Inventiones math. 35, 197–232 (1976)

Inventiones mathematicae

Enriques' Classification of Surfaces in Char. *p*, III

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Introduction

This paper continues the extension to char. p of Enriques' classification of algebraic surfaces, begun in two previous papers [7, 2] and in particular deals with the special phenomena of char. 2 and 3. We have already seen that all surfaces can be divided into four classes by their "Kodaira dimension" κ:

$$\kappa = \operatorname{tr} \deg_k \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nK_X)) - 1$$

and that

a) $\kappa = -1 \Rightarrow X$ ruled,

b) $\kappa = \quad 0 \Rightarrow 4K_X \equiv 0$ or $6K_X \equiv 0$

c) $\kappa = \quad 1 \Rightarrow |nK_X|$, n large, is composite with a pencil $\pi\colon X \to B$, making X elliptic or, in char. 2 or 3, possibly quasi-elliptic,

d) $\kappa = 2 \Rightarrow |nK_X|$, n large, defines a birational morphism onto a model X_0 with rational double points; the surface X is of "general type".

Moreover, the surfaces with $\kappa = 0$ are divided into four classes:

$$\begin{array}{lll} B_1 = 0, & B_2 = 22, & \chi(\mathcal{O}_X) = 2, \\ B_1 = 0, & B_2 = 10, & \chi(\mathcal{O}_X) = 1, \\ B_1 = 4, & B_2 = 6, & \chi(\mathcal{O}_X) = 0, \\ B_1 = 2, & B_2 = 2, & \chi(\mathcal{O}_X) = 0. \end{array}$$

We have seen in Part II [2] that the third class consists in the abelian varieties of dimension 2. The first class we call K3-surfaces, the second class Enriques' surfaces. In the fourth class, the Albanese variety is an elliptic curve E and the Albanese mapping

$$\pi\colon X \to E$$

has either all fibres non-singular elliptic or rational with a cusp. These surfaces we call hyperelliptic or quasi-hyperelliptic depending on the fibres of π. In [2], we have classified the hyperelliptic surfaces too.

The goals of the present paper are to analyze Enriques' surfaces and quasi-hyperelliptic surfaces. This is largely a question of the "pathologies" of characteristic 2 and 3, since quasi-hyperelliptic surfaces only exist in this case and if char. $\neq 2$ then we also saw in [2] that Enriques surfaces had the property

$$K \not\equiv 0, \quad 2K \equiv 0.$$

Such surfaces were investigated by the unpublished thesis of M. Artin, where he showed that Enriques' classical results in char. 0 extended to all finite characteristics $p \neq 2$. However, in char. 2, we shall see that Enriques surfaces are quite varied! Here is an outline of the paper: in § 1, we shall study the formal geometry of "cuspidal fibrations", i.e., fibrations like those of quasi-elliptic surfaces. In § 2, we shall give a classification of quasi-hyperelliptic surfaces. In § 3, we look at Enriques' surfaces and we see that there are three types in char. 2 and we give examples of each type. In § 4, we shall show that every Enriques surface is elliptic or quasi-elliptic and we shall prove that $\rho = B_2$ for such surfaces.

The study of special low characteristics can be one of two types: amusing or tedious. It all depends on whether the peculiarities encountered are felt to be meaningful variations of the general picture fitting in with standard principles, such as the failure of Sard's lemma, the loss of roots of unity, etc., or are felt instead to be accidental and random, due for instance to numerological interactions between combinations of exponents and the characteristic, of interest only for the sake of having uniform theorems applicable in all characteristics. Our hope is that the geometric aspects of char. 2 and 3 studied here are not entirely of the latter kind. For example, we find quite striking the following points, which we list in the hope that they will supply some motivation for taking the plunge into char. 2 and 3:

1) that a smooth surface X fibred in curves with cusps in char. 2 defines canonically three usually distinct families of subspaces $L_P^{(i)} \subset T_{X,P}$ for all cusps P: cf. Figure 2, § 1.

2) the structure of the automorphism groupscheme of the rational line with a cusp at ∞, and the orbit of ∞ itself under this group: cf. § 2.

