SOME FOOTNOTES TO THE WORK OF C. P. RAMANUJAM

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Reprinted from
C. P. RAMANUJAM—A TRIBUTE
Studies in Mathematics, No. 8
Tata Institute of Fundamental Research, Bombay
1978

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THIS PAPER consists of a series of remarks, each of which is connected in some way with the work of Ramanujam. Quite often, in the last few years, I have been thinking on some topic, and suddenly I realize—Yes, Ramanujam thought about this too—or—This really links up with his point of view. It is uncanny to see how his ideas continue to work after his death. It is with the thought of embellishing some of his favourite topics that I write down these rather disconnected series of results.

I

The first remark is a very simple example relevant to the purity conjecture (sometimes called Lang's conjecture) discussed in Ramanujam's paper [10]. The conjecture was—let

$$f: X^n \to Y^m$$

be a proper map of an n-dimensional smooth variety onto an m-dimensional smooth variety with all fibres of dimension $n - m$. Assume the characteristic is zero. Then show

$$\{y \in Y | f^{-1}(y) \text{ is singular}\}$$

has codimension one in Y. When $n = m$, this result is true and is known as "purity of the branch locus"; when $n = m + 1$, it is also true and was proven by Dolgačev, Simha and Ramanujam. When $n = m + 2$, Ramanujam describes in [10] a counter-example due to us jointly. Here is another counter-example for certain large values of $n - m$.

We consider the following very special case for f. Start with $Z^r \subset \mathbf{P}^m$ an arbitrary subvariety. Let $\check{\mathbf{P}}^m$ be the dual projective space—the space of hyperplanes in $\mathbf{P}^m$. The dual variety $\check{Z} \subset \check{\mathbf{P}}^m$ is, by definition the Zariski-closure of the locus of hyperplanes H such that, at some smooth point $x \in Z^r$, $T_{x,H} \supset T_{x,Z}$. It is apparently well known, although I don't know a reference, that in characteristic 0,

$$\check{\check{Z}} = Z.$$

Consider the special case where Z is smooth and spans $\mathbf{P}^m$. Then we don't need to take the Zariski-closure in the above definition and, in fact, the definition of $\check{Z}$ can be reformulated like this:

Let

$$I \subset \mathbf{P}^m \times \check{\mathbf{P}}^m$$

be the universal family of hyperplanes, i.e., if $(X_0, \dots, X_m)$, resp. $(\xi_0, \dots, \xi_m)$ are coordinates in $\mathbf{P}^m$, resp. $\check{\mathbf{P}}^m$, then I is given by

$$\Sigma \xi_i X_i = 0.$$

Let

$$X = I \cap (Z^r \times \check{\mathbf{P}}^m).$$

Note that I and $Z^r \times \check{\mathbf{P}}^m$ are smooth subvarieties of $\mathbf{P}^m \times \check{\mathbf{P}}^m$ of codimension 1 and $m - r$ respectively. One sees immediately that they meet transversely, so X is smooth of dimension $m + r - 1$. Consider

$$p_2 \colon X \to \check{\mathbf{P}}^m.$$

Its fibres are the hyperplane sections of Z, all of which have dimension $r - 1$. Thus p_2 is a morphism of the type considered in the conjecture. In this case

$$\{\xi \in \check{\mathbf{P}}^m \mid p_2^{-1}(\xi) \text{ singular}\} = \{\xi \in \check{\mathbf{P}}^m \mid \text{if } \xi \text{ corresponds to } H \subset \mathbf{P}^m, \text{ then } Z.H \text{ is singular}\}$$

$$= \check{Z}.$$

Thus the conjecture would say that the dual $\check{Z}$ of a smooth variety Z spanning $\mathbf{P}^m$ is a hypersurface.

I claim this is false, although I feel sure it can only be false in very rare circumstances. In fact, I don't know any cases other than the following example where it is false.† Simply take

$Z^r = $ [Grassmannian of lines in $\mathbf{P}^{2k}$, $k > 1$].

Here $r = 2(2k-1)$, $m = k(2k+1)-1$ and the embedding i: $Z^r \subset \mathbf{P}^m$ is the usual Plücker embedding. In vector space form, let

V = a complex vector space of dimension $2k+1$

Z = set of 2-dimensional subspaces $W_2 \subset V$

$\mathbf{P}^m$ = set of 1-dimensional subspaces $W_1 \subset \Lambda^2 V$

i = map taking W_2 to $W_1 = \Lambda^2 W_2$.

Note that we may identify

$\check{\mathbf{P}}^m$ = set of 1-dimensional subspaces $W_1' \subset \Lambda^2 V^*$, where

$\Lambda^2 V^*$ = space of skew-symmetric 2-forms A: $V \times V \to \mathbf{C}$.

