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An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg deVries equation and related non-linear equations

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A remarkable "dictionary" was discovered by I.M. Krichever [7], following suggestions in the work of Zaharov-Shabat [11], where they attempted to find a common formalism for the inverse scattering method of integrating certain non-linear partial differential equations. Subsequently, a characteristic p analog of this dictionary was discovered by V.G. Drinfeld, and a matrix analog was worked out by P. van Moerbeke and myself. Preceding this stage, a legion of authors have worked previously in the hyperelliptic-degree 2 operator case: much of this can be traced through the recent articles [3], [5] and [10]. It is not entirely inaccurate to say that initial insight behind this and related discoveries was the work of the first electronic computer! This lecture is a report on these 3 dictionaries with only a brief discussion of their applications. (see Added in proof)

This dictionary is a one-one correspondence between 2 types of superficially totally unrelated sorts of data: on one side of the dictionary, one has an algebraic curve, one or more points on it, and a vector bundle over it; on the other side, one has commutative subring of some big non-commutative ring of operators. This correspondence seems to me remarkable for many reasons. Firstly, it appears, as mentioned above, in at least 3 quite distinct cases. Secondly, it enables one, generally in terms of theta functions, to construct solutions both to equations formed from operators in these commutative subrings and to equations formed from flows in the space of all operators in the big non-commutative ring. Thirdly, it gives a new parametrization of the moduli space of the curves involved and/or their jacobians, vector bundle moduli spaces, etc.. We will discuss this in more detail below.

To make the idea precise, we state here the results in the simple case where the bundle is a line bundle, for all 3 types of operators:

(I) *Difference Operator case.* Let k be any field. Let $M^{d}_{\infty}(k)$ be the ring of finite difference operators over k, i.e., maps $A: \prod_{-\infty}^{+\infty} k \rightarrow \prod_{-\infty}^{+\infty} k$ given by

$$A(x)_n = \sum_{m=n+N_1}^{n+N_2} A_{nm} x_m, \qquad \text{all } n \in \boldsymbol{Z}.$$

If $[N_1, N_2]$ is the smallest interval such that $A_{nm}=0$ if $m-n \notin [N_1, N_2]$, we say $[N_1, N_2]$ is the support of A. If moreover $A_{n,n+N_1}\neq 0$ and $A_{n,n+N_2}\neq 0$ for all $n \in \boldsymbol{Z}$, we say that A is *properly bordered*. Then there is a natural bijection between sets of data as follows:

Data A. a) *X a complete curve over k (i.e., X reduced and irreducible, one-dimensional, proper over k).*

b) *$P, Q \in X$, smooth k-rational points,*

c) *$\mathscr{F}$ torsion-free rank 1 sheaf on X such that*

$$\chi(\mathscr{F})=0$$
$$h^1(\mathscr{F}(nP-nQ))=0, \qquad \text{all } n \in \boldsymbol{Z}.$$

Data B. *Commutative subrings $R\subset M_\infty^\Delta(k)$, with $k\subset R$ and such that $\exists\, A, B \in R$ which are properly bordered, with supports $[a_1, a_2]$, $[b_1, b_2]$ such that $(a_1, b_1)=1$, $(a_2, b_2)=1$ and $a_2b_1<a_1b_2$; two subrings $R_1, R_2\subset M_\infty^\Delta(k)$ being identified, however, if for some invertible element:*

$$\Lambda=(\lambda_n\delta_{nm}), \qquad \lambda_n \in k^*,$$

we have

$$R_1=\Lambda\circ R_2\circ \Lambda^{-1}.$$

(II) *Differential Operator case (Krichever).* Let k be any field of characteristic zero. Let $k[[t]][d/dt]$ be the ring of formal linear ordinary differential operators over k. Then there is a natural bijection between sets of data as follows:

Data A. a) *X a complete curve over k,*

b) *$P \in X$, smooth k-rational point, and an isomorphism*

$$T_{x,p}\cong k,$$

c) *$\mathscr{F}$ torsion-free rank 1 sheaf on X such that*

$$h^0(\mathscr{F})=h^1(\mathscr{F})=0.$$

Data B. *Commutative subrings $R\subset k[[t]][d/dt]$, with $k\subset R$ and such that $\exists A, B \in R$, operators of form:*

$$A=\left(\frac{d}{dt}\right)^{\alpha}+a_1(t)\left(\frac{d}{dt}\right)^{\alpha-1}+\cdots+a_\alpha(t)$$
$$B=\left(\frac{d}{dt}\right)^{\beta}+b_1(t)\left(\frac{d}{dt}\right)^{\beta-1}+\cdots+b_\beta(t)$$

with $(\alpha, \beta)=1$; two subrings $R_1, R_2\subset k[[t]][d/dt]$ being identified, however, if for some $u(t) \in k[[t]]$, $u(0)\neq 0$, we have

$$R_1 = u(t) \circ R_2 \circ u(t)^{-1}.$$

(III) *Field Operator case (Drinfeld).* Let k be any field, $\sigma \in \mathrm{Aut}\,(k)$ an automorphism of infinite order and let k_0 be the fixed field. Let $k\{\sigma\}$ be the ring of maps $A: k \to k$ of the form

$$A(x) = \sum_{i=0}^{N} a_i \sigma^i(x).$$

Then there is a natural bijection between sets of data as follows:

Data A. a) *X_0 a complete curve over k_0,*

b) *$P_0 \in X_0$ a smooth k_0-rational point,*

c) *$\mathscr{F}$ torsion-free rank 1 sheaf on $X \overset{\text{def}}{=} X_0 \times_{k_0} k$ such that $h^0(\mathscr{F}) = h^1(\mathscr{F}) = 0$,*

d) *an isomorphism:*

$$r: (1_{X_0} \times \sigma)^* \mathscr{F} \xrightarrow{\approx} \mathscr{F}(P_0 - P_1)$$

for some smooth point $P_1 \in X$, $P_1 \neq P_0$. (Here $1_{X_0} \times \sigma: X \to X$ is the map given by $1_{X_0}: X_0 \to X_0$ and σ: Spec $k \to$ Spec k.)

Data B. *Commutative subrings $R \subset k\{\sigma\}$, with $k_0 \subset R$ and such that $\exists\, A, B \in R$, operators of form*

$$A = a_n \sigma^n + \cdots + a_0, \qquad a_n \neq 0$$
$$B = b_m \sigma^m + \cdots + b_0, \qquad b_m \neq 0$$

with $(n, m) = 1$; two subrings $R_1, R_2 \subset k\{\sigma\}$ being identified, however, if for some $a \in k^*$, we have:

$$R_1 = a \cdot R_2 \cdot a^{-1}.$$

§ 1. Difference operator case

Let me first explain, in the rank 1 case, how one goes from Data A to Data B. This construction will give the essence of everything that follows and we can sketch the generalizations fairly rapidly. We consider the infinite sequence of sheaves:

$$\begin{tikzcd}[column sep=tiny]
 & \mathscr{F}(2Q-P) & & \mathscr{F}(Q) & & \mathscr{F}(P) & & \mathscr{F}(2P-Q) & & \mathscr{F}(3P-2Q) & \\
\cdots \arrow[ur, hook] & & \mathscr{F}(Q-P) \arrow[ul, hook] \arrow[ur, hook] & & \mathscr{F} \arrow[ul, hook] \arrow[ur, hook] & & \mathscr{F}(P-Q) \arrow[ul, hook] \arrow[ur, hook] & & \mathscr{F}(2P-2Q) \arrow[ul, hook] \arrow[ur, hook] & & \cdots \arrow[ul, hook]
\end{tikzcd}.$$

Those on the bottom row have no H^0 or H^1. Therefore those on the top row have a 1-dimensional H^0, and if

$$s_n \in H^0(X, \mathscr{F}((n+1)P - nQ))$$

is a non-zero section, s_n generates 1-dimensional vector spaces:

$$\mathscr{F}((n+1)P-nQ)/\mathscr{F}(nP-nQ)\cong\mathscr{F}((n+1)P-nQ)\otimes_{\mathcal{O}_X}K(P)$$

and

$$\mathscr{F}((n+1)P-nQ)/\mathscr{F}((n+1)P-(n+1)Q)\cong\mathscr{F}((n+1)P-nQ)\otimes_{\mathcal{O}_X}K(Q).$$

The first follows from the sequence:

$$\begin{aligned}0\to H^0(\mathscr{F}(nP-nQ))\to H^0(\mathscr{F}((n+1)P-nQ))&\\ \to H^0(\mathscr{F}((n+1)P-nQ))\otimes K(P)\to H^1(\mathscr{F}(nP-nQ))&\end{aligned}$$

and the second from the similar sequence with $K(Q)$. As a result, it follows that $\{s_n\}_{n\in Z}$ is a k-basis of the infinite-dimensional vector space $M=\Gamma(X-P-Q,\mathscr{F})$. In fact, starting with any $s\in\Gamma(X-P-Q,\mathscr{F})$, let k,l be least so that s extends to

$$s\in\Gamma(X,\mathscr{F}(kP+lQ)).$$

Then for a suitable $a\in k$, as_{k-1} and s will have the same pole at P, i.e.,

$$s-as_{k-1}\in\Gamma(X,\mathscr{F}((k-1)P+l(Q))).$$

Similarly, for suitable $b\in k$, bs_{-l} and s will have the same pole at Q, i.e.,

$$s-as_{k-1}-bs_{-l}\in\Gamma(X,\mathscr{F}((k-1)P+(l-1)Q)).$$

Continuing in this way, we eventually find a section of $\mathscr{F}$. Since $H^0(\mathscr{F})=(0)$, this is zero and s is written as a combination of the $\{s_n\}$. Now let

$$R=\Gamma(X-P-Q,\mathcal{O}_X).$$

Clearly M is an R-module, so for all $a\in R$, $n\in Z$, we can write:

$$a\cdot s_n=\sum_{m=n-N_1}^{n+N_2}A_{nm}s_m.$$

In fact, it is easy to see that N_1,N_2 may be taken to be the order of poles of a at P and Q:

$$(a)=N_1Q+N_2P-D,\qquad\text{some } D\geq 0 \text{ supported on } X-P-Q,$$

and that in this case $A_{n,n-N_1}\neq 0$ and $A_{n,n+N_2}\neq 0$, all n. (If a had a zero at either P or Q, N_1 or N_2 can be taken to be negative and the matrix A is upper or lower triangular.) Now consider the map:

$$\begin{aligned}R&\longrightarrow M_\infty^\Delta(k)\\ a&\longmapsto A.\end{aligned}$$

We may check that it is a homomorphism as follows:

$$\begin{aligned}(ab)s_n &= b\Big(\sum_k A_{nk}s_k\Big)\\ &= \sum_k A_{nk}(bs_k)\\ &= \sum_k A_{nk}\Big(\sum B_{km}s_m\Big)\\ &= \sum_m \Big(\sum_k A_{nk}B_{km}\Big)s_m.\end{aligned}$$

Note that the only choice we made in defining this map was that of the $\{s_n\}$. If s_n is replaced by $\lambda_n s_n$, $\lambda_n \in k^*$, then the matrix A is replaced by $(A')_{nm}=\lambda_n\lambda_m^{-1}A_{nm}$, i.e., $A'=\Lambda A\Lambda^{-1}$. Finally, for all N_1, N_2 sufficiently large, there are functions $a \in R$ with poles at P and Q of order exactly N_2, N_1: so the image of R in $M_\infty^\Delta(k)$ has the properties required in Data B. Incidentally, if X is smooth, it is an arduous task, but not deep, to give *explicit* formulae for the entries A_{nm} in terms of theta functions associated to X. This can be done following the methods of Fay [4].

The spectral properties of the rings R which we get in this way are very simple and help to understand how to reconstruct $(X, P, Q, \mathscr{F})$ from R. Since all the operators $A \in R$ commute, you can expect to find simultaneous eigenvectors $\vec{x}$ for all $A \in R$, at least over suitable extension fields $K \supset k$. We put no convergence restriction on $\vec{x}$, but seek vectors $\vec{x} \in \prod_{-\infty}^{+\infty} K$ for some field $K \supset k$, such that

$$A\vec{x}=\lambda_A\cdot\vec{x}, \qquad \text{all } A \in R.$$

In this case, the eigenvalues λ_A together give a homomorphism $\lambda\colon R \to K$, hence define a K-valued point of $X-P-Q$. The following holds:

Proposition. *Let* Data $\{X, P, Q, R\}$ *define* $R \subset M_\infty^\Delta(k)$ *as above. Let* $K \supset k$ *be a field,* $\lambda\colon R \to K$ *be a K-valued point of* $X-P-Q$, *lying over* $x \in X-P-Q$ (x *defined by the prime ideal* Ker λ). *Then there is an isomorphism between*

a) *the eigenspace* $\{\vec{x} \in \prod_{-\infty}^{+\infty} K \mid A\vec{x}=\lambda_A\cdot\vec{x},\ \text{all } A \in R\}$

and

b) $\mathrm{Hom}_R(\mathscr{F}_x/m_x\mathscr{F}_x, K)$. (*Here K is an R-module via* λ).