3) the fact that there are three types of Enriques' surfaces in char. 2, whose canonical double covering has structure group μ_2, $\mathbb{Z}/2\mathbb{Z}$, α_2, and which are at least sometimes deformable into each other: cf. § 3.

4) the intricate way in which Enriques' argument about the existence of reducible divisors in linear systems on Enriques' surfaces must be adapted in char. 2: cf. § 4.

5) the fact that isolated "pathological examples" (e.g. Igusa's surfaces) become quite natural when viewed in a broader perspective: cf. § 2.

§ 1. Remarks on the Differential Geometry of Cuspidal Fibres

In this section we study the germ (in the formal power series sense) of a general map $f\colon X \to B$, where X is a non-singular surface, B a non-singular curve, such that every fibre of f has a cusp. More precisely, if t (resp. x, y) are local coordinates on B (resp. X), then f is given by a formal power series $t = f(x, y)$, and we seek to understand

a) whether f can be put in normal form by suitable choices of the uniformizing parameters,

b) what invariants can be defined,

c) what is the differential geometry underlying the failure of Sard's lemma.

Starting with $f\colon X \to B$, let

$$\Sigma = \{P \in X \mid f \text{ is not smooth at } P\}$$

and let Σ_0 be a one-dimensional locally closed subset of Σ such that at each point $P \in \Sigma_0$, $f^{-1}(f(P))$ has an ordinary cusp at P.

Proposition 1. *We have*

(a) Σ_0 *is an étale cover of* $B^{(p^{-1})}$, $\quad p = \operatorname{char}(k)$.

(b) $(\Sigma_0 \cdot f^{-1}(f(P)))_P = \operatorname{char}(k)$ *for all* $P \in \Sigma_0$.

Proof. Let t be a local coordinate on B at $f(P)$ and let x, y be local coordinates on X at P. We can choose x, y such that

$$t = (\text{unit})(x^2 + y^3);$$

if $\operatorname{char}(k) \neq 2$, then $\dfrac{\partial t}{\partial x}$ vanishes to order 1 and $\dfrac{\partial t}{\partial x} = 0$ defines a germ of a curve non-singular at P and passing through the nearby cusps, i.e. $\dfrac{\partial t}{\partial x} = 0$ is a local equation for Σ_0. If instead $\operatorname{char}(k) = 2$, a similar argument shows that

$$\frac{\partial t}{\partial y} = (\text{unit}) \cdot \sigma^2$$

where $\sigma = 0$ is a local equation for Σ_0, and σ vanishes to exact order 1 at P, i.e. Σ_0 is non-singular. Now non-singular branches and ordinary cusps can meet only with multiplicities 2 or 3, hence

$$\dim(\mathcal{O}_{\Sigma_0, P}/\mathfrak{m}_{B, f(P)} \cdot \mathcal{O}_{\Sigma_0, P}) = (\Sigma_0 \cdot f^{-1}(f(P)))_P = 2 \quad \text{or} \quad 3.$$

On the other hand, $\Sigma_0 \to B$ factorizes as $\Sigma_0 \xrightarrow{\tau} B^{(p^{-n})} \to B$ for some n, where τ is separable and now

$$\dim(\mathcal{O}_{\Sigma_0, P}/\mathfrak{m}_{B, f(P)} \cdot \mathcal{O}_{\Sigma_0, P}) \begin{cases} = p^n & \textit{generically} \\ > p^n & \textit{if } \tau \textit{ ramifies over } \tau(P) \end{cases}$$

This proves Proposition 1, plus the old result of Tate [13] that such a line of cusps is only possible in characteristic 2 or 3. Q.E.D.

Now it is convenient to treat separately the cases of characteristic 2 and 3, which are very different. We consider first the simpler case $\mathrm{char}(k)=3$.