Write $[W_2] \in Z$ for the point defined by W_2, and $H_A \subset \mathbf{P}^m$ for the hyperplane defined by a 2-form A. Then it is immediate from the definitions that

$$i([W_2]) \in H_A \Leftrightarrow \mathrm{res}_{W_2} A \text{ is zero.}$$

To determine when moreover,

$$i_* (T_{W_2, Z}) \subset T_{i(W_2), H_A},$$

let $v_1, v_2 \in W_2$ be a basis, and make a small deformation of W_2 by taking $v_1 + \epsilon v_1'$, $v_2 + \epsilon v_2'$ to be a basis of $\tilde{W}_2 \subset V \otimes \mathbf{C}[\epsilon]$. Then $\tilde{W}_2$ represents a tangent vector t to Z at $[W_2]$ and

$$i_*(t) \subset T_{i(W_2), H_A} \Leftrightarrow A(v_1 + \epsilon v_1', v_2 + \epsilon v_2') \equiv 0 \pmod{\epsilon^2}$$

$$\Leftrightarrow A(v_1', v_2) + A(v_1, v_2') = 0,$$

Thus:

$$i_*(T_{W_2, Z}) \subset T_{i(W_2), H_A} \Leftrightarrow \text{for all } v_1', v_2' \in V,$$

$$A(v_1, v_2') + A(v_1', v_2) = 0$$

$$\Leftrightarrow W_2 \subset (\text{nullspace of } A).$$

†M. Reid has indicated to me another set of examples: Suppose

$$Z = \mathbf{P}(E)$$

where E is a vector bundle of rank s on y^t, so $r = s + t - 1$, and the fibres of $\mathbf{P}(E)$ are embedded linearly. Then if $s > t + 2$, $\check{Z}$ is not a hypersurface.

Therefore

$$H_A \text{ is tangent to } i(Z) \Leftrightarrow \dim(\text{nullspace}) \geqslant 2.$$

Now the nullspace of A has odd dimension, and if it is 3, one counts the dimension of the space of such A as follows:

$$\begin{aligned}\dim\begin{pmatrix}\text{space of } A\text{'s with}\\ \dim(\text{nullspace}) = 3\end{pmatrix} &= \dim\begin{pmatrix}\text{space of}\\ W_3 \subset V\end{pmatrix} + \dim \Lambda^2 (V/W_3)\\ &= 3(2k-2) + \frac{(2k-2)\,(2k-3)}{2}\\ &= 2k^2 + k - 3.\end{aligned}$$

Thus dim $\check{Z} = m - 3$, and codim $\check{Z} = 3$! (Compare this with Buchsbaum-Eisenbud [3], where it is shown that $\check{Z} \subset \check{\mathbf{P}}^m$ is a "universal codimension 3 Gorenstein scheme".)

II

The second remark concerns the Kodaira Vanishing Theorem. We want to show that Ramanujam's strong form of Kodaira Vanishing for surfaces of Char. 0 is a consequence of a recent result of F. Bogomolov. In particular, this is interesting because it gives a new completely algebraic proof of this result, and one which uses the Char. 0 hypothesis in a new way (it is used deep in Bogomolov's proof, where one notes that if $V^3 \to F^2$ is a ruled 3-fold and $D \subset V^3$ is an irreducible divisor meeting the generic fibre set-theoretically in one point, then D is birational to F). Ramanujam's result [11] is this: let F be a smooth surface of Char. 0, D a divisor on F.

Then

$$\left.\begin{array}{l}(D^2) > 0\\ (D.C) \geqslant 0, \text{ all curves } C \subset F\end{array}\right\} \Rightarrow H^1(F, \mathcal{O}(-D)) = (0). \qquad (1)$$

Bogomolov's theorem is that if F is a smooth surface of char. 0, E a rank 2 vector bundle on F, then

$$C_1(E)^2 > 4C_2(E) \Rightarrow E \text{ is unstable, meaning } \exists \text{ an extension}$$
$$0 \to L(D) \to E \to I_Z L \to 0,$$
$$I_Z = \text{ideal sheaf of a 0-dim. subscheme } Z \subset F,$$
$$L \text{ invertible sheaf, } D \text{ a divisor}$$
$$D \in [\text{num. pos. cone, } (D^2) > 0,\ (D.H) > 0]. \tag{2}$$

(See Bogomolov [2], Reid [13]; another proof using reduction mod p instead of invariant theory has been found by D. Gieseker.)