Proof. In fact

$$\mathrm{Hom}_R(\mathscr{F}_x/m_x\mathscr{F}_x, K) \cong \mathrm{Hom}_R(\Gamma(X-P-Q, \mathscr{F}), K).$$

Using the basis $\{s_n\}$ of $\Gamma(X-P-Q, \mathscr{F})$, this comes out as:

$$\cong \left\{\begin{array}{l}\text{maps } s_n \mapsto x_n \in K \text{ such that for all } a \in R\\ \text{if } as_n=\sum A_{nm}s_m, \text{ then}\\ \qquad\qquad \sum A_{nm}x_m=\lambda(a)\cdot x_n\end{array}\right\}$$

$$=\text{Eigenspace for eigenvalue } \lambda.$$

This suggests how to go back from Data B to Data A. The idea is simply to take $X-P-Q$ to be Spec R and to complete it to X. For each point of $X-P-Q$, we consider the corresponding eigenspace, and "glue" these together into a bundle over $X-P-Q$. Then $\mathscr{F}$ is just the sheaf of functions on this bundle, linear on each fibre (e.g., generated by the functions $x \mapsto x_n$). If $k=C$, the 2 points at infinity on X have a spectral meaning in that their neighborhoods are given by the set of all eigenfunctions growing exponentially as $n \to +\infty$ or $\to -\infty$: $|x_n| \geq C|x_{n-1}|$ or $|x_{n+1}| \geq C|x_n|$, for C getting larger and larger. Over any k, the 2 points at infinity correspond to the 2 valuations on R given by assigning to each matrix A the least integers N_1, N_2 such that A is supported on $[-N_1, N_2]$.

It seems difficult to make the above rigorous by a direct attack. We take a much more algebraic approach as follows: define 2 filtrations on R:

$$R_n = \{A \in R \mid A \text{ supported on } (-\infty, n]\}$$
$$R^m = \{A \in R \mid A \text{ supported on } [-m, +\infty)\}.$$

Using the 2 given elements $A, B \in R$ where

$$A \in R_{a_2} \cap R^{a_1}$$
$$B \in R_{b_2} \cap R^{b_1}$$

and $(a_2, b_2)=1$, $(a_1, b_1)=1$, one proves:

Lemma. i) *Every $C \in R$ is properly bordered,*
ii) *for all n, $\dim R_{n+1}/R_n \leq 1$, equality if $n \gg 0$,*
iii) *for all n, $\dim R^{n+1}/R^n \leq 1$, equality if $n \gg 0$.*

To prove this, simply note that if C has support $[-c_1, c_2]$ then

$$AC = CA \Rightarrow a_{n,n+a_2} c_{n+a_2, n+a_2+c_2} = c_{n,n+c_2} \cdot a_{n+c_2, n+c_2+a_2}$$
$$\Rightarrow (c_{n,n+c_2}=0 \text{ iff } c_{n+a_2, n+a_2+c_2}=0).$$

So

$$\left.\begin{matrix} AC=CA \\ BC=CA \end{matrix}\right\} \Rightarrow (c_{n,n+c_2}=0 \text{ for one } n \text{ iff } c_{n,n+c_2}=0 \text{ for all } n)$$

hence C is properly bordered. Now if C, C' have support $[*, c_2]$, then some combination $\alpha C + \beta C'$ has one zero along $c_{n,n+c_2}$, hence it has support $[*, c_2-1]$. A similar argument appllies to the bottom border. Finally, monomials $A^i B^j$ give us all supports $[*, n]$, $n \geq n_0$ and $[-m, *]$, $m \geq m_0$.

Corollary. *R is an integral domain, the subring of R generated by A and B is isomorphic to $k[X, Y]/(f)$, f irreducible, and R mod this ring is finite-dimensional over k. In particular, R is a finitely generated k-algebra.*

Proof. The fact that every $C \in R$ is properly bordered shows R is a domain. The lemma shows $\dim R_n \cap R^m \leq n+m+1$, and a simple count of the set of monomials $A^i B^j$ in $R_n \cap R^n$ $n \gg 0$, shows their number grows like n^2: so A and B satisfy some identity. Finally, using the inequality $a_2 b_1 > a_1 b_2$, choose positive integers λ, μ such that

$$\frac{a_2}{a_1} > \frac{\lambda}{\mu} > \frac{b_2}{b_1}.$$

Using $(a_2, b_2) = (a_1, b_1) = 1$, one finds monomials $A^i B^j$ with $i \gg 0$, $0 \leq j < a_2$ in

$$(R_{\lambda n+k+1} \cap R^{\mu n}) - (R_{\lambda n+k} \cap R^{\mu n})$$

for all k, $0 \leq k \leq \lambda - 1$, $n \gg 0$, and likewise, taking $0 \leq i < b_1$, $j \gg 0$, in

$$(R_{\lambda(n+1)} \cap R^{\mu n+k+1}) - (R_{\lambda(n+1)} \cap R^{\mu n+k})$$

for all k, $0 \leq k \leq \mu - 1$, $n \gg 0$. Thus these monomials plus the subspace $R_{\lambda n_0} \cap R^{\mu n_0}$ span R as a k-vector space. Q.E.D.

We can now define $X - P - Q$ to be $\operatorname{Spec} R$. To define the whole of X, the most convenient way seems to be as Proj of a *graded* ring. As in the proof of the Corollary, fix $\lambda, \mu \geq 1$ such that

$$\frac{a_2 - 1}{a_1} \geq \frac{\lambda}{\mu} \geq \frac{b_2}{b_1 - 1},$$

and define

$$\mathscr{R}_n = R_{\lambda n} \cap R^{\mu n}$$

$$\mathscr{R} = \bigoplus_{n=0}^{\infty} \mathscr{R}_n.$$

In particular $\mathscr{R}$ contains

a) the element 1 in $\mathscr{R}_1 = R_\lambda \cap R^\mu$: we call this e,

b) $A^\lambda \in \mathscr{R}_{a_1}$,

c) $B^\mu \in \mathscr{R}_{b_1}$.

An argument like that above shows that $\mathscr{R}$ is generated, as a module over $k[e, A^\lambda, B^\mu]$, by a subspace $\mathscr{R}_0 \oplus \cdots \oplus \mathscr{R}_{n_0}$, $n_0 \gg 0$. Thus $\mathscr{R}$ is also a finitely generated domain over k. Define

$$X = \operatorname{Proj}(\mathscr{R}).$$

X contains the affine piece $e \neq 0$, which is, by definition:

$$\operatorname{Spec}\left(\mathscr{R}\left[\frac{1}{e}\right]\right)_0$$

($_0$ signifies the degree 0 component), and

$$\mathscr{R}\left[\frac{1}{e}\right]_0 = \varinjlim_{\text{mult by } e} (R_{\lambda n} \cap R^{\mu n}) = R.$$

To see what we have put at infinity, note that X is covered by the 3 affine pieces $e \neq 0$, $A^\lambda \neq 0$ and $B^\mu \neq 0$. Since $A^\lambda B^\mu \in R_{\lambda(a_2+b_1-1)} \cap R^{\mu(a_2+b_1-1)}$, we get *in* $\mathscr{R}$:

$$A^\lambda \cdot B^\mu = e \cdot C, \qquad C \in \mathscr{R}_{a_2+b_1-1}.$$

Thus outside the affine $e \neq 0$, X has points $e = A^\lambda = 0$, $B^\mu \neq 0$ and points $e = B^\mu = 0$, $A^\lambda \neq 0$. I claim there is exactly one of each, and that it is a smooth k-rational point. To see this, check first that the direct systems

$$\begin{array}{ccccccc}
\mathscr{R}_0 & \longrightarrow & \mathscr{R}_{a_2} & \xrightarrow{A^\lambda} \cdots \longrightarrow & \mathscr{R}_{ka_2} & \xrightarrow{A^\lambda} & \cdots \\
\| & & \| & & \| & & \\
R_0 \cap R^0 & & R_{\lambda a_2} \cap R^{\mu a_2} & & R_{k\lambda a_2} \cap R^{k\mu a_2} & & \\
\cap & & \cap & & \cap & & \\
R_0 & \longrightarrow & R_{\lambda a_2} & \xrightarrow{A^\lambda} \cdots \longrightarrow & R_{k\lambda a_2} & \xrightarrow{A^\lambda} & \cdots
\end{array}$$

have the same direct limit, so the affine ring of $A^\lambda \neq 0$ is:

$$\mathscr{R}\left[\frac{1}{A^\lambda}\right]_0 = \varinjlim_n (R_{\lambda n a_2}, \text{ mult. by } A^\lambda) = \begin{pmatrix}\text{ring of fractions} \\ C/A^k,\ C \in R_{ka_2}\end{pmatrix}.$$

In this ring, the homogeneous ideal (e) defines the ideal of elements C/A^k, $C \in R_{ka_2-\lambda}$. Choose positive integers σ, τ such that

$$\sigma a_2 + \tau b_2 = ka_2 - 1$$

and set $C = A^\sigma B^\tau$. It follows that for all n,

$$\mathscr{R}\left[\frac{1}{A^\lambda}\right]_0 \cong k \cdot 1 \oplus k\left(\frac{C}{A^k}\right) \oplus \cdots \oplus k\left(\frac{C}{A^k}\right)^{n-1} \oplus \begin{Bmatrix}\text{ideal } C/A^k \\ C \in B_{ka_2-n}\end{Bmatrix},$$

thus

$$\left\{\text{Completion of } \mathscr{R}\left[\frac{1}{A^\lambda}\right]_0 \text{ in the } e\text{-adic topology}\right\} \cong k\left[\left[\frac{C}{A^k}\right]\right]$$

which proves our claim for the points $e = B^\mu = 0$. The other case is similar. Let P be the point $e = B^\mu = 0$ and let Q be the point $e = A^\lambda = 0$. Note that e vanishes to order λ at P and μ at Q. Thus

$$\mathcal{O}_X(1)\cong\mathcal{O}_X((e))=\mathcal{O}_X(\lambda P+\mu Q).$$

Incidentally, describing $X-P-Q=\operatorname{Spec} R$, then the valuations $f\mapsto \operatorname{ord}_P f$ and $f\mapsto \operatorname{ord}_Q f$, for $f\in R$, are easily seen to be just the upper and lower limits of support of f. Note that the ideal of P is

$$\bigoplus_{n=0}^{\infty} R_{\lambda n-1}\cap R^{\mu n}.$$

To get the the sheaf $\mathcal{F}$ on X, let M be the vector space of column vectors and consider it as a module over R. Filter it like R:

$$M_n=\{(a_i)\mid a_i=0,\ i>n\}$$
$$M^n=\{(a_i)\mid a_i=0,\ i<-n\}.$$

Introduce the graded $\mathcal{R}$-module:

$$\mathfrak{M}_n=M_{\lambda n}\cap M^{\mu n-1}$$
$$\mathfrak{M}=\bigoplus_{n=0}^{\infty}\mathfrak{M}_n.$$

One checks immediately that if $n\geq\max(a_2+1, b_1+1)$, then:

$$\mathfrak{M}_n=e\cdot\mathfrak{M}_{n-1}+A^{\lambda}\cdot\mathfrak{M}_{n-a_2}+B^{\mu}\cdot\mathfrak{M}_{n-b_1}.$$

It works like this:

$$\begin{array}{ccc}
 & [-\mu(n-1)+1, \cdots\cdots\cdots\cdots, \lambda(n-1)] & \\
 & \big\downarrow{\times e} & \\
[-\mu n+\underbrace{1, \cdots, \overbrace{\mu(n-1)+1, \cdots}}_{} & & \underbrace{\overbrace{\cdots, \lambda(n-1)}, \cdots, \lambda n}_{}] \\
\big\uparrow{\times B^{\mu}} & & \big\uparrow{\times A^{\lambda}} \\
[-\mu(n-b_1)+1, \cdots, 0] & & [0, \cdots, \lambda(n-a_2)]
\end{array}$$

Define $\mathcal{F}$ to be $\tilde{\mathfrak{M}}$. Since $\dim\mathfrak{M}_n=(\lambda+\mu)n$, $n\geq 0$, it follows that the Hilbert polynomial $\chi(\mathcal{F}(n))$ is $(\lambda+\mu)n$ for all n. In particular, $rk\mathcal{F}=1$ and $\chi(\mathcal{F})=0$. Finally, we may define related sheaves by:

$$\mathfrak{M}_n^{(a,b)}=M_{\lambda n+a}\cap M^{\mu n+b-1}$$
$$\mathfrak{M}^{(a,b)}=\bigoplus\mathfrak{M}_n^{(a,b)}$$
$$\mathcal{F}^{(a,b)}=\tilde{\mathfrak{M}}^{(a,b)}.$$

Since $\mathfrak{M}^{(a,b)}\subset\mathfrak{M}^{(a+1,b)}$, $\mathfrak{M}^{(a,b)}\subset\mathfrak{M}^{(a,b+1)}$, we get $\mathcal{F}^{(a,b)}\subset\mathcal{F}^{(a+1,b)}$, $\mathcal{F}^{(a,b)}\subset\mathcal{F}^{(a,b+1)}$, and it is easy to check that

$$\begin{array}{ccc} (\mathfrak{M}^{(a+1,b)}/\mathfrak{M}^{(a,b)})_n & \underset{n \text{ large}}{\cong} & M_{\lambda n+a+1}/M_{\lambda n+a} \\ & & \approx \Big\uparrow \text{mult. by } e_{a+1} \\ (\mathscr{R}/\text{ideal of } P)_n & \cong & R_{\lambda n}/R_{\lambda n-1} \end{array}$$

(where $e_k \in M$ is the k^{th} unit column vector). Therefore

$$\mathscr{F}^{(a+1,b)} \cong \mathscr{F}^{(a,b)}(P)$$

and by induction:

$$\mathscr{F}^{(a,b)} \cong \mathscr{F}(aP+bQ).$$

Moreover, if $a \geq -b$, then $e_{a+1} \in M_{a+1} \cap M^{b-1}$, hence

$$e_{a+1} \in \Gamma(X, \mathscr{F}^{(a+1,b)})$$

is a section that doesn't vanish at P. Then using the exact sequence

$$0 \rightarrow \mathscr{F}^{(a,b)} \rightarrow \mathscr{F}^{(a+1,b)} \rightarrow \boldsymbol{K}(P) \rightarrow 0$$

and the existence of the section e_{a+1} of $\mathscr{F}^{(a+1,b)}$, we find

$$H^1(\mathscr{F}^{(a,b)}) \xrightarrow{\sim} H^1(\mathscr{F}^{(a+1,b)}).$$

But if $a+b$ is large enough, H^1 is zero. So H^1 is zero whenever $a+b \geq 0$, i.e.,

$$H^1(X, \mathscr{F}(aP-aQ)) = (0), \qquad \text{all } a \in \boldsymbol{Z}.$$

This completes the construction of Data A. We leave it to the reader to verify that our maps between Data A and Data B are inverse to each other.

