \theta\left(\mathrm{e} + \sum_{\alpha=1}^{j} \int_{P_\alpha}^{Q_\alpha} \boldsymbol{\omega}\right) \cdot \prod_{1 \leqslant i < j \leqslant M+M'} \theta\left(\int_{R_i}^{R_j} \boldsymbol{\omega}\right) \cdot \prod_{i=1}^{M+M'} \theta\left(\int_{R_i}^{x} \boldsymbol{\omega}\right)$$

$$\times \prod_{-M'+k \leqslant \beta < \alpha \leqslant M+k} \theta\left(\int_{Q_\beta}^{P_\alpha} \boldsymbol{\omega}\right) \cdot \theta\left(\mathrm{e} + \sum_{\alpha=1}^{k-1} \int_{P_\alpha}^{Q_\alpha} \boldsymbol{\omega} + \int_{P_k}^{x} \boldsymbol{\omega}\right)$$

or

$$z(x) \cdot f_k(x) = \sum_{j=-M'+k}^{M+k} (-1)^j \frac{c_2}{c_1} \frac{\lambda_k}{\lambda_j}$$

$$\times \frac{\det\left\{\theta\left(\mathrm{e} + \sum_{\alpha=1}^{j-1} \int_{P_\alpha}^{Q_\alpha} \boldsymbol{\omega} + \int_{P_j}^{R_i} \boldsymbol{\omega}\right) \cdot \prod_{\alpha=-M'+k}^{j-1} \theta\left(\int_{Q_\alpha}^{R_i} \boldsymbol{\omega}\right) \cdot \prod_{\alpha=j+1}^{M+k} \theta\left(\int_{P_\alpha}^{R_i} \boldsymbol{\omega}\right)\right\}}{\prod_{l=-M'+k}^{M+k-1} \theta\left(\mathrm{e} + \sum_{\alpha=1}^{l} \int_{P_\alpha}^{Q_\alpha} \boldsymbol{\omega}\right) \cdot \prod_{1 \leqslant i < j \leqslant M+M'} \theta\left(\int_{R_i}^{R_j} \boldsymbol{\omega}\right) \cdot \prod_{-M'+k \leqslant \beta < \alpha \leqslant M+k} \theta\left(\int_{Q_\beta}^{P_\alpha} \boldsymbol{\omega}\right)} \cdot f_j(x).$$

Thus the operator C, up to a constant and suitably conjugated is given by

THEOREM 7.

$$c_{kj} = \frac{(-1)^j \det_{\substack{-M'+k \leqslant l \leqslant M+k \\ 1 \leqslant i \leqslant M+M' \\ l \neq j}} \theta\left(\mathrm{e} + \sum_{\alpha=1}^{l-1} \int_{P_\alpha}^{Q_\alpha} \boldsymbol{\omega} + \int_{P_l}^{R_i} \boldsymbol{\omega}\right) \cdot \prod_{\alpha=-M'+k}^{l-1} \theta\left(\int_{Q_\alpha}^{R_i} \boldsymbol{\omega}\right) \cdot \prod_{\alpha=l+1}^{M+k} \theta\left(\int_{P_\alpha}^{R_i} \boldsymbol{\omega}\right)}{\prod_{l=-M'+k}^{M+k-1} \theta\left(\mathrm{e} + \sum_{\alpha=1}^{l} \int_{P_\alpha}^{Q_\alpha} \boldsymbol{\omega}\right) \cdot \prod_{-M'+k \leqslant \beta < \alpha \leqslant M+k} \theta\left(\int_{Q_\beta}^{P_\alpha} \boldsymbol{\omega}\right)}.$$

§ 6. Almost periodic difference operators

Non-singular curves $\mathcal{R}$ with the properties listed in Theorem 1, but without the existence of a meromorphic function h, lead to *almost periodic difference operators*, in the following sense: for every $\varepsilon>0$, there is an integer $T>0$ such that for every interval $I(T)\subseteq \mathbf{Z}$ of length T you can find $\alpha\in I(T)$ with the property

$$|c_{k,k+i}-c_{k+\alpha,k+i+\alpha}|<\varepsilon \quad \forall k,\ k+i\in \mathbf{Z}.$$