To prove (2) $\Rightarrow$ (1), suppose D_1 is given satisfying the conditions of (1). Take any element $\alpha \in H^1(F, \mathcal{O}(-D_1))$ and via α, form an extension

$$0 \longrightarrow \mathcal{O}_F \xrightarrow{\mu} E \xrightarrow{\nu} \mathcal{O}_F(D_1) \longrightarrow 0.$$

Note that $C_1(E) = D_1$, $C_2(E) = 0$, $(D_1^2) > 0$, so E satisfies the conditions of (2). Therefore, by Bogomolov's theorem, E is unstable: this gives an exact sequence

$$0 \longrightarrow L(D_2) \xrightarrow{\sigma} E \xrightarrow{\tau} I_Z L \longrightarrow 0$$
$$D_2 \in (\text{num. pos. cone}).$$

Note that the subsheaf $\sigma(L(D_2))$ of E cannot equal the subsheaf $\mu(\mathcal{O}_F)$ in the definition of E, because this would imply, comparing the 2 sequences, that $D_2 \equiv -D_1$ whereas both D_1, D_2 are in the numerically positive cone. Therefore, the composition

$$L(D_2) \xrightarrow{\sigma} E \xrightarrow{\nu} \mathcal{O}_F(D_1)$$

is not zero, hence

$$L \cong \mathcal{O}_F(D_1 - D_2 - D_3),\ D_3 \text{ an effective divisor.}$$

Next, comparing Chern classes of E in its 2 presentations, we find

$$2C_1(L) + D_2 \equiv C_1(E) \equiv D_1 \tag{3a}$$
$$(C_1(L) + D_2).\ C_1(L) + \deg Z = C_2(E) = 0. \tag{3b}$$

By (3a), we find $D_1 - D_2 - 2D_3 \equiv 0$, hence $L \simeq \mathcal{O}_F(+D_3)$; by (3b), we find $(D_1 - D_3).D_3 \leqslant 0$. But

$$\det\begin{vmatrix} (D_1^2) & (D_1.D_3) \\ (D_1.D_3) & (D_3^2) \end{vmatrix} = (D_1^2)[(D_3^2)-(D_1.D_3)]+(D_1.D_3)[(D_1^2)-2(D_1.D_3)] + (D_1.D_3)^2$$

while $(D_3^2) - (D_1.D_3) \geqslant 0$ (by 3b)

$(D_1^2) - 2(D_1.D_3) = (D_1.D_2) > 0$ (since D_1, D_2 num. pos.)

$(D_1.D_3) \geqslant 0$ (by the assumptions on D_1).

On the other hand, this det is $\leqslant 0$ by Hodge's Index Theorem. Therefore $(D_1.D_3) = 0$ and det $= 0$. From the latter, D_3 is numerically equivalent to λD_1, $\lambda \in \mathbf{Q}$, hence $(D_1.D_3) = \lambda(D_1^2)$. Thus $\lambda = 0$ and since D_3 is effective, $D_3 = 0$. Therefore the subsheaf $\sigma(L(D_2))$ is isomorphic to $\mathcal{O}_F(D_1)$ and defines a splitting of the original exact sequence. Therefore the extension class $\alpha \in H^1(\mathcal{O}_F(-D_1))$ is 0, so $H^1(\mathcal{O}_F(-D_1)) = (0)$.

III

The last two remarks are applications of Kodaira's Vanishing Theorem. To me it is quite amazing how this cohomological assertion has such strong consequences, both for geometry and for local algebra. Here is a geometric application. This application is a link between the recent paper of Arakelov [1] (proving Shafarevich's finiteness conjecture on the existence of families of curves over a fixed base curve, with prescribed degenerations), and Raynaud's counter-example [12] to Kodaira Vanishing for smooth surfaces in char. p. What I claim is this (this remark has been observed by L. Szpiro also):

Proposition. *Let $p: F \to C$ be a proper morphism of a smooth surface F onto a smooth curve C over a field k of arbitrary characteristic. Let $E \subset F$ be a section of p and assume the fibres of p have positive arithmetic genus. Let F_0 be the normal surface obtained by blowing down all components of fibres of p not meeting E. Then:*

Kodaira's Vanishing Theorem $\Longrightarrow (E^2) \leqslant 0.$
for ample divisors on F_0

If Char $(k) = 0$, then Kodaira's Vanishing Theorem holds for F_0 (cf. [9]), so $(E^2) \leqslant 0$ follows. This result, and its refinement — $(E^2) < 0$ unless all the smooth fibres of p are isomorphic — are due to Arakelov [1], who proved them by a very ingenious use of the Weierstrass points of the fibres $p^{-1}(x)$. On the other hand, if char(k) $=p$, Raynaud has shown how to construct examples of morphisms p: $F \to C$ and sections $E \subset F$, where all the fibres of p are irreducible but singular (thus $F = F_0$), and $(E^2) > 0$. Thus Kodaira Vanishing is false for this F. If char$(k) = 2$ or 3, he finds in fact quasi-elliptic surfaces F of this type. This Proposition is, in fact, merely an elaboration of the last part of Raynaud's example.