We begin by finding a normal form for f. We have

Proposition 2. *Let* $\mathrm{char}(k)=3$ *and choose any local coordinate* t *on* B *at* $f(P)$. *Then in suitable power series coordinates* x, y *on* X *at* P, *we have*

$$t=x^2+y^3.$$

Proof. Choose x, y so that $x=0$ is a local equation for Σ_0. Then if $t=f(x, y)$ we have that x divides both $\frac{\partial t}{\partial x}$ and $\frac{\partial t}{\partial y}$, whence we can write t in the form

$$t=g(y)^3+x^2 r(x, y)$$

and since $t=0$ has an ordinary cusp at (0, 0) we have also $(dg/dy)(0)\neq 0$, $r(0,0)\neq 0$. If we change coordinates via

$$\tilde{y}=g(y), \qquad \tilde{x}=x\sqrt{r(x, y)},$$

t has the required form. Q.E.D.

By Proposition 2, we see that all such fibrations in $\mathrm{char}(k)=3$ are locally formally isomorphic. We can go a bit further and prove that they arise from smooth fibrations. Let us define

$$X^* = \textit{normalization of } X\times_B B^{(1/3)};$$

choosing x, y, t as in Proposition 2 then $s=t^{1/3}$ is a local coordinate on $B^{(1/3)}$ and $X\times_B B^{(1/3)}$ is given locally by $s^3=x^2+y^3$, i.e. $(s-y)^3=x^2$. If we put $r=\frac{x}{s-y}$ then $r^3=x$, so r is integral over $k[[x, y, s]]$ and r, s are local coordinates on X which is therefore non-singular along Σ_0.

Thus we have a diagram

$$\begin{tikzcd} \Sigma_0 \arrow[r, hook] \arrow[d, "\varphi"] & X^* \arrow[r, "\pi"] \arrow[d, "f^*"] & X \arrow[d, "f"] \\ B^{(1/3)} \arrow[r, equal] & B^{(1/3)} \arrow[r, "F"] & B \end{tikzcd}$$

where φ is étale, f^* is smooth along Σ_0, f has ordinary cusps along Σ_0, and F is the Frobenius morphism.

Since the map π is given by $x=r^3$, $y=s-r^2$ we have that if L is the line sub-bundle of T_{X^*} generated locally by $\frac{\partial}{\partial r}+2r\frac{\partial}{\partial s}$, then interpreting the sections of L as derivations of $\mathcal{O}_{X^*}$, we have:

$$\mathcal{O}_X=\ker(L\colon \mathcal{O}_{X^*}\to\mathcal{O}_{X^*});$$

the map π is purely inseparable of degree 3, and is said, in the theory of purely inseparable descent, to be the result of "dividing by the rank 1 distribution L".

We may make a "picture" of the map $t=x^2+y^3$ in char. 3. Notice that the line of cusps $x=0$ meets each fibre $t_0=x^2+y^3$ at its cusp with intersection multiplicity 3 and thus has the same tangent cone at this intersection. Thus we have:

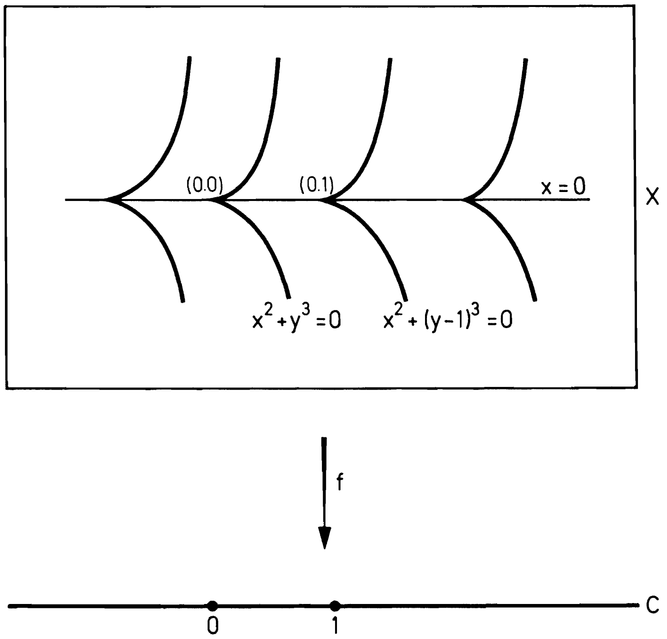

Fig. 1

The case of characteristic 2 is considerably more elaborate. The direct approach to putting the power series $t=f(x, y)$ in normal form does not work. The reason is that the local structure of the map along the line of cusps defines an invariant, which is alternatively a differential η on the line of cusps Σ_0 itself, or a differential ω on the curve $\Sigma_0^{(p)}$ which is the étale covering of the base B which parametrizes the set of cusps in the fibres (see Prop. 1): these are related by the usual p-linear isomorphism between differentials on a curve Y and those on $Y^{(p)}$. Because of this invariant, one must first choose t so that ω is in some standard form and, in fact, there are an infinite number of cases according to the multiplicity of zero of η at $t=0$. The most direct way to see why such an invariant exists is to examine the versal deformation of the cusp $y^2+x^3=0$. One calculates immediately that this definition is given in different characteristics by:

$$y^2+x^3+t_1+t_2x=0 \qquad \text{(char.} \neq 2, 3),$$
$$y^2+x^3+t_1+t_2x+t_3x^2=0 \qquad \text{(char.} =3),$$
$$y^2+x^3+t_1+t_2x+t_3y+t_4xy=0 \qquad \text{(char.} =2).$$

Inside the parameter space (t_1, t_2) (resp. (t_1, t_2, t_3) or $(t_1, \ldots, t_4)$) the set of points where the fibre still has a cusp is given by:

$$t_1 = t_2 = 0 \quad \text{only (char.} \neq 2, 3)$$
$$t_2 = t_3 = 0, \quad t_1 \text{ arbitrary (char.} = 3)$$
$$t_3 = t_4 = 0, \quad t_1, t_2 \text{ arbitrary (char.} = 2).$$

It follows that if

$$f \colon X \to B$$

is any formal family of curves with cusps, $B = \mathrm{Spec}(\mathcal{O})$, $\mathcal{O}$ an equicharacteristic complete local ring, then X is given by the complete local ring:

$$\mathcal{O}[[x, y]]/(y^2 + x^3) \qquad \text{(char.} \neq 2, 3),$$
$$\mathcal{O}[[x, y]]/(y^2 + x^3 + a), \qquad \text{some } a \in \mathfrak{m} \text{ (char.} = 3),$$
$$\mathcal{O}[[x, y]]/(y^2 + x^3 + a + bx), \qquad \text{some } a, b \in \mathfrak{m} \text{ (char.} = 2).$$

This at first suggests that $a \in \mathfrak{m}$ (char. $= 3$) and $a, b \in \mathfrak{m}$ (char. $= 2$) are invariants of the map f. However this presentation of f is not unique for 2 reasons: 1st there are many ways to identify the fibre $f^{-1}(0)$ with $\mathrm{Spec}\, k[[x, y]]/(y^2 + x^3)$, and 2nd, after such an identification is chosen, the t_i-space does not *represent* the deformation functor but is merely its versal hull. At least we can say that a (or a and b) in $\mathfrak{m}/\mathfrak{m}^2$ depend only on the identification of $f^{-1}(0)$ with $y^2 + x^3 = 0$. For example, for any $\lambda \in \mathcal{O}^*$, the two curves

$$y^2 + x^3 + \lambda^6 a + (\lambda^4 b) x = 0,$$
$$y^2 + x^3 + a + b \cdot x = 0$$

are isomorphic via $y \mapsto \lambda^3 y$, $x \mapsto \lambda^2 x$. Thus, in char. 3, a can be replaced by $\lambda^6 a$ for any $\lambda \in \mathcal{O}^*$ and there is not much useful information that can be extracted from a. However, in char. 2, b^3/a^2 is at least invariant under this substitution.