Considering the Jacobian Jac $(\mathcal{R})$ as a moduli space for divisor classes of degree $g-1$, the Theta-divisor $\Theta\subset$ Jac $(\mathcal{R})$ is the subvariety of positive divisors in Jac $(\mathcal{R})$ of order $g-1$. Whenever one considers a regular divisor $\mathcal{D}$, the corresponding sequence of meromorphic functions f_k and the associated sequence of regular divisors

$$\mathcal{D}_k=(f_k)+\mathcal{D}+\sum_1^k P_i-\sum_0^{k-1} Q_i, \quad k\in\mathbf{Z},$$

then

$$\mathcal{L}(\mathcal{D}_k-Q_k)=\{0\}$$

or what is the same

$$\{\mathcal{D}_k\}\notin Q_k+\Theta.$$

where $\{\mathcal{D}_k\}\in$ Jac $(\mathcal{R})$ is the point corresponding to $\mathcal{D}_k$.

Now, we define a *uniformly* regular divisor $\mathcal{D}$ with regard to the same sequence to be a regular divisor, with the property that

$$\mathcal{L}(\mathcal{D}'-Q_k)=\{0\}$$

for every k $(1\leqslant k\leqslant n)$ and for every:

$$\{\mathcal{D}'\}\in\overline{\bigcup_{p\in\mathbf{Z}}\{\mathcal{D}_{k+pn}\}}$$

or, equivalently, for every k $(1\leqslant k\leqslant n')$,

$$(\Theta+Q_k)\cap\overline{\bigcup_{p\in\mathbf{Z}}\{\mathcal{D}_{k+pn}\}}=\varnothing.$$

As we shall see later, there are many such *uniformly* regular divisors.

THEOREM 8. *Let $\mathcal{R}$ be a non-singular curve with points $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_{n'}$; let z be a function on $\mathcal{R}$ subject to*

$$(z)=-M_1\sum_1^n P_i-M_1'\sum_1^{n'} Q_i+d,$$

where d is a positive divisor not containing the P_i's and Q_i's. Then every uniformly regular divisor $\mathcal{D}$ determines an almost periodic difference operator C.

Proof. Let ν be the l.c.m. of n and n'. Observe that in Jac $(\mathcal{R})$,

$$\mathcal{D}_{k+\nu}-\mathcal{D}_k \equiv \frac{\nu}{n}\sum_1^n P_i - \frac{\nu}{n}\sum_1^{n'} Q_i.$$

Let $\mathfrak{D}$ be the divisor on the right hand side. The divisors $\ldots \mathcal{D}_{k-\nu}, \mathcal{D}_k, \mathcal{D}_{k+\nu}, \ldots$ form a linear sequence of points in Jac $(\mathcal{R})$. The transformation obtained by adding $\mathfrak{D}$ to a given point on the torus is periodic or almost periodic. If this transformation would be periodic, it would imply the existence of a meromorphic function h having for divisors some multiple of $\mathfrak{D}$. If we fix some measure of distance on Jac $(\mathcal{R})$, then at least we can say that for any $\varepsilon > 0$, there is a positive integer T such that for every interval $I(T)$ of length T there is an integer $P \in I(T) \cap \mathbf{Z}$ with the property that in Jac $(\mathcal{R})$

$$\begin{aligned}\left|\left(\mathcal{D}+\sum_1^k P_\alpha - \sum_0^k Q_\alpha\right) - \left(\mathcal{D}+\sum_1^{k+p\nu} P_\alpha - \sum_0^{k+p\nu} Q_\alpha\right)\right| &= |(\mathcal{D}_k - Q_k) - (\mathcal{D}_{k+p\nu} - Q_k)| \\ &= |\mathcal{D}_k - \mathcal{D}_{k+p\nu}| \\ &= |p\mathfrak{D}| < \varepsilon \quad \forall k \in \mathbf{Z}.\end{aligned}$$