Proof of Proposition: Suppose $(E^2) > 0$. Let p_0: $F_0 \to C$ be the projection and let E stand for the image of E in F_0 too. Consider divisors on F_0 of the form

$$H = E + p_0^{-1}(\mathfrak{A}), \deg \mathfrak{A} > 0.$$

Then $(H^2) > 0$ and $(H.C) > 0$ for all curves C on F_0, so H is ample by the Nakai-Moisezon criterion. On the other hand, let's calculate $H^1(F_0, \mathcal{O}(-H))$. We have

$$0 \to H^1(C, p_{0,*}\mathcal{O}(-H)) \to H^1(F_0, \mathcal{O}(-H)) \to H^0(C, R^1 p_{0,*}\mathcal{O}(-H)) \to 0.$$

Clearly

$$p_{0,*}\mathcal{O}(-H) = (0) \text{ and } R^1 p_{0,*}\mathcal{O}(-H) \simeq (R^1 p_{0,*}\mathcal{O}(-E)) \otimes \mathcal{O}_C(-\mathfrak{A}).$$

Now using the sequences:

$$0 \to \mathcal{O}_{F_0}(-E) \to \mathcal{O}_{F_0} \to \mathcal{O}_E \to 0$$
$$0 \to \mathcal{O}_{F_0} \to \mathcal{O}_{F_0}(E) \to \mathcal{O}_E((E^2)) \to 0$$

we find

$$\begin{array}{ccccccccc} p_{0,*}\mathcal{O}_{F_0} & \xrightarrow{\alpha} & p_{0,*}\mathcal{O}_E & \longrightarrow & R^1 p_{0,*}\mathcal{O}(-E) & \xrightarrow{\beta} & R^1 p_{0,*}\mathcal{O}_{F_0} & \longrightarrow & 0 \\ \| & & \| & & & & & & \\ \mathcal{O}_C & & \mathcal{O}_C & & & & & & \end{array}$$

so α and β are isomorphisms, and (using the fact that the genus of the fibres is positive):

$$\begin{array}{ccccccc} 0 \longrightarrow p_{0,*}\, \mathcal{O}_{F_0} & \overset{\gamma}{\longrightarrow} & p_{0,*}\, \mathcal{O}(E) & \longrightarrow & p_{0,*}\, \mathcal{O}_E((E^2)) & \overset{\delta}{\longrightarrow} & R^1 p_{0,*}\, \mathcal{O}_{F_0} \\ \| & & \| & & & & \\ \mathcal{O}_C & & \mathcal{O}_C & & & & \end{array}$$

so γ is an isomorphism, and δ is injective. Now via the isomorphism resp: $E \to C$, let the divisor class (E^2) on E correspond to the divisor class $\mathfrak{A}$ on C. Then $p_{0,*}\, \mathcal{O}_E((E^2)) \cong \mathcal{O}_C(\mathfrak{A})$, and we see that

$$\mathcal{O}_C(\mathfrak{A}) \subset R^1 p_{0,*} \mathcal{O}_{F_0} \simeq R^1 p_{0,*} \mathcal{O}_{F_0}(-E)$$

hence

$$\mathcal{O}_C \subset R^1 p_{0,*}\, \mathcal{O}_{F_0}(-H)$$

hence

$$H^1(C, \mathcal{O}(-H)) \neq (0).$$

Q.E.D.

IV

The last remark is an application of Kodaira's Vanishing Theorem to local algebra. It seems to me remarkable that such a global result should be useful to prove local statements about the non-existence of local rings, but this is the case. The question I want to study is that of the *smoothability of non-Cohen-Macauley surface singularities.* In other words, given a surface F, $P \in F$ a non-CM-singular point, when does there exist a flat family of surfaces F_t parametrized by $k[[t]]$ such that $F_0 = F$ while the generic F_t is smooth. More locally, the problem is:

Given a complete non-CM-purely† -2-dimensional local ring $\mathcal{O}$ without nilpotents, when does there exist a complete 3-dimensional local ring $\mathcal{O}'$ and a non-zero divisor $t \in \mathcal{O}'$ such that

a) $\mathcal{O} \simeq \mathcal{O}'/t\mathcal{O}'$

b) $\mathcal{O}_{\wp}$ regular for all prime ideals $\wp \subset \mathcal{O}$ with $t \notin \wp$.