The dictionary can be greatly extended. Here is one much more general correspondence:

Data A′. a) *X a one-dimensional scheme, proper over k, without embedded components. Let*

$$\boldsymbol{R} = \bigoplus_{\substack{\eta \in X \\ \text{generic}}} \mathcal{O}_{\eta, X}$$

be its total ring of fractions,

b) *$S, T \subset X$ disjoint finite closed subsets meeting every component of X. Let*

$$\begin{aligned} \mathcal{O}_S &= \{f \in \boldsymbol{R} \mid f \in \mathcal{O}_{x,X}, \quad \text{all } x \in S\} \\ \mathcal{O}_T &= \{f \in \boldsymbol{R} \mid f \in \mathcal{O}_{x,X}, \quad \text{all } x \in T\}, \end{aligned}$$

c) *$\mathscr{F}$ a coherent sheaf on X such that $\chi(\mathscr{F}) = 0$ and $\mathscr{F}$ has no zero-dimensional associated points,*

d) *a flag of $\mathcal{O}_S$-modules*

$$\mathcal{F}_S = K_0 \supset K_1 \supset K_2 \supset \cdots \supset K_\alpha = f \cdot K_0,$$

$f \in \mathcal{O}_S$ a non-zero divisor zero at every $x \in S$, $\dim_k (K_l/K_{l+1}) = 1$, and a flag of $\mathcal{O}_T$-modules:

$$\mathcal{F}_T = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_\beta = g \cdot L_0$$

$g \in \mathcal{O}_T$ a non-zero divisor zero at every $x \in T$, $\dim_k (L_l/L_{l+1}) = 1$.

We put 2 requirements on this: first

$$\mathcal{O}_S = \{a \in \boldsymbol{R} \mid aK_l \subset K_l, 0 \leq l \leq \alpha\}$$
$$\mathcal{O}_T = \{a \in \boldsymbol{R} \mid aL_l \subset L_l, 0 \leq l \leq \beta\}.$$

Secondly, if we define K_l, L_l for all $l \in \boldsymbol{Z}$ by

$$K_{k+\alpha} = f \cdot K_k, \; L_{l+\beta} = g \cdot L_l$$

and sheaves $\mathcal{F}^{(k,l)}$ by

$$\mathcal{F}^{(k,l)} = \begin{cases} K_k \text{ at } S \\ L_l \text{ at } T \\ \mathcal{F} \text{ elsewhere} \end{cases}$$

then

$$h^0(\mathcal{F}^{(k,-k)}) = 0.$$

Data B′. *Commutative subrings $R \subset M^d_\infty(k)$ with $k \subset R$ such that $\exists\, A, B \in R$ where A is properly bordered above with support $[a_1, a_2]$ and B is properly bordered below with support $[b_1, b_2]$ and $a_2 b_1 < a_1 b_2$; two subrings R_1, R_2 being identified if*

$$R_1 = \Lambda \circ R_2 \circ \Lambda^{-1}$$

(Λ diagonal) as before.

To go from A′ to B′, as before we just choose

$$s_n \in H^0(X, \mathcal{F}^{(n+1,-n)})$$

and verify that $\{s_n\}$ is a k-basis of $H^0(X-S-T, \mathcal{F})$. Hence $R = H^0(X-S-T, \mathcal{O}_X)$ acts as a ring of matrices on the $\{s_n\}$ and this is data B'.

To go from B' to A', define $R_n, R^n, \mathcal{R}$ and $M_n, M^n, \mathfrak{M}^{(a,b)}$ as before. Instead of the lemma above, the argument only shows:

$$\dim R_{n+1}/R_n \leq a_2, \qquad \text{all } n$$
$$\dim R^{n+1}/R^n \leq b_1, \qquad \text{all } n.$$

One then proves as before that R is a finite module over $k[A, B]$ and that A, B satisfy some non-zero identity. Similarly $\mathscr{R}$ and $\mathfrak{M}^{(a,b)}$ are finite modules over $k[e, A^\lambda, B^\mu]$. Set $X = \operatorname{Proj} \mathscr{R}$, $\mathscr{F}^{(a,b)} = \widetilde{\mathfrak{M}}^{(a,b)}$. As before, the open set $e \neq 0$ in X is just $\operatorname{Spec} R$. Moreover we have

$$A^\lambda \cdot B^\mu = e \cdot C, \qquad C \in \mathscr{R}_{a_2+b_1-1}$$

so the divisor $e=0$ breaks up into S defined by $e=B^\mu=0$ and T defined by $e=A^\lambda=0$.

The open set $A^\lambda \neq 0$ is Spec of the ring

$$R^{(A)} = \varinjlim_n (R_{\lambda n a_2}, \text{ mult by } A^\lambda) = \left\{\begin{matrix}\text{ring of fractions} \\ C/A^k, \ C \in R_{k a_2}\end{matrix}\right\},$$

and $\mathscr{F}^{(a,b)}$ on this open set comes from the module:

$$M_a^{(A)} = \varinjlim_n (M_{\lambda n a_2 + a}, \text{ mult by } A^\lambda) = \left\{\begin{matrix}\text{module of fractions} \\ m/A^k, \ m \in M_{k a_2 + a}\end{matrix}\right\}.$$

These are independent of b, have isomorphic localizations on the complement of S (the set S is defined in this open piece by $e=0$, which becomes $1/A \neq 0$ in this affine ring). Therefore these define a flag $\{K_l\}$ as required. Moreover

$$\frac{1}{A} \cdot \left\{\begin{matrix}\text{module of} \\ m/A^k, \ m \in M_{k a_2 + a}\end{matrix}\right\} = \left\{\begin{matrix}\text{module of} \\ m/A^k, \ m \in M_{k a_2 + a - a_2}\end{matrix}\right\}$$

so the α in Data A' is a_2 and the f is the function defined by $1/A$ in the above affine ring. $\{K_l\}$, β and g are defined similarly. The calculation of χ and vanishing of h^0's goes through as in the special case. Note that $X-S-T$ is the affine $e=0$, so each component X_i of X meets either S or T. But if, for instance, only S met X_i, then $\chi(\mathscr{F}^{(k,-k)}|_{X_i})$ would go to $+\infty$ as $k \to \infty$, so $h^0(\mathscr{F}^{(k,-k)}) \to \infty$ which is not so. Thus both S and T meet all X_i. Finally, note that if $C \in R$, then $C \in R_l$ iff $C \cdot (M)_a \subset M_{a+l}$, all a; hence:

$$R^{(A)} = \left\{ f \in R\left[\frac{1}{A}\right] \,\middle|\, f \cdot M_a^{(A)} \subset M_a^{(A)}, \text{ all } a \right\};$$

from which the requirement

$$\mathcal{O}_S = \{ f \in \boldsymbol{R} \mid f \cdot K_l \subset K_l, \text{ all } l \}$$

follows directly.

There is one particularly nice case of the dictionary. This relates to arbitrary *periodic* difference operators A: we say A has *period* n if

$$A_{k+n, l+n} = A_{k,l}, \qquad \text{all } k, l \in \boldsymbol{Z}.$$

If S is the shift operator:

$$S_{k,l}=\delta_{k,l+1},$$

then to say A has period n is equivalent to saying

$$A\cdot S^n=S^n\cdot A.$$

Therefore, if A is any properly bordered periodic difference operator, with support $[-a_1, a_2]$, such that $(n, a_1)=(n, a_2)=0$, the ring $R=k[A, S^n, S^{-n}]$ is an example of Data B. It is easy to see that the corresponding curves X are those such that

a) $nP\equiv nQ$

b) there is a function f on X with poles a_1Q+a_2P.

Another interesting case is when $k=\boldsymbol{C}$ and we strengthen the hypothesis $h^1(\mathscr{F}(nP-nQ))=0$. Suppose:

a) *X is a smooth curve of genus g.*

b) *$P, Q\in X$. Let $\phi: X\to \mathrm{Pic}^1(X)$ be the canonical map and let $\alpha=\phi(P)-\phi(Q)\in \mathrm{Pic}^0(X)$.*

c) *$\mathscr{F}$ is an invertible sheaf on X of degree $g-1$ defining a point $[\mathscr{F}]\in \mathrm{Pic}^{g-1}(X)$.*

Let

$$\Sigma=\overline{\{[\mathscr{F}]+n\alpha\}_{n\in Z}}\subset \mathrm{Pic}^{g-1}(X)$$

where $\overline{}$ denotes closure in the complex topology. Let $\Theta\subset \mathrm{Pic}^{g-1}(X)$ be the theta divisor, i.e., the set of divisor classes with $h^1>0$. Then assume

$$\Sigma\cap\Theta=\phi.$$

As we have remarked, periodic matrices arise when Σ is finite. However, whenever $\Sigma\cap\Theta=\phi$, I claim that the matrices $A\in M^{\Delta}_{\infty}(\boldsymbol{C})$ that arise will be almost periodic. By definition this means that:

$$\begin{aligned}&\forall\varepsilon>0, \exists N_1<N_2<N_3<\cdots \quad \text{such that}\\ &|A_{k,l}-A_{k+N_i,l+N_i}|<\varepsilon, \quad \text{all } k,l\\ &\text{and } N_{i+1}-N_i \text{ are bounded.}\end{aligned}$$

The most interesting aspect of the dictionary, however, is to analyze what it does to the *Jacobian flows*. Again take $k=\boldsymbol{C}$ and consider the general Data A′, B′ correspondence. The Jacobian variety $\mathrm{Pic}^0(X)$ acts on Data A′ by

$$\mathscr{F}\longmapsto\mathscr{F}\otimes L$$

L an invertible sheaf on X, at least "generically", i.e., for most L, $\mathscr{F}\otimes L$ will still satisfy the vanishing hypothesis. This means that the tangent space to $\mathrm{Pic}^0(X)$, which is canonically $H^1(\mathcal{O}_X)$, defines a vector space of commuting vector fields on the manifold of Data A′, although, when you integrate these into flows, they may be incomplete. It is very interesting to express these vector fields in terms of Data B′. The result is this: let $R \subset M_\infty^\Delta(C)$ be an instance of Data B′. Regard R as a fixed abstract ring, but consider deformations of its embedding in $M_\infty^\Delta(C)$:

$$\phi_t : R \mapsto M_\infty^\Delta(C).$$

Fix one element $C \in R$. For any $X \in M_\infty^\Delta(C)$, let

$$(X_+)_{ij} = \begin{cases} X_{ij}, & i<j \\ 0, & i \geq j \end{cases}; \quad (X_0)_{ij} = \begin{cases} X_{ii}, & i=j \\ 0, & i \neq j \end{cases}; \quad (X_-)_{ij} = \begin{cases} 0, & i \leq j \\ X_{ij}, & i>j \end{cases}.$$

Then the flows in Data B′ are defined by the differential equations:

$$\frac{d}{dt}\phi_t(A) = [\phi_t(C)_+, \phi_t(A)].$$

This is not hard to prove:

a) describe $H^1(\mathcal{O}_X)$ by Cech co-cycles via the covering $\mathcal{U} = \{X-S, X-T\}$, giving:

$$\begin{aligned} R &= \Gamma((X-S) \cap (X-T), \mathcal{O}_X) \\ &= Z^1(\mathcal{U}\,;\, \mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_X). \end{aligned}$$

Corresponding to $c \in R$, we get the tangent vector to $\mathrm{Pic}^0(X)$ described by the invertible sheaf L on $X \times_k k[\varepsilon]$:

$$L \cong \begin{cases} \mathcal{O}_X \otimes_k k[\varepsilon] \text{ on } X-S,\ X-T \\ \text{glued by mult. by } 1+\varepsilon c. \end{cases}$$

b) Let $\mathcal{F}^{(k,l)}$, $s_k \in \Gamma(\mathcal{F}^{(k+1,-k)})$ define $\phi_0 : R \to M_\infty^\Delta(k)$. Deform these to

$$\begin{aligned} &\mathcal{F}^{(k,l)} \otimes_{\mathcal{O}_X} L \text{ on } X \times_k k[\varepsilon] \\ &s_k^* \in \Gamma(\mathcal{F}^{(k+1,-k)} \otimes_{\mathcal{O}_X} L). \end{aligned}$$

Via $L \xrightarrow{\sim} \mathcal{O}_X \otimes k[\varepsilon]$ on $X-S$ (resp. $X-T$), write

$$\begin{aligned} s_k^* &= s_k + \varepsilon s_k',\ s_k' \in \Gamma(X-S, \mathcal{F}^{(k+1,-k)}) \\ &= s_k + \varepsilon s_k'',\ s_k'' \in \Gamma(X-T, \mathcal{F}^{(k+1,-k)}) \end{aligned}$$

where

$$s_k + \varepsilon s_k' = (1+\varepsilon c)(s_k + \varepsilon s_k''),$$

or

$$s_k' - s_k'' = cs_k.$$

c) If $\phi_0(c) = C_{k,l}$, then by definition

$$cs_k = \sum C_{k,l} s_l.$$

Write $C = C_+ + C_0 + C_-$. Then

$$cs_k = \underbrace{\sum_{k<l} C_{k,l} s_l}_{(C_+ s)_k} + \underbrace{\sum_{k \geq l} C_{k,l} s_l}_{((C_0 + C_-)s)_k.}$$

Since

$$(C_+ s)_k \in \Gamma\Big(\sum_{l>k} \mathscr{F}^{(l+1,-l)}\Big) \subset \Gamma(X-S, \mathscr{F}^{(k+1,-k)})$$

$$((C_0 + C_-)s)_k \in \Gamma\Big(\sum_{l \leq k} \mathscr{F}^{(l+1,-l)}\Big) \subset \Gamma(X-T, \mathscr{F}^{(k+1,-k)})$$

we may define

$$s_k' = (C_+ s)_k, \qquad s_k'' = -((C_0 + C_-)s)_k$$

and get s_k^* with the required property.

d) To determine the change in the matrices $\phi(f)$ associated to $f \in R$, we must write:

$$f \cdot s_k^* = \sum (\phi_0(f)_{kl} + \varepsilon \phi_1(f)_{kl})(s_l^*).$$

This works out to say

$$f \cdot s_k' = \sum \phi_1(f)_{kl} s_l + \sum \phi_0(f)_{kl} s_l'$$

or

$$([C_+, \phi_0(f)]s)_k = \sum (C_+)_{kl} f \cdot s_l - \sum \phi_0(f)_{kl} (C_+)_{ln} s_n = \sum \phi_1(f)_{kn} s_n$$

as required.