Consider the closure $\overline{\{p\mathfrak{D}\}}$ of the sequence of points $p\mathfrak{D}$ in Jac $(\mathcal{R})$: this will be the union of a finite number of cosets of the real subtorus $P \subset$ Jac $(\mathcal{R})$. We wish to prove first that, with a suitable choice of A- and B-periods on $\mathcal{R}$, P is contained in the real sub-torus given by the A-periods alone. In fact, our hypothesis that $\mathcal{D}$ is uniformly regular means that certain cosets of P are disjoint from Θ. This means that the cohomology class of Θ restricted to P is zero. But we have

$$H_1(P, \mathbf{Z}) \subset H_1(\text{Jac}(\mathcal{R}), \mathbf{Z}) \cong H_1(\mathcal{R}, \mathbf{Z})$$

and the cohomology class of Θ on Jac $(\mathcal{R})$ is just given by the 2-form: intersection product $(a, b) \to (\mathrm{a}\cdot\mathrm{b})$ on $H_1(\mathcal{R}, \mathbf{Z})$. Thus this triviality means $H_1(P, \mathbf{Z})$ is an isotropic subspace. Any maximal isotropic subspace of $H_1(\mathcal{R}, \mathbf{Z})$ can be taken as the set of A-periods, so this proves our assertion. This choice of A-periods means that θ is a *periodic* function in the P-coordinates, hence for any $\mathbf{e}$, the values

$$\theta\left(\mathbf{e} + \sum_{\alpha=1}^{l} \int_{P_\alpha}^{Q_\alpha} \boldsymbol{\omega}\right)$$

for *all* l are equal to their values in some compact fundamental domain mod periods. This plus the explicit formula for c_{ij} in the last section proves Theorem 8.

THEOREM 9. *Consider a curve* $\mathcal{R}$, $2n$ *points* $P_1, \ldots, P_n$, $Q_1, \ldots, Q_n$ *on* $\mathcal{R}$, *a meromorphic function* z *having the properties above with* $M_1 = M_1'$. *Let* $\tilde{}$ *be an antiholomorphic involution for which* $\mathcal{R}\setminus\mathcal{R}_{\mathbf{R}} = \mathcal{R}_+ \cup \mathcal{R}_-$ *where* $\mathcal{R}_{\mathbf{R}} = \{p \in \mathcal{R} \mid \tilde{p} = p\}$ *such that* $\tilde{P}_i = Q_i$ *with* $P_i \in \mathcal{R}_+$ *and* $Q_i \in \mathcal{R}_-$ *and such that* $\overline{z(p)} = z(\tilde{p})$. *Consider a divisor* $\mathcal{D}$ *having the property that*

$$\mathcal{D} + \tilde{\mathcal{D}} - P_n - Q_n$$

is the divisor of some differential on $\mathcal{R}$, *which is real positive on* $\mathcal{R}_{\mathbf{R}}$. *Then* $\mathcal{D}$ *determines a self-adjoint, almost periodic difference operator* C, *whose* L^2*-spectrum is the set of values of* z *on* $\mathcal{R}_{\mathbf{R}}$.

LEMMA. *Any divisor* $\mathcal{D}$ *on the curve* $\mathcal{R}$ *satisfying the conditions of Theorem 9 is uniformly regular.*

Proof. Recall from the proof of Theorem 3 that

$$\mathcal{D}^{(k)} + \tilde{\mathcal{D}}^{(k)} = P_k + Q_k + (\omega_k),$$

for some 1-form ω_k, with $\omega_k \geqslant 0$ on $\mathcal{R}_{\mathbf{R}}$. Therefore all the divisors $\mathcal{D} = \mathcal{D}^{(k+pn)}$, $p \in \mathbf{Z}$, satisfy:

$$\mathcal{D}' + \tilde{\mathcal{D}}' = P_k + Q_k + (\omega), \quad \omega \geqslant 0 \quad \text{on} \quad \mathcal{R}_{\mathbf{R}}.$$

Passing to the limit of any sequence, it follows that this condition still holds. But by the argument in Theorem 3, any such ω must have a non-zero residue at Q_k, hence Q_k doesn't occur in the divisor $\mathcal{D}'$, hence $\mathcal{D}' - Q_k \notin \Theta$.