† i.e. $\mathcal{O}/I$ is 2-dimensional for all minimal prime ideals of $I \subset \mathcal{O}$

First of all, let

$$\mathcal{O}^* = \bigoplus_{\substack{I \subset \mathcal{O} \\ \text{minimal} \\ \text{prime ideals}}} \begin{pmatrix} \text{integral closure of } \mathcal{O}/I \\ \text{in its fraction field} \end{pmatrix}$$

and let

$$\tilde{\mathcal{O}} = \left\{ a \in \mathcal{O}^* \,\middle|\, \begin{matrix} m^n a \subset \mathcal{O} \text{ for some } n \geqslant 1 \\ m = \text{maximal ideal in } \mathcal{O} \end{matrix} \right\}$$

$$= \Gamma\,(\text{Spec } \mathcal{O}\text{-closed pt., } \mathcal{O})$$

Note that $\tilde{\mathcal{O}}$ is a finite $\mathcal{O}$-module and $m^n . \tilde{\mathcal{O}} \subset \mathcal{O}$ for some large n, so that $\tilde{\mathcal{O}}/\mathcal{O}$ is an $\mathcal{O}$-module of finite length. Moreover, it is easy to see that $\tilde{\mathcal{O}}$ is a semi-local Cohen-Macauley ring. It has been proven by Rim [14] (cf. also Hartshorne [6], Theorem 2.1, for another proof) that:

$$\mathcal{O} \text{ smoothable} \Rightarrow \tilde{\mathcal{O}} \text{ local.}$$

The result we want to prove is:

THEOREM. *Assume* char $(\mathcal{O}/m) = 0$, Spec $\mathcal{O}$ *has an isolated singularity at its closed point and that* $\mathcal{O}$ *is smoothable, so that, by the remarks above,* $\tilde{\mathcal{O}}$ *is a normal local ring. Let* π: $X^* \to \text{Spec } \mathcal{O}$ *be a resolution and let*

$$p_a\,(\tilde{\mathcal{O}}) = l\,(R^1\pi_*\, \mathcal{O}_X)$$

be the genus of the singularity $\tilde{\mathcal{O}}$. *Then*

$$l\,(\tilde{\mathcal{O}}/\mathcal{O}) \leqslant p_a\,(\tilde{\mathcal{O}}).$$

Actually, for our applications, we want to know this result for rings $\mathcal{O}$ where Spec $\mathcal{O}$ has ordinary double curves too, with a suitable definition of p_a. We will treat this rather technical generalization in an appendix.

For example, the theorem shows:

COROLLARY. *Let* $\tilde{\mathcal{O}} = k[[x, y]]$, char $k = 0$. *Let* $I \subsetneq (x, y)$ *be an ideal of finite codimension. Then if* $\mathcal{O} = k + I$, $\mathcal{O}$ *is not smoothablé.*

On the other hand, if $F \subset \mathbf{P}^n$ is an elliptic ruled surface and $\mathcal{O}'$ is the completion of the local ring of the cone over F at its apex, then $\mathcal{O}'$ is a normal 3-dimensional ring which is not Cohen-Macauley. If $C = V(t) = (F.\ \mathbf{P}^{n-1})$ is a generic hyperplane section of F, then $t \in \mathcal{O}'$ and $\mathcal{O} = \mathcal{O}'/t\mathcal{O}'$ is the completion of the local ring of the cone over C at its apex. Now C is an elliptic curve, but embedded by an incomplete linear system — in fact, C is a projection of an elliptic curve $\widetilde{C}$ in $\mathbf{P}^n$ from a point not on $\widetilde{C}$ — this follows from the exact sequence:

$$0 \to H^0(\mathcal{O}_F) \to H^0(\mathcal{O}_F(1)) \to H^0(\mathcal{O}_C(1)) \to \underset{\substack{\| \\ \mathbf{C}}}{H^1(\mathcal{O}_F)} \to 0.$$

Let $\widetilde{\mathcal{O}}$ be the completion of the local ring of the cone over $\widetilde{C}$ at its apex. Then $\widetilde{\mathcal{O}}$ is a normal 2-dimensional ring, in fact an "elliptic singularity", i.e., $p_a(\widetilde{\mathcal{O}}) = 1$; moreover $\widetilde{\mathcal{O}} \supset \mathcal{O}$ and $\dim \widetilde{\mathcal{O}}/\mathcal{O} = 1$. This shows that there are smoothable singularities $\mathcal{O}$ with

$$l\,(\widetilde{\mathcal{O}}/\mathcal{O}) = p_a\,(\widetilde{\mathcal{O}}) = 1.$$

Proof of Theorem. Let $\mathcal{O} \simeq \mathcal{O}'/t\mathcal{O}'$ give the smoothing of $\mathcal{O}$. The proof is based on an examination of the exact sequence of local cohomology groups:

$$(*) \ \dots \longrightarrow H^1_{\{x\}}(\mathcal{O}') \longrightarrow H^1_{\{x\}}(\mathcal{O}) \longrightarrow H^2_{\{x\}}(\mathcal{O}') \xrightarrow{\ t\ } H^2_{\{x\}}(\mathcal{O}') \xrightarrow{\ \alpha\ } H^2_{\{x\}}(\mathcal{O}) \longrightarrow \dots$$

where $x \in \operatorname{Spec} \mathcal{O} \subset \operatorname{Spec} \mathcal{O}'$ represents the closed point.