Note that as $[\phi_t(C), \phi_t(A)] = 0$ and $[\phi_t(C)_0, \phi_t(A)]$ generates a flow in the direction of equivalent subrings, the flow in Data B′ may also be written:

$$\frac{d}{dt}\phi_t(A) = \frac{1}{2}[\phi_t(C)_+ - \phi_t(C)_-, \phi_t(A)].$$

To see the connection with the Toda lattice equations, as promised in the title, take $A = C$ to be an n-periodic symmetric matrix with support $[-1, +1]$ ("tridiagonal"). Let A be

$$\begin{pmatrix} & & \cdots\cdots\cdots\cdots & & & \\ e^{a_{n-1}-a_n} & b_n & e^{a_n-a_1} & 0 & 0 & 0 \\ 0 & e^{a_n-a_1} & b_1 & e^{a_1-a_2} & 0 & 0 \\ 0 & 0 & e^{a_1-a_2} & b_2 & e^{a_2-a_3} & 0 \\ 0 & 0 & 0 & e^{a_2-a_3} & b_3 & e^{a_3-a_4} \\ & & \cdots\cdots\cdots\cdots & & & \end{pmatrix},$$

$$\text{where} \quad \sum_{i=1}^{n} a_i=0 \quad \sum_{i=1}^{n} b_i=0.$$

Then one readily calculates that the flow is given by

$$\dot{a}_k=b_k$$
$$\dot{b}_k=e^{a_{k-1}-a_k}-e^{a_k-a_{k+1}}$$

which are the Toda lattice equations describing a set of n particles on a circle, each pair being connected by a spring-like force that tends to keep the $(k+1)^{\text{st}}$ ahead of the k^{th}, rising exponentially if they get closer or even get in the wrong order, but relaxing exponentially as they get farther apart in the right order.

It appears that particularly nice solutions of these equations arise by taking X to be a singular curve whose smooth model is $\boldsymbol{P}^1$. Apparently, if X has p ordinary double points we get the so-called p-soliton solutions of these non-linear equations. And if X is "unicursal", i.e., the map $\boldsymbol{P}^1 \to X$ is bijective, hence $\mathrm{Pic}^0(X)$ is an additive group, then we apparently get solutions in which the entries A_{ij} of the matrix are rational functions of i, j. This has not been fully worked out as yet.

§ 2. Differential operator case (Krichever)

Again let us start with Data A. Our first goal is to construct a deformation of the sheaf $\mathscr{F}$ to a sheaf $\mathscr{F}^*$ over $X \times_k k[[t]]$, plus a differential operator

$$\nabla : \mathscr{F}^* \longrightarrow \mathscr{F}^*(P)$$

such that:

1) $\nabla(as)=a\cdot\nabla s+\dfrac{\partial a}{\partial t}\cdot s, \qquad \forall a \in \mathcal{O}_X \otimes_k k[[t]],\ s \in \mathscr{F}^*.$

Moreover, if $z \in \mathfrak{M}_{P,X}-\mathfrak{M}^2_{P,X}$ is a local coordinate so that $\partial/\partial z \in T_{P,X}$ is the basis given by $T_{P,X} \cong k$, then we require:

2) $\nabla(s)=\dfrac{s}{z}+(\text{section of } \mathscr{F}^*).$

If $k=\boldsymbol{C}$, we can describe $\mathscr{F}^*$ analytically in a very simple way: let $U \subset X$ be a small complex neighborhood of P in which z is still a local coordinate. Define $\mathscr{F}^*$ on $X\times \boldsymbol{C}$ to be $\mathscr{F}\otimes\mathcal{O}_C$ on $U\times \boldsymbol{C}$ and on $(X-P)\times \boldsymbol{C}$, but glue $\mathscr{F}^*$ to itself on $(U-P)\times \boldsymbol{C}$ by the transition function $e^{t/z}$. Define ∇ on sections of $\mathscr{F}^*$ on $(X-P)\times \boldsymbol{C}$ to be $\partial/\partial t$. Since

$$e^{-t/z}\frac{\partial}{\partial t}(e^{t/z}f(z,t))=\frac{1}{z}f(z,t)+\frac{\partial}{\partial t}f(z,t),$$

∇ extends to an operator from $\mathscr{F}^*$ to $\mathscr{F}^*(P)$ as required. To do this algebraically, we do the same thing regarding $e^{t/z}$ as a formal power series in t. This gives us a formal sheaf on the formal completion of $X\times_k A^1_k$ along $X\times(0)$. By Grothendieck's formal existence theorem, it defines a sheaf $\mathscr{F}^*$ on $X\times_k k[[t]]$.

Then $H^i(X\times_k k[[t]],\mathscr{F}^*)=(0)$, $i=0,1$, so the map:

$$\begin{array}{c} H^0(X\times_k k[[t]],\mathscr{F}^*(P))\rightarrow H^0(\mathscr{F}^*(P)/\mathscr{F}^*) \\ \hphantom{H^0(X\times_k k[[t]],\mathscr{F}^*(P))\rightarrow{}}\wr\Vert \\ \hphantom{H^0(X\times_k k[[t]],\mathscr{F}^*(P))\rightarrow{}}k[[t]] \end{array}$$

is an isomorphism. Let s_0 be a generator of this $k[[t]]$-module. Define

$$s_n\in H^0(X\times_k k[[t]],\mathscr{F}^*((n+1)P))$$

by

$$s_n=\nabla^n(s_0).$$

Note that $s_n=s_0/z^n+$(lower terms), hence $s_0,\cdots,s_n$ are a $k[[t]]$-basis of $H^0(X\times_k k[[t]],\mathscr{F}^*((n+1)P))$. Now let $R=\Gamma(X-P,\mathcal{O}_X)$. Then for every $a\in R$, if $a=\alpha/z^n+$(lower terms) at P, then $as_0\in H^0(\mathscr{F}^*((n+1)P))$ so

$$\begin{aligned} as_0&=\alpha s_n+\sum_{i=0}^{n-1}a_i(t)s_i\\ &=\left(\alpha\nabla^n+\sum_{i=0}^{n-1}a_i(t)\nabla^i\right)s_0. \end{aligned}$$

Define an embedding of R in $k[[t]][d/dt]$ by taking a to:

$$D(a)=\alpha\left(\frac{d}{dt}\right)^n+\sum_{i=0}^{n-1}a_i(t)\left(\frac{d}{dt}\right)^i.$$

It is easy to verify that this is a homomorphism and that if s_0 is changed to $u(t)\cdot s_0$, $u(0)\neq 0$, then $D(a)$ is replaced by

$$u(t)\circ D(a)\circ u(t)^{-1}$$

so we get an equivalent ring.

To see intuitively how to go backwards from R to $(X,P,\mathscr{F})$, consider as in § 1 the spectral properties of the differential operators in R. Let $K\supset k$ be an extension field and look for formal power series $f(t)=\sum a_i t^i$, $a_i \in K$, such that

$$Df=\lambda_D f, \qquad \text{all } D \in R.$$

Then $D \mapsto \lambda_D$ is a homomorphism $R \to K$, hence a K-valued point of $X-P$ and the following holds:

Proposition. *Let* Data $(X, P, \mathscr{F})$ *define* $R \subset k[[t]][d/dt]$ *as above. Let* $K \supset k$ *be a field,* $\lambda: R \to K$ *be a K-valued point of* $X-P$ *lying over* $x \in X-P$ *(x defined by the prime ideal* Ker λ*). Then there is an isomorphism between:*

a) *the eigenspace* $\{f \in K[[t]] \mid Df=\lambda(D)\cdot f \text{ all } D \in R\}$,

and

b) $\mathrm{Hom}_R(\mathscr{F}_x/m_x\mathscr{F}_x, K)$ *(K an R-module via* λ*).*

Proof. Start with $\phi: \mathscr{F}_x/m_x\mathscr{F}_x \to K$. This is the same as an R-linear map:

$$\phi: \bigoplus_{n=0}^{\infty} s_n \cdot k = \Gamma(X-P, \mathscr{F}) \to K.$$

Any such map extends uniquely to an $R[[t]]$-linear map

$$\phi^*: \bigoplus_{n=0}^{\infty} s_n \cdot k[[t]] = \Gamma((X-P) \times k[[t]], \mathscr{F}^*) \to K[[t]]$$

such that

$$\phi^*(\nabla a) = \frac{d}{dt}\phi^*(a).$$

Such a ϕ^* is determined by the value $f(t)=\phi^*(s_0)$ and conversely, given $f(t)$, ϕ^* must map

$$\sum s_n \cdot a_n(t) \mapsto \sum a_n(t)\left(\frac{d}{dt}\right)^n f(t).$$

For this to be R-linear, however, means:

$$\phi^*(a \cdot s_0) = \lambda(a)\phi^*(s_0), \qquad \text{all } a \in R,$$

i.e., if $a \cdot s_0 = (\sum a_i(t)\nabla^i)s_0$, then

$$\sum a_i(t)\left(\frac{d}{dt}\right)^i f(t) = \lambda(a)\cdot f(t)$$

which means that f is in the λ-eigenspace of R. Q.E.D.

Now to go backwards from R to $(X, P, \mathscr{F})$, the intuitive picture is this: $X-P$

is just $\operatorname{Spec} R$. $D \mapsto \deg D$ is a valuation on R and thus X is just $\operatorname{Spec} R$ plus one point P such that

$$\mathcal{O}_{P,X} = \left\{ \frac{D_1}{D_2} \,\middle|\, D_1, D_2 \in R,\ \deg D_1 \leq \deg D_2 \right\}.$$

To get $\mathscr{F}$, associate to each point of $X-P$ the corresponding eigenspace of R and "glue" these into a bundle. $\mathscr{F}$ is to be the sheaf of functions on this bundle, linear in each fibre, generated by the functions $f \mapsto f^{(n)}(0)$.

Alternatively, if M is the vector space of singular distributions on the t-line supported at $t=0$, M is an R-module and $\mathscr{F}$ on $X-P$ is just $\tilde{M}$.

However, rather than following this approach, it seems easier to use a more abstract approach better suited to generalizations. Starting with $R \subset k[[t]][d/dt]$, and $A, B \in R$ as in Data B, let:

$$R_n = \{D \in R \mid \deg D \leq n\}.$$

Then we have:

Lemma. i) *For all* $D \in R$, $D = \alpha(d/dt)^n + (\text{lower terms})$, $\alpha \in k$.
ii) *For all* n, $\dim R_{n+1}/R_n \leq 1$, *equality holding if* n *is large.*

Proof. Let $A, B \in R$ be the given operators of relatively prime degree. If $\deg A = \alpha$, calculate the term of degree $a+n+1$ in $DA-AD$ and we find that if $D = a_n(t)(d/dt)^n + (\text{lower terms})$, then $\alpha \cdot a_n'(t) = 0$, hence $a_n(t)$ is a constant. Thus $\dim R_{n+1}/R_n \leq 1$. The monomials $A^i B^j$ give us operators of arbitrary sufficiently large degree.

Introduce the graded ring:

$$\mathscr{R} = \sum_{n=0}^{\infty} R_n.$$

Then as in § 1, we have:

Corollary. *R is a finite $k[A]$-module. $\mathscr{R}$ is a finite $k[e, A]$-module where $e \in \mathscr{R}_1$ represents the operator* 1, *$A \in \mathscr{R}_\alpha$ represents A. Hence R and $\mathscr{R}$ are finitely generated integral domains over k of transcendence degree* 1 *and* 2 *respectively.*

We now define

$$X = \operatorname{Proj}(\mathscr{R}).$$

The affine open $e \neq 0$ is given by:

$$\begin{aligned} \begin{pmatrix} \text{open subset} \\ e \neq 0 \end{pmatrix} &= \operatorname{Spec}\left(\mathscr{R}\left[\frac{1}{e}\right]\right)_0 \\ &\cong \operatorname{Spec} R, \end{aligned}$$

and the affine open $A \neq 0$ is given by:

$$\begin{pmatrix}\text{open subset}\\ A\neq 0\end{pmatrix} = \operatorname{Spec}\left(\mathscr{R}\left[\frac{1}{A}\right]_0\right)$$
$$\cong \operatorname{Spec}\{\text{ring of fractions } C/A^k,\ \deg C \leq k\alpha\}.$$

As in § 1, if $C = A^i B^j$ has degree $k\alpha - 1$, then the completion of this last ring in the e-adic topology is just $k[[z]]$, z being the local coordinate corresponding to C/A^k. So $e=0$ consists in one smooth k-rational point P, and the sheaf $\mathcal{O}_X(1)$ is just $\mathcal{O}_X(P)$.