Now assume $S = \mathrm{Spec}\, k[[t]]$, and X is regular. Then $a = a_1 t + \cdots$, $a_1 \neq 0$, hence $b^3/a^2 \in \mathfrak{m}$. We define the value of the invariant ω at the point $t = 0$ to be:

$$\omega|_{t=0} = d(b^3/a^2)|_{t=0}; \quad \text{i.e.,} \quad = b^3/a^2 \in (\mathfrak{m}/\mathfrak{m}^2).$$

Why is this independent of the choice of representation of f via a and b? Note that it only depends on the values of a and b mod $\mathfrak{m}^2$. Moreover, we claim that *all* changes in the formal isomorphism of $f^{-1}(0)$ with $y^2 + x^3 = 0$ leave b^3/a^2 mod $\mathfrak{m}^2$ fixed. This is most easily seen if we recall that the tangent space to the versal deformation space of $g = 0$ is intrinsically described as:

$$T = \mathrm{Hom}(g \cdot k[[x, y]], k[[x, y]]/(g, g_x, g_y)).$$

The automorphisms of $y^2 + x^3 = 0$ are generated by

$$x \mapsto x + \lambda^2 x^n,$$
$$y \mapsto \begin{cases} y + \lambda x^{m+1} + \lambda^2 x^{2m-1} y + \lambda^3 x^{3m}, & n = 2m \\ y + \lambda y x^m + \lambda^2 y x^{2m} + \lambda^3 y x^{3m}, & n = 2m + 1 \end{cases}$$

all $n \geqq 2$, which induce

$$y^2+x^3 \mapsto \begin{cases} (y^2+x^3)(1+\lambda^4 x^{4m-2}), & n=2m \\ (y^2+x^3)(1+\lambda^2 x^{2m})^3, & n=2m+1. \end{cases}$$

In our case

$$k[[x,y]]/(g, g_x, g_y) \cong k \cdot 1 + k \cdot x + k \cdot y + k \cdot xy;$$

thus these automorphisms induce the identity on T for $n \geqq 4$, and for $n=2$ and 3, they still induce the identity on the subspace

$$T_0 = \begin{bmatrix} \text{set of homomorphisms } \phi \text{ such that} \\ \phi(y^2+x^3) \in k \cdot 1 + k \cdot x \end{bmatrix}$$

where our deformations lie. This proves that $(a, b) \bmod \mathfrak{m}^2$ can only be changed to $(\lambda^6 a, \lambda^4 b) \bmod \mathfrak{m}^2$, hence $b^3/a^2 \bmod \mathfrak{m}^2$ is independent of all choices.

Still assuming $S = \operatorname{Spec} k[[t]]$, X regular, we can extend our definition of $\omega|_{t=0} \in \mathfrak{m}/\mathfrak{m}^2$ to a definition of $\omega \in \Omega^1_{\mathcal{O}/k}$. First make a base change to K = algebraic closure of $k((s))$, and then note that the substitution

$$\begin{aligned} & t+s=\tilde{t}, \\ & y+\sqrt{a(s)}+\sqrt[4]{b(s)} \cdot (x+\sqrt{b(s)})=\tilde{y}, \\ & x+\sqrt{b(s)}=\tilde{x} \end{aligned}$$

carries our surface

$$0=y^2+x^3+a(t)+b(t) \cdot x$$

into a new surface

$$0=\tilde{y}^2+\tilde{x}^3+\tilde{a}(\tilde{t})+\tilde{b}(\tilde{t}) \cdot \tilde{x}$$

so that the cusp $\tilde{x}=\tilde{y}=\tilde{t}=0$ on the new surface is the image of the generic cusp on the original surface. One now calculates in a page or so that if we define ω for the original surface everywhere by:

$$\omega = \frac{\dot{b}^3}{\dot{a}^2 + b \cdot \dot{b}^2}\, dt = d(\log(\dot{a}^2 + b \cdot \dot{b}^2)) \tag{$*$}$$

(Notation: $\dot{c} = dc/dt$)

then the translation $\tilde{t}=t+s$ carries the generic value of this ω to the value $\omega|_{\tilde{t}=0}$ for the new surface. Thus $(*)$ is also independent of the choice of a, b to represent f. Noting that $\Omega^1_{\mathcal{O}/k}$ is locally free of rank 1, one also has the more intrinsic formula:

$$\omega = \frac{(db)^{\otimes 3}}{da^{\otimes 2} + b \cdot (db)^{\otimes 2}}.$$

This formula is very suggestive of the usual formula for the j-invariant of the elliptic curve

$$0 = y^2 + x^3 + a + bx$$

in char. $\neq 2, 3$. This suggests strongly that if our map f arises by specializing an elliptic fibration from char. 0, with the line of cusps "swallowing" the elliptic curves, then ω is likely to be a "ghost" of the j-invariant of that elliptic fibration. This is a good question to study.