What can we say about each of these groups ?

(a) $H^1_{\{x\}}(\mathcal{O}')$ is zero since $\mathcal{O}'$ is an integrally closed ring of dimension 3, hence has depth at least 2.

(b) To compute $H^1_{\{x\}}(\mathcal{O})$, use

$$H^0(\operatorname{Spec} \mathcal{O}, \mathcal{O}) \to H^0(\operatorname{Spec} \mathcal{O} - \{x\}, \mathcal{O}) \to H^1_{\{x\}}(\mathcal{O}) \to 0$$

which gives us:

$$H^1_{\{x\}}(\mathcal{O}) \simeq \widetilde{\mathcal{O}}/\mathcal{O}.$$

(c) As for $H^2_{\{x\}}(\mathcal{O}')$, it measures the degree to which $\mathcal{O}'$ is not Cohen-Macauley. A fundamental fact is that it is of finite length—cf. Theoreme de finitude, p. 89, in Grothendieck's seminar [15].

(d) As for $H^2_{\{x\}}(\mathcal{O})$, we can say at least:

$$\begin{aligned} H^2_{\{x\}}(\mathcal{O}) &\simeq H^1(\operatorname{Spec}\mathcal{O} - \{x\}, \mathcal{O}) \\ &\simeq H^1(\operatorname{Spec}\widetilde{\mathcal{O}} - \{x\}, \widetilde{\mathcal{O}}) \\ &\cong H^2_{\{x\}}(\widetilde{\mathcal{O}}) \end{aligned}$$

but unfortunately this group is huge: it is not even an $\widetilde{\mathcal{O}}$-module of finite type.

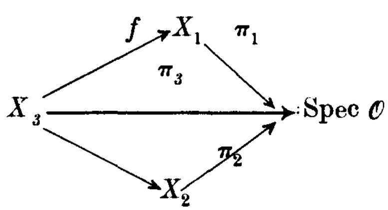
However, for any local ring $\mathcal{O}$ with residue characteristic O and with isolated singularity, we can define, by using a resolution of Spec $\mathcal{O}$, important subgroups:

$$H^i_{\{x\},\,\mathrm{int}}(\mathcal{O}).$$

Namely, let $\pi\colon X \to \operatorname{Spec}\mathcal{O}$ be a resolution and set

$$H^i_{\{x\},\,\mathrm{int}}(\mathcal{O}) = \operatorname{Ker}[\pi^* : H^i_{\{x\}}(\mathcal{O}) \to H^i_{\pi^{-1}x}(\mathcal{O}_X)].$$

This is independent of the resolution, as one sees by comparing any 2 resolutions $\pi_i : X_i \to \operatorname{Spec}\mathcal{O}$, $i = 1,2$, via a 3rd:



and using the Leray spectral sequence

$$H^p_{\pi_1^{-1}(x)}(X_1, R^q f_*(\mathcal{O}_{X_3})) \Rightarrow H^\bullet_{\pi_3^{-1}(x)}(X_3, \mathcal{O}_{X_3})$$

plus Matsumura's result $R^q f_* \mathcal{O}_{X_3} = (0)$, $q > 0$ when X_1 and X_3 are smooth and characteristic zero. Moreover, when $\mathcal{O} \simeq \mathcal{O}'/I$, then the restriction map

$$H^i_{\{x\}}(\mathcal{O}') \longrightarrow H^i_{\{x\}}(\mathcal{O})$$

gives

$$H^i_{\{x\}, \mathrm{int}}(\mathcal{O}') \longrightarrow H^i_{\{x\}, \mathrm{int}}(\mathcal{O})$$

because we can find resolutions fitting into a diagram:

$$\begin{array}{ccc} X & \hookrightarrow & X' \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{O} & \hookrightarrow & \operatorname{Spec} \mathcal{O}'. \end{array}$$

Next, we prove using the Kodaira Vanishing Theorem and following Hartshorne and Ogus ([16], p. 424) :

LEMMA. *Assume x is the only singularity of $\mathcal{O}$,* dim $\mathcal{O} = n$ *and* π: $X \to \operatorname{Spec} \mathcal{O}$ *is a resolution. Then*

$$H^i_{\{x\}, \mathrm{int}}(\mathcal{O}) \simeq H^i_{\{x\}}(\mathcal{O}), \ 0 \leqslant i \leqslant n-1$$

and

$$H^i_{\{x\}, \mathrm{int}}(\mathcal{O}) \simeq R^{i-1}\pi_*(\mathcal{O}_X)_x, \ 2 \leqslant i \leqslant n.$$