Next, let M denote the big ring $k[[t]][d/dt]$, but considered now as a module over various rings by left and right multiplication. Define:

$$M_n = \{D \in M \mid \deg D \leq n\}$$
$$\mathfrak{M} = \bigoplus_{n=0}^{\infty} M_n.$$

We consider M as a $k[[t]]\otimes_k R$-module by letting $k[[t]]$ act by left-multiplication and R by right multiplication. Similarly, we consider $\mathfrak{M}$ as a $k[[t]]\otimes_k \mathscr{R}$-module. It is immediate that these modules are finitely generated: let $\mathscr{F}^* = \tilde{\mathfrak{M}}$ be corresponding sheaf over $X \times_k k[[t]]$. We have canonical maps:

$$\phi_n : M_n \to \Gamma(X \times_k k[[t]], \mathscr{F}^*(n)) = \Gamma(X \times_k k[[t]], \mathscr{F}^*(nP)).$$

Let s_n be the image of $(d/dt)^n$. For $n \gg 0$, this is an isomorphism and $H^1(\mathscr{F}^*(nP)) = (0)$. For each n, s_n generates $\mathscr{F}^*(n)/\mathscr{F}^*(n-1)$. Hence by descending induction on n, ϕ_n is an isomorphism for all $n \geq 0$ and $H^1(\mathscr{F}^*(nP)) = (0)$. Also, by the Hilbert polynomial, $\mathscr{F}^*$ is a rank 1 sheaf.

Next, consider $(d/dt)_{\text{left-mult.}} : \mathfrak{M} \to \mathfrak{M}[1]$. It induces a map

$$\nabla : \mathscr{F}^* \to \mathscr{F}^*(P)$$

and from the identities

$$\left(\frac{d}{dt}\right)_{\text{left}} \cdot D_{\text{right}} = D_{\text{right}} \cdot \left(\frac{d}{dt}\right)_{\text{left}}$$

$$\left(\frac{d}{dt}\right)_{\text{left}} \cdot a(t)_{\text{left}} = a'(t)_{\text{left}} + a(t)_{\text{left}} \cdot \left(\frac{d}{dt}\right)_{\text{left}}$$

$$\left(\frac{d}{dt}\right)_{\text{left}} = \left(\frac{d}{dt}\right)_{\text{right}} + (\text{operator from } \mathfrak{M} \text{ to } \mathfrak{M}),$$

$$A^k_{\text{right}} = C_{\text{right}} \cdot \left(\frac{d}{dt}\right)_{\text{right}} + (\text{lower order operator})$$

we deduce that ∇ satisfies:

$$\nabla(as) = \frac{\partial a}{\partial t} \cdot s + a \cdot \nabla(s)$$

$$\nabla(s) = \frac{s}{z} + (\text{section of } \mathscr{F}^*).$$

We are now essentially back where we started: I claim that any pair $(\mathscr{F}^*, \nabla)$ with these properties is constructed as a deformation of $\mathscr{F}$ on X as in the beginning of this section. We omit this verification.

Extensions of this Dictionary to rank d sheaves $\mathscr{F}$ and commutative rings R in which all operators have degrees divisible by d can be made. However an additional complication arises from the possibility that the sheaf $\mathscr{F}$ may be unstable. We have only worked out the "generic case" where the bundles involved are all stable, and, moreover, have not characterized the rings of this generic type. However we can give a procedure for constructing certain rings of commuting operators from vector bundles. We need some definitions: let X be a curve over k (char $k=0$), and assume for simplicity that X is smooth, and irreducible of genus $g>0$. Consider the set of all stable rank r bundles with parabolic structure at P: i.e.,

$$E_0 \subset E_1 \subset \cdots \subset E_r = E_0(P)$$

$$\chi(E_i) = i$$

E_i stable locally free of rank r for all i, meaning for all non-zero subsheaves $F \subsetneqq E_i$,

$$\chi(F) < \frac{\mathrm{rk} F}{r} \cdot \chi(E_i).$$

The set of all these forms a smooth quasi-projective moduli space V^r of dimension $r^2(g-1)+1+r(r-1)/2$. To each point $\{E_*\} \in V^r$, we may associate an infinite flag of bundles by requiring

$$E_{i+r} = E_i(P), \qquad \text{all } i \in \boldsymbol{Z}.$$

Let

$$\mathrm{End}^k(E_*)$$

be the sheaf which is just $\mathrm{End}(E_0)$ on $X-P$ where, near P, the endomorphism λ is required to satisfy

$$\lambda(E_i) \subset E_{i+k}.$$

Then it is well known that the tangent bundle T_V can be identified canonically via:

$$T_{V,\{E_*\}} \cong H^1(X, \mathrm{End}^0(E_*)).$$

Look at the exact sequence:

$$0 \to \mathrm{End}^0(E_*) \to \mathrm{End}^1(E_*) \to \bigoplus_{k=0}^{r-1} \mathrm{Hom}\left(\frac{E_k}{E_{k-1}}, \frac{E_{k+1}}{E_k}\right) \to 0.$$

Lemma. $H^0(\mathrm{End}^1(E_*)) = k$.

Proof. In fact, take any

$$\lambda : E_0|_{X-P} \to E_0|_{X-P}$$

which extends to P so that $\lambda(E_k) \subset E_{k+1}$, all k. If $\lambda(E_k) \subset E_k$ for some k, then since E_k is stable, $\lambda = \alpha \cdot \mathrm{id.}$, some $\alpha \in k$. If not, then in a neighborhood $U \subset X$ of P, choose

$$e_1 \in \Gamma(U, E_{-r+1}) - \Gamma(U, E_{-r}).$$

Then $e_1, \lambda e_1, \cdots, \lambda^{r-1} e_1$ will have non-zero image in the quotients E_{-r+1}/E_{-r}, $E_{-r+2}/E_{-r+1}, \cdots, E_0/E_{-1}$, hence will give a basis of $E_0/E_0(-P)$. Thus they are a basis of E_0 in some smaller $U_1 \subset U$. Likewise, if z is a local coordinate at P, then $(1/z)e_1, \lambda e_1, \cdots, \lambda^{r-1} e_1$ are a basis of E_1 near P; so in terms of this basis, λ is given by a matrix

$$\lambda = \begin{bmatrix} 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 0 \\ 0 & 0 & 0 \cdots 1 \\ a_1/z & a_2 & a_3 \cdots a_r \end{bmatrix}, \qquad a_i \in \Gamma(U_1, \mathcal{O}_X).$$

Since $\lambda(E_0) \not\subset E_0$, $a_1(P) \neq 0$.

Now $\det(\lambda)$ is a rational function on X with poles only at P. Then the above shows that it has a simple pole at P, and this is impossible since $g > 0$. Q.E.D.

Now taking cohomology, we find:

$$0 \to \bigoplus_{k=0}^{r-1} \mathrm{Hom}\left(\frac{E_k}{E_{k-1}}, \frac{E_{k+1}}{E_k}\right) \to T_{V,\{E_*\}} \to H^1(\mathrm{End}^1(E_*)) \to 0.$$

Globally, this defines r sub-line bundles $L_i \subset T_V$, hence a rank r distribution:

$$\bigoplus_{i=1}^{r-1} L_i \subset T_V.$$

Now define:

Data A′ (*smooth stable case*):

a) *X a complete smooth curve over k,*

b) *$P \in X$ a k-rational point,*

c) *ϕ: Spec $k[[t]] \to V^r$ a morphism such that*

$$\dot{\phi}(\partial/\partial t) \in \bigoplus_{i=1}^{r-1} L_i$$

$$\dot{\phi}(\partial/\partial t)|_{t=0} \notin \Big(\bigoplus_{k \text{ omitted}} L_i\Big) \qquad \textit{for any } 0 \le k \le r-1$$

$$\phi(0) \in (\textit{open set where } h^0(E_0) = h^1(E_0) = 0).$$

Data B′. *Commutative subrings $R \subset k[[t]][d/dt]$ such that $r \mid \deg C$, all $C \in R$ and $\exists\ A, B \in R$ of form*

$$A = a_0(t)\Big(\frac{d}{dt}\Big)^n + \cdots + a_n(t)$$

$$B = b_0(t)\Big(\frac{d}{dt}\Big)^m + \cdots + b_m(t)$$

where $(n, m) = r$, $a_0(0) \neq 0$, $b_0(0) \neq 0$, modulo

$$R \sim u(t) \circ R \circ u(t)^{-1}, \qquad u(0) \neq 0.$$

We claim merely that every piece of Data A′ defines a piece of Data B′. To see this, let ϕ define the family of vector bundles with parabolic structure $\{E_k\}$ over $X \times \operatorname{Spec} k[[t]]$. Let $\vec{e}_\alpha$ be a basis of $E_0|_{U_\alpha}$ where $\{U_\alpha\}$ is a sufficiently fine covering of $X \times \operatorname{Spec} k[[t]]$ and assume that if $P \in U_\alpha$, then

$$\frac{e_{\alpha,1}}{z}, \ldots, \frac{e_{\alpha,i}}{z}, e_{\alpha,i+1}, \ldots, e_{\alpha,r}$$

is a basis of $E_i|_{U_\alpha}$, $0 \le i \le r$. Let

$$\vec{e}_\alpha = A_{\alpha\beta} \cdot \vec{e}_\beta$$

on $U_\alpha \cap U_\beta$. Then

$$\dot{\phi}(\partial/\partial t) \in H^1(X \times \operatorname{Spec} k[[t]], \operatorname{End}^0(E_*))$$

is given by the 1-co-cycle:

$$A_{\alpha\beta}^{-1} \cdot \frac{\partial A_{\alpha\beta}}{\partial t}$$

and the assumption that this lies in $\oplus L_i$ means that it dies in $H^1(\operatorname{End}^1)$, i.e.,

$$(*) \qquad A_{\alpha\beta}^{-1} \frac{\partial A_{\alpha\beta}}{\partial t} = A_{\alpha\beta}^{-1} D_\alpha A_{\alpha\beta} - D_\beta$$

$$D_\alpha \in \Gamma(U_\alpha, \operatorname{End}^1(E_*)).$$

Note that if $P \notin U_\alpha$, D_α is a matrix of regular functions and if $P \in U_\alpha$, D_α has the form

$$(**)\qquad D_\alpha=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,r-1} & a_{1r}/z \\ a_{21} & a_{22} & \cdots & a_{2,r-1} & a_{2,r} \\ za_{31} & a_{32} & \cdots & a_{3,r-1} & a_{3,r} \\ \vdots & & & \vdots & \vdots \\ za_{r1} & za_{r2} & \cdots & a_{r,r-1} & a_{rr} \end{pmatrix}.$$

Then define ∇ by $\nabla\vec{e}_\alpha=D_\alpha\vec{e}_\alpha$: (*) shows that ∇ is a global differential operator and (**) shows that $\nabla(E_k)\subset E_{k+1}$. Moreover, note that $a_{21}(P)\neq 0, \cdots, a_{r,r-1}(P)\neq 0$, $a_{1,r}(P)\neq 0$. For if one of these were zero, then $\dot{\phi}(\partial/\partial t)|_{t=0}$ would die already in H^1 of $\mathrm{End}^1(E_*)\cap\mathrm{End}(E_k)$ and hence $\dot{\phi}(\partial/\partial t)|_{t=0}$ would lie in $\oplus L_i$ (k omitted) contrary to assumption. Therefore, the *polar part* of ∇:

$$\bar{\nabla}:\bigoplus_{k=0}^{q-1} E_k/E_{k-1}\to\bigoplus_{k=0}^{r-1} E_{k+1}/E_k$$

is an isomorphism.

Now choose a generator s_0 of Γ $(X\times\mathrm{Spec}\,k[[t]], E_1)$. By the 3rd assumption on ϕ, s_0 generates E_1/E_0. Define $s_n\in\Gamma(E_n)$ by

$$s_n=\nabla^n s_0.$$

Then $\{s_n\}$ are a $k[[t]]$-basis of $\Gamma((X-P)\times\mathrm{Spec}\,k[[t]], E_0)$, hence the ring R is defined as before via:

$$\forall a\in\Gamma(X-P,\mathcal{O}_X),\quad \text{if}$$
$$a\cdot s_0=\sum a_i(t)\cdot s_i=(\sum a_i(t)\nabla^i)s_0,$$

then let

$$D(a)=\sum a_i(t)\left(\frac{d}{dt}\right)^i$$

and let

$$R=\mathrm{Image}\,(D).$$

In the case where $r=1$, V^r reduces to the Jacobian and ϕ reduces to a 1-parameter group on the Jacobian. Interestingly enough, if $r>1$, the distribution $\oplus L_i$ is contained in the tangent space to the fibres of the map

$$\pi: V\to(\text{Jacobian})$$
$$E_*\mapsto\Lambda^r E_0,$$

hence the curve ϕ is a curve on one of the *rational* varieties $\pi^{-1}(pt.)$. Thus whereas an explicit description of the operators in the rings R may be expected to involve the theta function when $r=1$, the operators must be very different (perhaps more elementary?) when $r>1$. The differential geometry of the highly non-integrable distribution $\oplus L_i$ on V has not yet been studied as far as I know.

Suppose $r=1$, X is smooth and $k=\boldsymbol{C}$. As in the difference operator case, we may strengthen the hypothesis $h^i(\mathscr{F})=0$ as follows:

Let $\phi: \boldsymbol{R}\to\mathrm{Pic}^0(X)$ be the 1-parameter group given by the line bundles $e^{t/z}$ defined above.

Let $\Sigma\subset\mathrm{Pic}^{g-1}(X)$ be the closure in the complex topology of the locus of points $[\mathscr{F}]+\phi(t)$, $t\in\boldsymbol{R}$.