Let's return to the algebraic setting of the beginning of this section:

$$f: X \to B,$$

where X is a smooth algebraic surface, B a smooth algebraic curve and $\Sigma_0 \subset X$ is a line of cusps. In this case, we get a closely related invariant:

$$\eta \in \Gamma(\Sigma_0, \Omega^1_{\Sigma_0}).$$

To define η, we proceed as follows:

i) let $\sigma = 0$ be a local equation of Σ_0 at a point $P \in \Sigma_0$,

ii) choose a local coordinate t near $f(P) \in B$

iii) then as we saw in the proof of Proposition 1,

$$dt = \sigma^2 \cdot \alpha$$

for some differential α on X, regular near P.

iv) Let C be Cartier's operator on differentials. Since $\sigma\, C(\alpha) = C(\sigma^2 \alpha) = C(dt) = 0$, it follows $C(\alpha) = 0$, hence $\alpha = dx$ for some function x defined near P.

v) Now $d(t + \sigma^2 x) = dt + \sigma^2 \cdot dx = 0$, so

$$t = y^2 + \sigma^2 x \tag{**}$$

for some function y defined near P.

vi) Then the fact that $t = 0$ has an ordinary cusp at P implies easily that $x, y \in \mathfrak{m}_p/\mathfrak{m}_p^2$ are independent, hence x, y are local coordinates on X near P.

vii) Let $\frac{\partial}{\partial y}$ be the derivation of $\mathcal{O}_X$ near P such that $\frac{\partial}{\partial y}(x) = 0$, $\frac{\partial}{\partial y}(y) = 1$.

Define

$$\eta = \left. \frac{\left(\frac{\partial \sigma}{\partial y}\right)^2 dx}{x \left(\frac{\partial \sigma}{\partial y}\right)^2 + 1} \right|_{\Sigma_0} = d\left(\log\left(x\left(\frac{\partial \sigma}{\partial y}\right)^2 + 1\right)\right)\Bigg|_{\Sigma_0}$$

To see that η is independent of all choices (i.e., the choice of σ, t and x), we relate it to the previously defined ω. We recall that for any smooth curve C, there is a natural p-linear isomorphism between the differentials on C and those on $C^{(p)}$ given by

$$a \cdot db \leftrightarrow a^p \cdot d(b^p),$$
$$a, b \in \mathcal{O}_C;\ a^p, b^p \in \mathcal{O}_{C^{(p)}}.$$

If $\omega = a \cdot db$, write $\omega^{(p)} = a^p \cdot d(b^p)$. Then we have:

Proposition 3. *Let*

$$f\colon \hat{X} \to \hat{B}$$

be the formal completion of f near P. By Proposition 1, f sets up an isomorphism

$$B \cong (\Sigma_0)^{(2)},$$

and under this isomorphism,

$$\omega = \eta^{(2)}. \tag{***}$$

Proof. It suffices to check this at one point. Starting with (**), let $\sigma = \sigma_1 x + \sigma_2 y$ where $\sigma_i \in \mathcal{O}_{X,P}$. Note that $\sigma_1(P) \neq 0$ since $y^2 + \sigma^2 x = 0$ has an ordinary cusp at the origin. Then if we set

$$\tilde{x} = x \cdot \sqrt[3]{\frac{\sigma_1^2}{1 + x\sigma_2^2}},$$

is easily rewritten: (**)

$$0 = y^2 + \tilde{x}^3 + t\left(1 + \tilde{x}\sigma_2^2 \cdot \sqrt[3]{\frac{1 + x\sigma_2^2}{\sigma_1^2}}\right)^{-1}$$
$$= y^2 + \tilde{x}^3 + t + \frac{t\sigma_2^2}{\sigma_1^{2/3}} \cdot \tilde{x} + t\,\phi(y, \tilde{x}) \cdot \tilde{x}^2.$$