PROOF. Because $\mathcal{O}$ has an isolated singularity, we may assume $\mathcal{O} \simeq \hat{\mathcal{O}}_{x, X_0}$, where X_0 is an n-dimensional projective variety with x its only singular point. We may assume our resolution is global:

$$\pi : X \to X_0.$$

Let $\hat{X} = X \times_{X_0} \operatorname{Spec} \mathcal{O}$ and let I be the injective hull of $\mathcal{O}_{x, X_0}/m_{x, X_0}$ as $\mathcal{O}_{x, X_0}$-module. Then according to Hartshorne's formal duality theorem (cf. [7], p. 94), for all coherent sheaves $\mathcal{F}$ on X, the 2 $\mathcal{O}$-modules

$$H^i_{\pi^{-1}x}(\mathcal{F}), \ \operatorname{Ext}^{n-i}_{\mathcal{O}_{\hat{X}}}(\hat{\mathcal{F}}, \Omega^n_{\hat{X}})$$

are dual via Hom $(-, I)$. In particular,

$$H^i_{\pi^{-1}x}(\mathcal{O}_X), \ H^{n-i}(\Omega^n_{\hat{X}})$$

are dual. But

$$H^{n-i}(\Omega^n_{\hat{X}}) \simeq R^{n-i}\pi_*(\Omega^n_X) \otimes_{\mathcal{O}_{X_0}} \mathcal{O}$$

and it has been shown by Grauert and Riemenschneider [5] that $R_i\pi_*(\Omega^n_X) = (0)$, $i > 0$. (This is a simple consequence of Kodaira's Vanishing Theorem because if L_0 is an ample invertible sheaf on X_0

with $H^i(X_0, L_0 \otimes \pi_* \Omega^n_X) = (0)$, $i > 0$, then by the Leray Spectral Sequence:

$$H^i(X, \pi^* L_0 \otimes \Omega^n_X) \simeq H^0(X_0, L_0 \otimes R^i \pi_* \Omega^n_X)$$

$$\updownarrow \text{ dual}$$

$$H^{n-i}(X, (\pi^* L_0)^{-1})$$

and Kodaira's Vanishing Theorem applies to all invertible sheaves M such that $\Gamma(X, M^n)$ is base point free and defines a birational morphism, $n \gg 0$ (cf. [9]).) Recapitulating, this shows $H^{n-i}(\Omega^n_{\hat{X}}) = (0)$, $i < n$, hence $H^i_{\pi^{-1}x}(\mathcal{O}_X) = (0)$, $i < n$, hence $H^i_{\{x\}\text{int}}(\mathcal{O}) \xrightarrow{\approx} H^i_{\{x\}}(\mathcal{O})$ is an isomorphism.

To get the second set of isomorphisms, we use the Leray Spectral Sequence:

$$H^p_{\{x\}}(X_0, R^q \pi_* \mathcal{O}_X) \Rightarrow H^\bullet_{\pi^{-1}x}(X, \mathcal{O}_X).$$

The only non-zero terms occur for $p = 0$ or $q = 0$, so we get a long exact sequence

$$\ldots \to H^0_{\{x\}}(R^{i-1}\pi_* \mathcal{O}_x) \to H^i_{\{x\}}(\pi_* \mathcal{O}_x) \to H^i_{\pi^{-1}x}(\mathcal{O}_X) \to H^0_{\{x\}}(R^i \pi_* \mathcal{O}_x)$$
$$\to H^{i+1}_{\{x\}}(\pi_* \mathcal{O}_X) \to \ldots$$

Using the first part, plus the isomorphism:

$$\begin{aligned} H^i_{\{x\}}(\pi_* \mathcal{O}_X) &\simeq H^{i-1}(\text{Spec}\, \mathcal{O} - \{x\}, \pi_* \mathcal{O}_X) \\ &\simeq H^{i-1}(\text{Spec}\, \mathcal{O} - \{x\}, \mathcal{O}) \\ &\simeq H^i_{\{x\}}(\mathcal{O}), \; i \geq 2, \end{aligned}$$

we get the results. *Q.E.D.*

We now go back to the sequence (*). It gives us:

$$0 \longrightarrow \widetilde{\mathcal{O}}/\mathcal{O} \longrightarrow H^2_{\{x\},\text{int}}(\mathcal{O}') \xrightarrow{t} H^2_{\{x\},\text{int}}(\mathcal{O}') \longrightarrow R^1 \pi_*(\mathcal{O}_X)_x \longrightarrow \cdots$$

where $\pi: X \to \text{Spec}\, \mathcal{O}$ is a resolution. Therefore

$$\begin{aligned} l(\widetilde{\mathcal{O}}/\mathcal{O}) &= l(\ker \text{ of } t \text{ in } H^2_{\{x\},\text{int}}(\mathcal{O}')) \\ &= l(\text{Coker of } t \text{ in } H^2_{\{x\},\text{int}}(\mathcal{O}')) \\ &\leq l(R^1 \pi_*(\mathcal{O}_X)_x) = p_a(\widetilde{\mathcal{O}}). \end{aligned}$$ *Q.E.D.*