Assume

$$\Sigma\cap\Theta=\phi.$$

In this case, it is to be expected that the coefficients of the differential operators that arise are almost periodic functions of t. On the other hand, if $r=1$, $k=\boldsymbol{C}$ and X is rational with double points, these coefficients should be rational functions of exponential functions $e^{\lambda_i t}$; and if X is unicursal, these coefficients should be rational functions of t.

The most interesting aspect of the dictionary, however, is its effect on the *Jacobian flows*. Let $k=\boldsymbol{C}$ and consider the general Data A′→Data B′ mapping (or for singular curves X, we may consider the Data A→Data B mapping). For every invertible sheaf L, L acts on V^r by $E_*\mapsto E_*\otimes L$, hence it acts on the set of all possible ϕ. As in § 1, this means that the tangent space $H^1(\mathcal{O}_X)$ to $\mathrm{Pic}^0(X)$ acts as a space of commuting vector fields on the manifold of all possible Data A′'s. These vector fields are very beautiful when expressed in terms of Data B′. The result is this: let $R\subset\boldsymbol{C}[[t]][d/dt]$ be an instance of Data B′. Regard R as a fixed abstract ring, but consider deformations of its embedding in $\boldsymbol{C}[[t]][d/dt]$:

$$D_s: R\to\boldsymbol{C}[[t]]\left[\frac{d}{dt}\right].$$

Fix one element $b\in R$ and some $l\geq 1$; let $k=\mathrm{order}\ D_s(b)$. We shall define, in a minute, for every ordinary differential operator D of order k whose leading coefficient $a(t)(d/dt)^k$ satisfies $a(0)\neq 0$, an approximate $(l/k)^{\mathrm{th}}$ power $(D^{l/k})_+$ of D. Then the flows in Data B′ are defined by the differential equation of Lax type*:

$$(\natural)\qquad \frac{d}{ds}D_s(a)=[(D_s(b)^{l/k})_+, D_s(a)],\qquad \text{all } a\in R.$$

What is $(D^{l/k})_+$? We may introduce a formal symbol $(d/dt)^{-1}$ subject to the commutation relations:

* I am told that this description of the Jacobian flows has been discovered also by Gel'fand and collaborators.

$$(d/dt)^{-1}\cdot a = a\cdot(d/dt)^{-1} - (d/dt)^{-1}\cdot\frac{da}{dt}\cdot(d/dt)^{-1},$$

or, solving inductively:

$$\left(\frac{d}{dt}\right)^{-1}\cdot a = \sum_{k=0}^{\infty}(-1)^k\frac{d^k a}{dt^k}\cdot\left(\frac{d}{dt}\right)^{-k-1}.$$

We get this way a very large non-commutative ring of formal operators whose elements we write:

$$D = \sum_{i=-\infty}^{+k} a_i(t)\left(\frac{d}{dt}\right)^i, \qquad a_i(t)\in \boldsymbol{C}[[t]].$$

If the a_i are replaced by C^∞ functions of t, this may be interpreted as the ring of pseudo-differential operators in t mod the ideal of C^∞-integral operators. Let PsD $\{t\}$ denote our formal ring. Then we have the following lemma:

Lemma. *Let $D\in \boldsymbol{C}[[t]][d/dt]$, $D=a_0(t)(d/dt)^n+\cdots+a_n(t)$, $a_0(0)\neq 0$. Then, up to an n^{th} root of 1, D has a unique n^{th} root $D^{1/n}\in$ PsD $\{t\}$. Moreover the commutator subring Z_D of D in PsD $\{t\}$ is the commutative ring of operators*:

$$\sum_{i=-\infty}^{+k} a_i D^{i/n}, \qquad a_i\in \boldsymbol{C}.$$

Proof. The main point is the calculation:

$$\left[D, c(t)\left(\frac{d}{dt}\right)^m\right] = (na_0(t)c'(t) - mc(t)a_0'(t))\left(\frac{d}{dt}\right)^{n+m-1} + \text{lower terms}.$$

From this it follows by easy induction that Z_D has, mod scalars and lower order terms, a unique element of each degree $m\in \boldsymbol{Z}$, and that it has the form $a_0(t)^{m/n}(d/dt)^m$ +(lower order terms). If $E\in Z_F$ has degree 1 and $E'\in Z_D$ has degree -1, it follows that $E\cdot E'=c+N$, $\deg N<0$, $c\in \boldsymbol{C}$, $c\neq 0$. Therefore

$$E^{-1} = E'\cdot\frac{1}{c}\cdot\left(\sum_{i=0}^{\infty}(-1)^i N^i/c^i\right)\in Z_D,$$

hence,

$$Z_D\supset\left\{\text{ring of Laurent series } \sum_{i=-\infty}^{+k} c_i E^i \text{ in } E\right\},$$

hence "$=$" holds here because each side has one new element in each degree. Thus Z_D is commutative. Finally, D itself is in Z_D so

$$D = \sum_{i=-\infty}^{+n} c_i E^i, \qquad c_i\in \boldsymbol{C},\ c_n\neq 0,$$

and, in a ring of Laurent series, such an element has a unique n^{th} root (up to a root of unity):

$$D = \sqrt[n]{c'_n} \cdot E \cdot \left(1 + \frac{c_{n-1}}{c_n} E^{-1} + \frac{c_{n-2}}{c_n} E^{-2} + \cdots \right)^{1/n}$$

where the last term can be expanded by the binomial theorem. Q.E.D.

Definition. For all D as above, set

$$D^{k/n} = (D^{k/n})_+ + (D^{k/n})_-$$

where $(D^{k/n})_+ \in \boldsymbol{C}[[t]][d/dt]$, and $(D^{k/n})_- \in \mathrm{PsD}\,\{t\}$ has negative degree.

To prove equation ($\natural$), we first extend the isomorphism

$$\Gamma(X-P, \mathcal{O}_X) \xrightarrow{\approx} R \subset \boldsymbol{C}[[t]]\left[\frac{d}{dt}\right]$$
$$a \longmapsto D(a)$$

to an isomorphism:

$$\begin{array}{l} \hat{K}_{P,X} \xrightarrow{\approx} Z_R \subset \mathrm{PsD}\,\{t\} \\ \Big\| \text{def} \\ \text{fraction field} \\ \text{of } \hat{\mathcal{O}}_{P,X} \end{array} .$$

To do this, for all k, let

$$\hat{E}_k = E_k \otimes (\hat{\mathcal{O}}_{P,X} \otimes \boldsymbol{C}[[t]])$$

and note that ∇ extends to an isomorphism:

$$\nabla : \hat{E}_k \to \hat{E}_{k+1}.$$

Then define

$$s_{-k} \in E_{-k+1}$$

by

$$s_{-k} = \nabla^{-k} s_0$$

and note that for all k:

$$\hat{E}_k \cong \left\{ \text{module of elements } \sum_{i=-\infty}^{k-1} a_i(t) \cdot s_i \right\}.$$

In particular, for all $a \in \hat{K}_{P,X}$, if a has a k-fold pole at P, then $as_0 \in \hat{E}_{k+1}$, so

$$as_0 = \left(\sum_{i=-\infty}^{k} a_i(t)\nabla^i\right)s_0.$$

Set

$$D(a) = \sum_{i=-\infty}^{k} a_i(t)\left(\frac{d}{dt}\right)^i \in \mathrm{PsD}\,\{t\}.$$

With this preparation, we can easily check (♮):

a) describe $H^1(\mathcal{O}_X)$ by the acyclic resolutions:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(nP) \to \mathcal{O}_X(nP)/\mathcal{O}_X \to 0, \qquad n \gg 0,$$

giving

$$\begin{aligned} H^1(\mathcal{O}_X) &\cong \varinjlim_n \operatorname{coker}\left\{H^0(\mathcal{O}_X(nP)) \to \frac{\mathcal{O}_X(nP)}{\mathcal{O}_X}\right\} \\ &\cong \frac{\hat{K}_{P,X}}{\hat{\mathcal{O}}_{P,X} + \Gamma(X-P, \mathcal{O}_X)} \quad . \end{aligned}$$

Then $c \in \hat{K}_{P,X}$ defines a tangent vector to $\mathrm{Pic}^0(X)$ described by an invertible sheaf L on $X \times_C C[\varepsilon]/(\varepsilon^2)$ by:

$$L \cong \begin{cases} \mathcal{O}_X \otimes_C C[\varepsilon] \text{ on } X-P, \text{ over } \operatorname{Spec} \hat{\mathcal{O}}_{P,X} \\ \text{glued by mult. by } 1+\varepsilon c. \end{cases}$$

b) Via L, we may deform E_* to $E_* \otimes_{\mathcal{O}_X} L$ over $X \times \operatorname{Spec} C[[t, \varepsilon]]/(\varepsilon^2)$, and we may extend ∇ to

$$\nabla : E_k \otimes_{\mathcal{O}_X} L \to E_{k+1} \otimes_{\mathcal{O}_X} L$$

by $(a \otimes b) = (\nabla a) \otimes b$, for any section b of L over $X \times \operatorname{Spec} C[\varepsilon]$, i.e., b not depending on t. Moreover, we may lift $s_0 \in \Gamma(X \otimes C[[t]], E_1)$ to a section s_0^* of $E_1 \otimes_{\mathcal{O}_X} L$ given by

$$\begin{aligned} s_0^*|_{X-P} &= (s_0 + \varepsilon s_0') \in \Gamma((X-P) \times C[[t, \varepsilon]]/(\varepsilon^2), E_1) \qquad \text{via } L|_{X-P} \cong \mathcal{O}_X|_{X-P} \\ s_0^*|_{\mathcal{O}_{X,P}} &= (s_0 + \varepsilon s_0'') \in \hat{E}_1 \otimes C[\varepsilon]/(\varepsilon^2) \qquad \text{via } L|_{\mathcal{O}_{X,P}} \cong \hat{\mathcal{O}}_{X,P} \end{aligned}$$

where

$$(s_0 + \varepsilon s_0'') = (1 + \varepsilon c)(s_0 + \varepsilon s_0')$$

or

$$s_0'' = c s_0 + s_0'.$$

c) But we may write

$$cs_0 = \sum_{i=-\infty}^{k} c_i(t)s_i.$$

If we *define*

$$s_0'' = \sum_{i=-\infty}^{-1} c_i(t)s_i \qquad s_0' = -\sum_{i=0}^{k} c_i(t)s_i$$

then $s_0^* = (s_0 + \varepsilon s_0', s_0 + \varepsilon s_0'')$ gives the required lifting of the section s_0. Define $s_i^* = \nabla^i s_0^*$.

d) To determine the change in the differential operators $D(a)$, $a \in \Gamma(X-P, \mathcal{O}_X)$, we must solve

$$as_0^* = \sum_{i=0}^{k} (a_i(t) + \varepsilon a_i'(t))s_i^*.$$

Expanding this over $X-P$, it says:

$$as_0' = \sum_{i=0}^{l} a_i'(t)\nabla^i s_0 + \sum_{i=0}^{l} a_i(t)\nabla^i s_0'.$$

But

$$\begin{aligned} as_0' &= -\left(\sum_{j=0}^{k} c_j(t)\nabla^j\right)(as_0) \\ &= -\left(\sum_{j=0}^{k} c_j(t)\nabla^j\right)\left(\sum_{i=0}^{l} a_i(t)\nabla^i\right)s_0 \end{aligned}$$

while

$$\sum_{i=0}^{l} a_i(t)\nabla^i s_0' = -\left(\sum_{i=0}^{l} a_i(t)\nabla^i\right)\left(\sum_{j=0}^{k} c_j(t)\nabla^j\right)s_0.$$

Thus, if we set

$$(*) \qquad \sum_{i=0}^{l} a_i'(t)\nabla^i = \left[\sum_{i=0}^{l} a_i(t)\nabla^i, \sum_{j=0}^{k} c_j(t)\nabla^j\right]$$

we have a solution (the higher degree terms in the commutator are zero because $\sum_{j=-\infty}^{k} c_j(t)\nabla^j$ commutes with $\sum_{i=0}^{l} a_i(t)\nabla^i$).

e) But

$$\sum_{j=0}^{k} c_j(t)\left(\frac{d}{dt}\right)^j = D(c)_+,$$

the differential operator part of the pseudo-differential operator assigned to c. If we choose $b \in \Gamma(X-P, \mathcal{O}_X)$ with an h-fold pole at P and let $c = b^{k/h}$, then

$$\sum_{j=0}^{k} c_j(t)\left(\frac{d}{dt}\right)^j = (D(b)^{k/h})_+,$$

so (*) reads:

$$\frac{d}{d\varepsilon}D(a) = [D(a), (D(b)^{k/h})_+]$$

as required.

To see the connection of the general theory with the Korteweg-deVries equation, as promised in the title, we take $D(a)=D(b)$ to be the second order operator

$$D=\left(\frac{d}{dt}\right)^2 + a(t),$$

and take $k=3$, $h=2$. Then one can solve mechanically for $D^{1/2}$, finding:

$$D^{1/2}=\left(\frac{d}{dt}\right)+\frac{a(t)}{2}\left(\frac{d}{dt}\right)^{-1}-\frac{a'(t)}{4}\left(\frac{d}{dt}\right)^{-2}+\frac{a''(t)-a(t)^2}{8}\left(\frac{d}{dt}\right)^{-3}$$
$$+\frac{6a(t)\cdot a'(t)-a'''(t)}{16}\left(\frac{d}{dt}\right)^{-4}+\cdots$$

whence

$$D^{3/2}=\left(\frac{d}{dt}\right)^3+\frac{3a(t)}{2}\left(\frac{d}{dt}\right)+\frac{3a'(t)}{4}+\frac{a''(t)+3a(t)^2}{8}\left(\frac{d}{dt}\right)^{-1}+\cdots$$

and

$$[D, (D^{3/2})_+] = -\tfrac{1}{4}(a'''(t)+6a(t)\cdot a'(t)).$$

Therefore, if

$$D_s=\left(\frac{d}{dt}\right)^2 + a(s, t),$$

the Jacobian flow is given by:

$$\frac{\partial a}{\partial s} = -\frac{1}{4}\left(\frac{\partial^3 a}{\partial t^3}+6a\cdot\frac{\partial a}{\partial t}\right)$$

which, (up to coefficients which can be normalized away) is the Kortweg-de Vries equation.