Proof of Theorem 9. From the Lemma it follows that $\mathcal{D}$ is uniformly regular; by Theorem 3 (§ 2) and 8 it maps into a self-adjoint almost periodic difference operator C. Consider now the space of meromorphic functions

$$\mathcal{L} = \{f \mid (f) > -\mathcal{D} - \sum k_i P_i - \sum l_i Q_i \quad \text{with } k_i, l_i \in \mathbf{Z} \text{ arbitrary}\}.$$

Any $f \in \mathcal{L}$ maps into a sequence λ_n, zero for almost all n (Lemma 5, § 1) such that

$$f = \sum_{i=-\infty}^{\infty} \lambda_i f_i.$$

Let $\mathcal{D}$ not contain any point of $\mathcal{R}_{\mathbf{R}}$. Then using the inner product defined in Theorem 3,

$$\frac{1}{2\pi}\int_{\mathcal{R}_{\mathbf{R}}} |f|^2 \omega = \frac{1}{2\pi} \sum_n \sum_m \lambda_n \bar{\lambda}_m \int_{\mathcal{R}_{\mathbf{R}}} f_n f_m^* \omega = \sum |\lambda_n|^2 < \infty.$$

$\mathcal{L}$ is a space of complex-valued functions on $\mathcal{R}_{\mathbf{R}}$, separating points and closed under conjugation, so by the Stone–Weierstrass theorem the space $\mathcal{L}$ naturally completes to the space $L^2(\mathcal{R}_{\mathbf{R}})$ of L^2-complex valued functions on $\mathcal{R}_{\mathbf{R}}$; the space of almost everywhere vanishing sequences completes to $l^2(\mathbf{Z})=\{\{\lambda_n\} \mid \sum|\lambda_n|^2<\infty\}$. A basis for the space $L^2(\mathcal{R}_{\mathbf{R}})$ is given by the functions f_k. This defines now a unitary transformation from $l^2(\mathbf{Z})$ to $L^2(\mathcal{R}_{\mathbf{R}})$. The difference operator C acts on $l^2(\mathbf{Z})$ as follows

$$(C\lambda)_n=\sum_{k=n-M}^{n+M}\lambda_k c_{kn}=\sum_{k=n-M}^{n+M}\overline{c_{nk}}\lambda_k$$

and C acts on $L^2(\mathcal{R}_{\mathbf{R}})$ as a multiplication operator. Indeed for $f=\sum\lambda_n f_n$

$$\begin{aligned} Cf &= \sum\lambda_n(Cf_n)\\ &=\sum\lambda_n z f_n\\ &= zf. \end{aligned}$$

This operator is bounded and self-adjoint, since $\mathcal{R}_{\mathbf{R}}$ does not contain P_i or Q_i. The spectrum of this operator is the range of z, defined on the cycles $\mathcal{R}_{\mathbf{R}}$.

If $\mathcal{D}$ contains a point of $\mathcal{R}_{\mathbf{R}}$, we may argue by a limiting process that the theorem still holds, or, noting that ω has zeroes where f has poles, we replace $L^2(\mathcal{R}_{\mathbf{R}})$ by the space of functions f on $\mathcal{R}_{\mathbf{R}}-\mathcal{D}\cap\mathcal{R}_{\mathbf{R}}$ such that

$$\int_{\mathcal{R}_{\mathbf{R}}}|f|^2\omega<\infty.$$

Then the proof goes through as before.

References

[1]. Abraham, R. & Marsden, J., *Foundations of Mechanics*. Benjamin, San Francisco, 1978.

[2]. Adler, M., On a trace functional for pseudo-differential operators and the symplectic structure of the Korteweg-De Vries type equations. *Inventiones Math.* (1979).

[3]. Adler, M. & van Moerbeke, P., The Kac-Moody extension for the classical groups and algebraic curves. To appear.

[4]. Altman, A. & Kleiman, S., Compactifying the Jacobian. *Bull. A.M.S.*, 82 (1976), 947–949.

[5]. —— Compactifying the Picard scheme. *Advances in Math.* To appear.

[6]. Burchnall, J. L. & Chaundy, T. W., Commutative ordinary differential operators. *Proc. London Math. Soc.*, 81 (1922), 420–440; *Proc. Royal Soc. London* (*A*), 118 (1928), 557–593 (with a note by H. F. Baker); *Proc. Royal Soc. London* (*A*), 134 (1931), 471–485.