APPENDIX

The purpose of this appendix is to make a rather technical extension of the result in §IV, which seems to be better for use in applications. Let X be an affine surface, reduced, with at most ordinary double curves, plus one point $P \in X$ about which we know nothing. Let

$$\tilde{X} = \operatorname{Spec} \Gamma(X - P, \mathcal{O}_X)$$

so that we get

$$\pi_1 : \tilde{X} \to X,$$

an isomorphism outside P, everywhere a finite morphism, with $\tilde{X}$ Cohen-Macauley. Our goal is to show that in certain cases X is not smoothable near P, i.e. $\nexists$ an analytic family

$$f: X' \to \Delta = \text{disc in the } t\text{-plane},$$

where $f^{-1}(0) \approx$ (neighborhood of P in X), and $f^{-1}(t)$ is smooth, $t \neq 0$. (Here we work in the analytic setting rather than the formal one to be able below to take an exponential.) We know that a necessary condition for X' to exist is that $\pi_1^{-1}(P)$ is one point $\tilde{P}$, so henceforth we assume this too. Next blow up $\tilde{X}$, but only at $\tilde{P}$ and at centers lying over $\tilde{P}$: it is not hard to see that we arrive in this way at a birational proper morphism

$$\pi_2 : X^* \to \tilde{X}$$

such that X^* has at most ordinary double curves and pinch points (points like $z^2 = x^2y$), these pinch points moreover lying over $\tilde{P}$. Define

$$p_a(\mathcal{O}_{\tilde{P}}) = \dim_{\mathbf{C}}[R^1\pi_{2,*}(\mathcal{O}_{X^*})_{\tilde{P}}].$$

It is easy to verify that this number is independent of the choice of X^*. (However, this would *not* be true if $\tilde{X}$ had cuspidal lines—in this case, there is no bound on dim $R^1\pi_*$ as you blow up $\tilde{X}$ more and more!) We claim the following

THEOREM. *If X is smoothable near P, then* $l(\mathcal{O}_{\tilde{P}}/\mathcal{O}_P) \leqslant p_a(\mathcal{O}_{\tilde{P}})$.

PROOF. We follow the same plan as in the case where $\mathcal{O}$ has an isolated singularity, except that, for an arbitrary local ring $\mathcal{O}$, we set

$$H^i_{\{x\},\,\mathrm{int}}(\mathcal{O}) = \bigcup_{\substack{\text{modifications} \\ \pi: X \to \mathrm{Spec}\,\mathcal{O}, \\ \text{where} \\ X - \pi^{-1}(x) \xrightarrow{\approx} \mathrm{Spec}\,\mathcal{O} - \{x\}}} [\ker: H^i_{\{x\}}(\mathcal{O}) \to H^i_{\pi^{-1}(x)}(\mathcal{O}_X)]$$

The proof is then the same as before except that we cite only the following case of the lemma:

THEOREM (Boutot [17]): *Let* $\mathcal{O}$ *be a normal excellent local k-algebra, with residue field k, and* char $(k) = 0$. *Then*:

$$H^2_{\{x\},\mathrm{int}}(\mathcal{O}) = H^2_{\{x\}}(\mathcal{O}).$$

This result is a Corollary of Proposition 2.6, Chapter V [17]. Since char$(k) = 0$, we may disregard "red" in that Proposition and apply it to the values of the functor on the dual numbers. It tells us that there is a blow-up $\pi : X \to \mathrm{Spec}\,(\mathcal{O})$ concentrated at the origin such that

$$\mathrm{Pic}_{X/k}(k[\epsilon]/(\epsilon^2)) \longrightarrow \mathrm{Pic}_{\mathrm{Spec}(\mathcal{O})-\{x\}}(k[\epsilon]/(\epsilon^2))$$

is an isomorphism. In other words, in the sequence:

$$\longrightarrow H^1(X, \mathcal{O}_X) \xrightarrow{\alpha} H^1(X - \pi^{-1}(x), \mathcal{O}_X) \xrightarrow{\beta} H^2_{\pi^{-1}(x)}(\mathcal{O}_X) \longrightarrow$$

α is surjective, hence β is zero.

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