Now whenever X is a hyperelliptic curve and $P \in X$ is a Weierstrass point, then there is a function a on X which a double pole at P only. Then with suitable normalization:

$$D(a)=\left(\frac{d}{dt}\right)^2+a_0(t).$$

Thus if we follow a Jacobian flow on Pic(X), we get a 1-parameter family of operators:

$$D_s(a)=\left(\frac{d}{dt}\right)^2+a_0(s,t)$$

where a_0 satisfies the Korteweg-de Vries equation. For smooth X, these solutions were discovered by McKean and van Moerbecke [9] and others; for singular X of type

$$y^2=xf(x)^2$$

these appear to be the n-soliton solutions of Kay and Moses [6]; and for a unicursal X of type

$$y^2=x^{2n+1},$$

these appear to be the rational solutions of Airault, McKean and Moser [1]. These connections have not yet been investigated in detail. (see Added in proof)

§ 3. **Field operator case** (Drinfeld)

As in the introduction, let k be a field, $\sigma \in \mathrm{Aut}(k)$ of infinite order, k_0 the fixed field. Generalizing the dictionary in the introduction, consider:

Data A′. a) *X_0 a reduced and irreducible complete curve over k_0. Let $X=X_0\times_{k_0}k$: we assume this is reduced* and irreducible.*

b) *$P_0\in X_0$ a regular closed point. Let $P=P_0\times_{k_0}k\subset X$.*

c) *A torsion-free sheaf $\mathscr{F}$ on X such that*

$$h^0(\mathscr{F})=h^1(\mathscr{F})=0.$$

d) *A maximal flag of subsheaves:*

$$\mathscr{F}=\mathscr{F}_0\supsetneqq\mathscr{F}_{-1}\supsetneqq\cdots\supsetneqq\mathscr{F}_{-d}=\mathscr{F}(-P),$$

where length$(\mathscr{F}_{k+1}/\mathscr{F}_k)=1$.

e) *A homomorphism of sheaves*

* As pointed out to me by J. Tate, "X reduced" is automatic because whenever k_0 is the fixed field of some σ, k is separable over k_0. (To see this, suppose on the contrary there were $x_1,\cdots,x_n\in k_0^{1/p}$ which are linearly independent over k_0 but dependent over k: $x_1+a_2x_2+\cdots+a_nx_n=0$, $a_i\in k$. Assume n is minimal too. Then $x_1+\sigma a_2\cdot x_2+\cdots+\sigma a_n\cdot x_n=0$ so $(\sigma a_2-a_2)x_2+\cdots+(\sigma a_n-a_n)x_n=0$, so either $x_2,\cdots,x_n$ are also dependent or $\sigma a_k=a_k$, all k, hence $a_k\in k_0$. Both cases are impossible.)

$$\alpha: (1_{X_0}\times\sigma)^*\mathscr{F}|_{X-P}\to\mathscr{F}|_{X-P}$$

on $X-P$, which is not surjective, such that, on X, α carries $\mathscr{F}_k$ to $\mathscr{F}_{k+1}$. Here σ: Spec $k\leftarrow$Spec k is induced by $\sigma: k\to k$, so that $(1_{X_0}\times\sigma)^*\mathscr{F}$ is a sheaf on X conjugate to $\mathscr{F}$ via σ.

Data B′. *A commutative subring $R\subset k\{\sigma\}$, with $R\supsetneqq k_0$ and $R\cap k=k_0$, modulo the identification*:

$$R\sim aRa^{-1},\qquad a\in k^*.$$

We claim these 2 sorts of Data are equivalent as before. Before proving this, however, we want to prove a remarkable observation of Drinfeld—that in Data A′, the assumption $h^0(\mathscr{F})=h^1(\mathscr{F})=0$ follows from the a priori weaker assumption that $\chi(\mathscr{F})=0$. To see this, first define $\mathscr{F}_n$ for all $n\in \boldsymbol{Z}$ by requiring

$$\mathscr{F}_{n+d}=\mathscr{F}_n(P).$$

Note that $\chi(\mathscr{F}_n)=n$. Since $\mathscr{F}_n\subset\mathscr{F}_{n+1}$, $h^0(\mathscr{F}_n)\le h^0(\mathscr{F}_{n+1})$ and $h^1(\mathscr{F}_n)\ge h^1(\mathscr{F}_{n+1})$. Let n_0 be the smallest n such that $h^0(\mathscr{F}_n)\neq 0$ and let $s_0\in\Gamma(\mathscr{F}_{n_0})$. Certainly $n_0\le 1$ because $h^0(\mathscr{F}_1)\ge\chi(\mathscr{F}_1)=1$. Consider the maps

$$\alpha: (1_{X_0}\times\sigma)^*\mathscr{F}_n\to\mathscr{F}_{n+1}.$$

Define inductively sections

$$s_n\in\Gamma(\mathscr{F}_{n+n_0})$$

by

$$s_n=\alpha((1_{X_0}\times\sigma)^*s_{n-1}).$$

By assumptions e, since $\chi(\mathscr{F}_{n+1})=\chi((1_{X_0}\times\sigma)^*\mathscr{F}_n)+1$, $l(\text{coker }\alpha)=1$ and $\alpha|_{X-P}$ is *not* surjective, α must be surjective at P. Thus

$$\mathscr{F}_{n+1}=\mathscr{F}_n+\alpha((1_{X_0}\times\sigma)^*\mathscr{F}_n).$$

From this it follows also that intersecting in $\mathscr{F}_{n+1}$:

$$\alpha((1_{X_0}\times\sigma)^*\mathscr{F}_{n-1})=\mathscr{F}_n\cap\alpha((1_{X_0}\times\sigma)^*\mathscr{F}_n).$$

Therefore the sequence of sections s_n satisfies the implications:

$$\begin{aligned}s_n\in\Gamma(\mathscr{F}_{n+n_0-1})&\Rightarrow\alpha(1_{X_0}\times\sigma)^*s_{n-1}\in\Gamma(\mathscr{F}_{n+n_0-1}\cap\alpha(1_{X_0}\times\sigma)^*\mathscr{F}_{n+n_0-1})\\&\Rightarrow s_{n-1}\in\Gamma(\mathscr{F}_{n+n_0-2}).\end{aligned}$$

Since $s_0\notin\Gamma(\mathscr{F}_{n_0-1})$, it follows that for all n, $s_n\notin\Gamma(\mathscr{F}_{n+n_0-1})$. But therefore the sections $\{s_0, s_1, \cdots, s_n\}$ are linearly independent. Thus

$$\begin{aligned} n+1 &\leq h^0(\mathscr{F}_{n+n_0}) \\ &= \chi(\mathscr{F}_{n+n_0}) \qquad \text{if } n \gg 0 \\ &= n+n_0. \end{aligned}$$

Thus $n_0 \geq 1$. Putting this together, $n_0=1$, i.e., $h^0(\mathscr{F}_0)=0$ as asserted.

Now to go from Data A′ to Data B′, construct $s_n \in \Gamma(\mathscr{F}_{n+1})$ as above. Notice that $h^1(\mathscr{F}_n) \leq h^1(\mathscr{F}_0)=0$ if $n \geq 0$ so by the argument just given, $\{s_0, \cdots, s_n\}$ is a basis of $\Gamma(\mathscr{F}_{n+1})$, all $n \geq 0$. Therefore $\{s_n\}_{n\geq 0}$ is a basis of $\Gamma(X-P, \mathscr{F})$. Now let $R=\Gamma(X_0-P_0, \mathscr{O}_{X_0})$ and consider the action of R on $\Gamma(X-P, \mathscr{F})$. This is given by:

$$f \cdot s_0 = \sum_{n=0}^{N} a_n(f) \cdot s_n,$$

for all $f \in R$, and suitable $a_n(f) \in k$. Define a map from R to $k\{\sigma\}$ by

$$f \mapsto \sum_{n=0}^{N} a_n(f) \sigma^n.$$

As in the previous cases, it is easy to see that this is an injective homomorphism and the image is an example of Data B′.

Concerning "eigenvalues" of the operators $R \subset k\{\sigma\}$, the corresponding problem is to look for solutions in some extension field $K \supset k$ of equations of the form

$$\sum_{i=0}^{n} a_i \cdot \sigma^i \xi = 0, \qquad a_i \in k.$$

We have the following result:

Proposition. *Let* Data $(X_0, P_0, \{\mathscr{F}_i\}, \alpha)$ *define* $R \subset k\{\sigma\}$ *as above. Let* $K \supset k$ *be a field and* $\sigma: K \to K$ *an extension of* σ *to* K. *Let* $x_0 \in X_0 - P_0$ *be a closed point with ideal* m_{x_0}. *Then there is an isomorphism between*

a) *the* k_0*-vector space of* $\xi \in K$ *such that* $\sum a_i \sigma^i \xi = 0$, *all* $\sum a_i \sigma^i \in R$ *corresponding to functions* $a \in m_{x_0}$

and

b) $\mathrm{Hom}_{(k,\sigma)}(\mathscr{F}/m_{x_0} \cdot \mathscr{F}, K)$.

Proof. Since $\Gamma(X-P, \mathscr{F}) = \oplus_{n=0}^{\infty} k s_n$ is a free $k\{\sigma\}$-module with basis s_0,

$$\begin{aligned} \mathrm{Hom}_{(k,\sigma)}(\mathscr{F}/m_{x_0} \cdot \mathscr{F}, K) &\cong (k, \sigma)\text{-maps } \lambda: \Gamma(X-P, \mathscr{F}) \to K \\ &\qquad \text{with } m_{x_0} \cdot \Gamma(X-P, \mathscr{F}) \subset \operatorname{Ker} \lambda \\ &\cong \text{elements } \xi \in K \text{ killed by } m_{x_0}. \end{aligned}$$

Q.E.D.

To go backwards, start from R. Define

$$R_n = \{x \in R \mid \deg x \leq n\}$$
$$\mathscr{R} = \bigoplus_{n=0}^{\infty} R_n$$
$$X_0 = \mathrm{Proj}\ \mathscr{R}.$$

Note that R is an integral domain and $x \mapsto \deg x$ is a valuation on R. Let s be the g.c.d. of the values $\{\deg x\}$, $x \in R$. For any elements $A_1, A_2 \in R_{ks}$, write them:

$$A_i = a_i \sigma^{ks} + (\text{lower terms}).$$

Using the commutativity of R, it follows that

$$\sigma^s(a_1/a_2) = a_1/a_2.$$

Let $k_2 \subset k$ be the fixed field of σ^s: then k_2 is a Galois extension of k_0 with group $\boldsymbol{Z}/s\boldsymbol{Z}$. Let $k_1 \subset k_2$ be the subfield generated by the ratios a_1/a_2. Then $s = d \cdot r$, and k_1 will be Galois of degree r, for some factorization of s. In particular, for all n:

$$\dim_{k_0} (R_{(n+1)s}/R_{ns}) \leq r$$

with equality for n large. Now let $e \in \mathscr{R}_1$ represent 1 and take some non-constant operator $A \in \mathscr{R}_{\alpha s}$ with $\deg A = \alpha s$. Then as before we see that $\mathscr{R}$ is a finite $k_0[e, A]$-module, hence it is a finitely generated k_0-algebra as well as an integral domain. Thus X_0 is a reduced and irreducible curve proper over $\mathrm{Spec}\, k_0$. The affine open set $e \neq 0$ is just:

$$\begin{pmatrix}\text{open subset} \\ e \neq 0\end{pmatrix} = \mathrm{Spec}\ \mathscr{R}\left[\frac{1}{e}\right]_0$$
$$\cong \mathrm{Spec}\, R$$

and the affine open set $A \neq 0$ is just:

$$\begin{pmatrix}\text{open subset} \\ A \neq 0\end{pmatrix} = \mathrm{Spec}\ \mathscr{R}\left[\frac{1}{A}\right]_0$$
$$\cong \mathrm{Spec}\left\{\begin{matrix}\text{ring of fractions} \\ C/A^k,\ \deg C \leq k\alpha s\end{matrix}\right\}.$$

Since k_0 is algebraically closed in R, it follows that $X \underset{\text{def}}{=} X_0 \times_{k_0} k$ is also irreducible, and, as remarked in the footnote earlier, k is separable over k_0 so X is also reduced. On the other hand the Cartier divisor $e = 0$ on X_0 is given by:

$$\begin{aligned}
V(e) &= \mathrm{Proj}\ (\mathscr{R}/e\mathscr{R}) \\
&= \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} R_n/R_{n-1}\right) \\
&= \mathrm{Proj}\ (\text{subring of } k_1[t] \text{ of finite codim., } \deg t = s) \\
&= \mathrm{Spec}\, k_1.
\end{aligned}$$

Since this is reduced and irreducible, $V(e)$ consists in one regular point P_0, with residue field k_1.