[7]. De Souza, M., Compactifying the Jacobian. Thesis at the Tata Institute (1973).

[8]. Dikii, L. A. & Gel'fand, I. M., Fractional Powers of Operators and Hamiltonian systems. *Funk. Anal. Priloz.*, 10 (1976).

[9]. Drinfeld, V. G., Elliptic modules. *Mat. Sbornik*, 94 (1974); translation 23 (1976), 561.

[10]. Dubrovin, B. A., Matveev, V. B. & Novikov, S. P., Non-linear equations of the Korteweg-de Vries type, finite zone linear operators, and Abelian manifolds. *Uspehi Mat. Nauk*, 31 (1976); *Russian Math. Surveys*, 31 (1976), 55–136.

[11]. Fay, J., *Theta functions on Riemann surfaces.* Springer Lecture Notes 352 (1973).

[12]. Kac, M. & van Moerbeke, P., On periodic Toda lattices. *Proc. Nat. Acad. Sci. U.S.A.*, 72 (1975), 1627–1629, and A complete solution of the periodic Toda problem. *Proc. Nat. Acad. Sci. U.S.A.*, 72 (1975), 2875–2880.

[13]. Kempf, G., Knudsen, P., Mumford, D. & Saint-Donat, B. *Toroidal embeddings I.* Berlin–Heidelberg–New York: Springer vol. 339 (1973).

[14]. Kostant, B., Quantization and unitary representation, *Lectures on Modern Analysis and Applications III*, Berlin–Heidelberg–New York, Springer vol. 170 (1970).

[15]. Krichever, I. M., Algebro-geometric construction of the Zaharov-Shabat equations and their periodic solutions. *Sov. Math. Dokl.*, 17 (1976), 394–397.

[16]. —— *Uspekhi Mat. Nauk*, (1978).

[17]. —— Methods of Algebraic geometry in the theory of non-linear equations. *Uspekhi Mat. Nauk*, 32: 6 (1977), 183–208. Translation: *Russian Math Surveys*, 32: 6 (1977), 185–213.

[18]. McKean, H. P. & van Moerbeke, P., The spectrum of Hill's equation. *Inventiones Math.*, 30 (1973), 217–274.

[19]. —— Sur le spectre de quelques operateurs et les varietes de Jacobi. *Sem Bourbaki*, 1975–76, No, 474, 1–15.

[20]. —— About Toda and Hill curves. *Comm. Pure Appl. Math.*, (1979) to appear.

[21]. McKean, H. P. & Trubowitz, E., The spectrum of Hill's equation, in the presence of infinitely many bands. *Comm. Pure. Appl. Math.*, 29 (1976), 143–226.

[22]. van Moerbeke, P., The spectrum of Jacobi matrices. *Inventiones Math.*, 37 (1976), 45–81.

[23]. —— About isospectral deformations of discrete Laplacians. *Proc. of a conference on non-linear analysis*, Calgary, June 1978. Springer Lecture Notes, 1979.

[24]. Mumford, D., An algebro-geometrical construction of commuting operators and of solutions to the Toda lattice equation, Korteweg-de Vries equation and related non-linear equations. *Proc. of a Conference in Algebraic Geometry*, Kyoto, 1977, publ. by Japan Math. Soc.

[25]. —— *Abelian varieties*, Tata Institute; Oxford University Press, (1970).

[26]. —— The Spectra of Laplace-like periodic partial difference operators and algebraic surfaces, to appear.

[27]. Rosenlicht, M., Generalized Jacobian varieties. *Ann. of Math.*, 59 (1954), 505–530.

[28]. Serre, J. P., *Groupes algebriques et Corps de Classes*, Paris. Hermann (1959).

[29]. Siegel, C. L., *Topics in complex function theory, vol. 2.* New York, Wiley (1971).

[30]. Zaharov, V. E. & Shabat, A. B., A scheme for integrating the non-linear equations of math. physics by the method of the inverse scattering problem I. *Funct. Anal. and its Appl.* 8 (1974) (translation 1975, p. 226).

Received August 28, 1978