For the next step, we rename the ring $k\{\sigma\}$ as M and regard it as a module over $k \otimes_{k_0} R$: namely, let k act by left multiplication and let R act by right multiplication. Moreover, let

$$M_n = \{x \in M \mid \deg x \leq n\}$$

$$\mathfrak{M} = \bigoplus_{n=0}^{\infty} M_n$$

$$\mathfrak{M}[n] = \mathfrak{M} \text{ with grading shifted by } n \ (\mathfrak{M}[n]_k = \mathfrak{M}_{n+k})$$

$$\mathscr{F}_{n+1} = \widetilde{\mathfrak{M}[n]} \text{ on } X.$$

It is easy to check that $\mathfrak{M}$ is a finitely generated $\mathscr{R}$-module (in fact, it is finitely generated over $k[e, A]$), so all the sheaves $\mathscr{F}_n$ are coherent. Multiplication by $e \in \mathscr{R}$ defines a degree-preserving injection:

$$e : \mathfrak{M}[n] \to \mathfrak{M}[n+1]$$

hence an injection

$$\mathscr{F}_{n+1} \hookrightarrow \mathscr{F}_{n+2},$$

which reduces to an isomorphism on the open set $e \neq 0$, i.e., on $X - P$. Moreover, coker (e) is a graded module, all of whose graded pieces are isomorphic to k, so $\mathscr{F}_{n+2}/\mathscr{F}_{n+1}$ is a sheaf isomorphic to k, hence has length 1. To check $\mathscr{F}_{n-s} \cong \mathscr{F}_n(-P)$, you have to be careful because $\mathscr{F}_n(-P)$ does not correspond to:

$$(\text{Graded ideal of } P) \cdot (\text{Graded module of } \mathscr{F}_n).$$

This is because $\mathscr{R}$ is not generated by elements of degree 1 so X does not carry an invertible sheaf $\mathscr{O}(1)$! You have to take a sufficiently large, sufficiently divisible l. Then working with degrees divisible by l:

$$\begin{aligned}\mathscr{F}_n(-P) &= (\text{graded ideal of } P) \cdot (\text{graded module of } \mathscr{F}_n)^{\sim} \\ &= \left[\bigoplus_{m=1}^{\infty} \operatorname{Im}(R_{ml-1} \to R_{ml})\right] \cdot \left[\bigoplus_{n=0}^{\infty} M_{n+ml}\right]^{\sim}.\end{aligned}$$

But if $s \mid l$, $l \gg 0$, then $R_{l-1} = R_{l-s}$ and $R_{l-s} \cdot M_k = M_{l-s+k}$ for all $k \gg 0$. Thus

$$\begin{aligned}\mathscr{F}_n(-P) &= \left[\bigoplus_{m=1}^{\infty} \operatorname{Im}(R_{ml-s} \to R_{ml})\right] \cdot \left[\bigoplus_{m=0}^{\infty} M_{n+ml}\right]^{\sim} \\ &= \left[\bigoplus_{m=0}^{\infty} R_{l-s} \cdot M_{n+(m-1)l}\right]^{\sim} \\ &= \left[\bigoplus_{m=0}^{\infty} M_{n+ml-s}\right]^{\sim} \\ &= \mathscr{F}_{n-s}.\end{aligned}$$

Now using the dictionary of FAC, we have:

$$(*)\qquad \left.\begin{matrix} M_n \cong \Gamma(X, \mathscr{F}_{n+1}) \\ H^1(X, \mathscr{F}_{n+1}) = (0) \end{matrix}\right\} \quad n \gg 0.$$

Moreover, comparing n and $(n-1)$, if $(*)$ holds for n and $n \geq 0$, then we have:

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \mathscr{F}_n) \arrow[r] & H^0(X, \mathscr{F}_{n+1}) \arrow[r] & H^0(X, \mathscr{F}_{n+1}/\mathscr{F}_n) \arrow[r] & H^1(X, \mathscr{F}_n) \arrow[r] & 0 \\ 0 \arrow[r] & M_{n-1} \arrow[r] \arrow[u, "\alpha"] & M_n \arrow[r] \arrow[u, "\wr"] & k(\sigma^n) \arrow[r] \arrow[u, "\beta"] & 0. \end{tikzcd}$$

Then since $\mathscr{F}_{n+1}/\mathscr{F}_n = (\mathfrak{M}[n]/\mathfrak{M}[n-1])^{\sim}$ and since $\sigma^n \in \mathfrak{M}[n]_0$ generates this module, it follows that β is an isomorphism. Thus the diagram shows that $(*)$ holds for $n-1$ also. Continuing down, it follows eventually that $h^0(\mathscr{F}_0) = h^1(\mathscr{F}_0) = 0$. A Corollary of this is that $\mathscr{F}_0$ is torsion free. Note incidentally that P consists of r distinct regular points, so the sheaves $\mathscr{F}_n$ must be locally free of rank $d = s/r$ in a neighborhood of P. Finally, left multiplication by σ gives a degree preserving map

$$\mathfrak{M}[n] \to \mathfrak{M}[n+1]$$

which is linear with respect to R, σ-semi-linear with respect to k. Thus it defines a homomorphism α as required. Over the affine piece $X-P$, all the $\mathscr{F}_n$ reduce to $\tilde{M}$ and α is again left multiplication by σ. Thus its cokernel is $\widetilde{M/\sigma M}$, which is just the sheaf k sitting in fact at the point of $\operatorname{Spec} R \otimes_{k_0} k$ defined by the ideal

$$\begin{gathered} \operatorname{Ker}\,[R \otimes_{k_0} k \xrightarrow{\phi} k] \\ \phi(a_0 + a_1\sigma + \cdots + a_n\sigma^n) \otimes b = a_0 b. \end{gathered}$$

The most interesting case of this dictionary is when k_0 is the finite field $\boldsymbol{F}_q$, $k \supset k_0$ is any extension, and $\sigma(x) = x^q$. In this case, Data A′ is essentially what Drinfeld calls a "Shtuka" and Data B′ is exactly what he calls an "Elliptic module" [2]. The point is that if $(X_0, P_0, \{\mathscr{F}_n\}, \alpha)$ is an example of Data A′, then the whole tower $\{\mathscr{F}_n\}$ is derived simply from the diagram:

$$(*)\qquad \begin{tikzcd} (1_{X_0} \times \sigma)^* \mathscr{F}_0 \arrow[dr, hook, "\alpha"] & \\ & \mathscr{F}_1 \\ \mathscr{F}_0 \arrow[ur, hook, "\beta"] & \end{tikzcd}$$

(β = given inclusion). In fact, we saw that

$$\alpha(1_{X_0} \times \sigma)^* \mathscr{F}_{-1} = \alpha(1_{X_0} \times \sigma)^* \mathscr{F}_0 \cap \mathscr{F}_0 \qquad \text{(intersection in } \mathscr{F}_1)$$

$$\alpha(1_{X_0}\times\sigma)^*\mathscr{F}_{-2}=\alpha(1_{X_0}\times\sigma)^*\mathscr{F}_{-1}\cap\mathscr{F}_{-1}\qquad\text{(intersection in }\mathscr{F}_1).$$
etc.

Now Drinfeld defines quite generally a *Shtuka* to be a pair of vector bundles $\mathscr{F}_0, \mathscr{F}_1$ on X, plus a diagram like $(*)$ such that $l(\operatorname{coker}\alpha)=l(\operatorname{coker}\beta)=1$. The support of coker (α) is called the zero of the shtuka and the support of coker (β) is called the pole. The shtuka arising from towers $\{\mathscr{F}_n\}$ are easily seen to be characterized by 2 properties:

1) Let $P=\{$the pole and all its conjugates over $\boldsymbol{F}_q\}$. Then the zero is disjoint from P.
2) Restricted to P, $\alpha^{-1}\cdot\beta$ defines a q^{-1}-semi-linear map of the $\boldsymbol{F}_q$-vector space $\mathscr{F}_0/m_P\mathscr{F}_0$ into itself. This map should be nilpotent.

The purpose of the twin tools of elliptic modules and shtuka in Drinfeld's papers is to set up a non-abelian reciprocity law, i.e., prove Langland's conjecture for the field $\boldsymbol{F}_q(X_0)$. I don't want to say anything about this except to indicate why, in the rank one case-Data A and B, the dictionary gives a new method of constructing explicitly the *abelian* extensions of the field $\boldsymbol{F}_q(X_0)$. Let us rephrase the idea of Data A once again, assuming now that $\mathscr{F}$ is an invertible sheaf on X. Let $\operatorname{Pic}^0(X_0)$ be the jacobian of X_0, considered as parametrizing invertible sheaves of degree 0. As usual, map the regular points $(X_0)_{\text{reg}}$ of X_0 to $\operatorname{Pic}^0(X)$ by taking y to the point representing the sheaf $\mathcal{O}_X(y-P_0)$: call this ψ. Let $\sigma\colon \operatorname{Pic}^0(X_0)\to\operatorname{Pic}^0(X_0)$ be the $\boldsymbol{F}_q$-morphism induced by pull-back by σ on sheaves: $\mathscr{F}\mapsto(1_{X_0}\times\sigma)^*\mathscr{F}$. Following Lang [8], we consider the diagram

$$\begin{array}{ccc}(Y_0)_{\text{reg}} & \longrightarrow & \operatorname{Pic}^0(X)\\ \downarrow & & \downarrow{\scriptstyle 1-\sigma}\\ (X_0)_{\text{reg}} & \xrightarrow{\psi} & \operatorname{Pic}^0(X)\end{array}$$

where $(Y_0)_{\text{reg}}$ is the fibre product. Now note

$$\begin{pmatrix}\text{A } k\text{-valued point}\\ y \text{ of } Y_{\text{reg}}\end{pmatrix}=\begin{pmatrix}\text{A pair }(\mathscr{F},x),\ \mathscr{F}\text{ an invertible sheaf}\\ \text{on } X \text{ of degre } 0,\ x\in X_{\text{reg}} \text{ such that}\\ (1-\sigma)([\mathscr{F}])=\psi(x)\end{pmatrix}$$
$$=\begin{pmatrix}\text{A pair }(\mathscr{F},x)\text{ where}\\ \mathscr{F}\otimes(1_{X_0}\times\sigma)^*\mathscr{F}^{-1}\cong\mathcal{O}_X(x-P_0)\end{pmatrix}.$$

Associating the sheaf $\mathscr{G}=\mathscr{F}((g-1)P_0)$ to $\mathscr{F}$, we carry the identification further:

$$=\begin{pmatrix}\text{A pair }(\mathscr{G},x),\ \mathscr{G}\text{ of degree } g-1,\ x\in X_{\text{reg}}\text{ such that}\\ (1_{X_0}\times\sigma)^*\mathscr{G}\cong\mathscr{G}(P_0-x)\end{pmatrix}.$$

Thus $(Y_0)_{\text{reg}}$ is the scheme classifying all possible examples of Data A′. But Lang's geometric class field theory states that the curve Y_0 is the maximal abelian covering of X_0, such that (1) it is unramified over $(X_0)_{\mathrm{reg}}$, (2) with certain bounds on the ramification over the singular points (these bounds getting as weak as you wish as the points get more singular) and (3) with no residue field extension over P_0. The dictionary now states that $(Y_0)_{\mathrm{reg}}$ is equally the scheme classifying all possible examples of Data B. But these are readily described by equations: write

$$\Gamma(X_0-P_0, \mathcal{O}_{X_0})\cong \boldsymbol{F}_q[Z_1, \cdots, Z_n]/(f_1, \cdots, f_k).$$

Let $\bar{Z}_i$ have a pole of order n_i at P_0. Then Data B is given by assigning:

$$Z_i \mapsto A_i = \sum_{j=0}^{n_i} a_{ij}\sigma^j$$

where $A_i \in k\{\sigma\}$ satisfy

$$(*)\qquad \begin{aligned} &A_iA_j=A_jA_i\\ &f_k(A_1, \cdots, A_n)=0\\ &a_{i,n_i}\neq 0. \end{aligned}$$

We may normalize this mod $R\sim aRa^{-1}$, e.g., by picking $\bar{Z}_1, \bar{Z}_2$ so that $(n_1, n_2)=1$ and $\bar{Z}_1^{n_2}/\bar{Z}_2^{n_1}(P_0)=1$, and then requiring

$$a_{1,n_1}=a_{2,n_2}=1.$$

Then the equations $(*)$, written out as equations in the a_{ij} define a scheme over k_0, which is precisely the affine piece of the abelian covering Y_0 over $(X_0)_{\mathrm{reg}}-P_0$. This is one of the simpler results in Drinfeld's extraordinary paper [2].

Appendix (added on Oct. 15, 1977)

Professor E. Coddington has kindly given me references to 3 very beautiful papers of J. L. Burchnall and T. W. Chaundy, all entitled "Commutative Ordinary Differential Operators", which appear in

Proceedings London Mathematical Society 21 (1922), p. 420
Proceedings Royal Society London (A), 118 (1928), p. 557
Proceedings Royal Society London (A), 134 (1931), p. 471.

It appears that virtually all the results described in § 2 are in fact due to them: in particular the correspondence given in the introduction between Data A and B for differential operators was established by them. Even more remarkably, they even recognized the fact that when the curve X has singularities, there are several classes of commutative rings of operators in Data B corresponding to which strata of the compactified Pic (X) $\mathscr{F}$ lies in: see their 3rd paper where the case of the curve X given by $x^m=y^n$, $(m, n)=1$, is analyzed at length. In their papers, and in a note by H. F. Baker following their 2nd paper, the explicit construction of these operators via theta-functions and related abelian functions is given in detail. The one point they do not explore is the infinitesimal deformation of a pair of commuting operators hence they were not led to a Lax equation or to the link with the Korteweg-deVries equation. Instead they discuss at length a procedure for relating 2 rings $R_1, R_2 \subset C[[t]][d/dt]$ namely via an auxiliary operator T such that: $R_1 \circ T = T \circ R_2$. It would seem that once this link is made, their work anticipates a large amount of the recent work on degenerate-spectrum Sturm-Liouville operators and exact solutions of the Korteweg-deVries equation.

Added in proof. 1) Krichever's work had been anticipated in some nearly forgotten papers of Burchnall and Chaundy in the 20's—cf. Appendix. Also at this point I would like to thank D. Kajdan for introducing me to these ideas and sharing his many insights. 2) cf. H. McKean, *Theta functions, Solitons, and Singular Curves*, to appear.

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