Therefore, to 1st order in t, this deformation is given by the homomorphism

$$y^2 + \tilde{x}^3 \mapsto t + \frac{t\,\sigma_2(0,0)^2}{\sigma_1(0,0)^{2/3}} \cdot \tilde{x} \in (k[[\tilde{x}, y]]/(\tilde{x}^2, y^2)).$$

Thus

$$\omega(0) = \frac{\sigma_2(0,0)^6}{\sigma_1(0,0)^2}\, dt|_{t=0}.$$

But now on Σ_0, at $(0,0)$, $\sigma_1(0,0)\,dx = \sigma_2(0,0)\,dy$, hence

$$\eta(0) = ((\sigma_2(0,0)^2\, dx|_{\sigma=0})|_{x=y=0}$$
$$= \left(\frac{\sigma_2(0,0)^3}{\sigma_1(0,0)}\, dy|_{\sigma=0}\right)\Big|_{x=y=0}$$

On Σ_0, the map f is given by $t = y^2$, hence $\eta^{(2)}(0)$ is exactly $\omega(0)$ as claimed. Q.E.D.

Corollary. *Suppose $f\colon X \to B$ is a quasi-elliptic surface,*

$$S = \{P \in B \mid f^{-1}(P) \text{ is reducible or multiple}\} \quad \text{and} \quad \Sigma_0 \subset X - f^{-1}(S)$$

is the line of cusps. Then the formal differential ω is actually an element $\Gamma(B - S, \Omega_B^1)$.

Proof. In this case, f defines an isomorphism of $\Sigma_0^{(2)}$ and $B - S$.

In the classification of quasi-elliptic surfaces in char. 2, this differential ω may play a role similar to the j-invariant in the classification of elliptic surfaces.

Using this invariant, we can now put cuspidal fibrations in normal forms. Recall that if C is Cartier's operator, and B is a non-singular curve, then C acts on the differentials on B and

i) $C\omega=0 \Leftrightarrow \omega=df$, some f,
ii) $C\omega=\omega \Leftrightarrow \omega=df/f$, some f.

From (ii), it is quite easy to deduce:

(iii) ω is regular at $P\in B$ and $C\omega=\omega$, then $\mathrm{ord}_0\,\omega=2k$ for some k, and for some formal parameter t on B at P,

$$\omega=d(\log(1+t^{2k+1}))=\frac{t^{2k}}{1+t^{2k+1}}\,dt.$$

Our normal form Theorem is this:

Proposition 4. *Let $f\colon X\to S=\mathrm{Spec}\,k[[t]]$ be a formal map of a non-singular surface X onto S and assume that all the fibres have ordinary cusps. Let ω be the invariant on S defined above. Then for some integer k, $\mathrm{ord}_0\,\omega=6k$, ($k=\infty$ if $\omega\equiv 0$). If k is finite, let t' be a new parameter on S such that*

$$\omega=d(\log(1+t'^{\,6k+1})),$$

then for suitable coordinates x, y on X, f is given by:

$$t'=y^2+x^2\cdot(x+y^{2k+1}),\quad \text{if } k \text{ is finite},$$
$$t'=y^2+x^3,\qquad \text{if } k \text{ is infinite}.$$

Proof. Since $\dfrac{\partial\sigma}{\partial x}(0)\neq 0$, we may take y and σ to be coordinates on X replacing σ by z, x by $\tau(y,z)$, the map is given by:

$$t=y^2+z^2\cdot\tau. \tag{$*$}$$

It is a simple calculation to show that η is now given by

$$\eta=d\cdot\log\left(\tau\left(\frac{\partial\tau}{\partial y}\right)^2+\left(\frac{\partial\tau}{\partial x}\right)^2\right)\Bigg|_{z=0}.$$

First, suppose $\omega\equiv 0$. Then $\eta\equiv 0$, hence

$$\left(\frac{\partial\tau}{\partial y}\right)^3(y,0)\cdot dy\equiv 0.$$

But this means that

$$\tau(y,z)=\tau_1(y)^2+z\cdot\tau_2(y,z).$$

Note that $\tau_2(0,0)\neq 0$. Substituting this in ($*$) and letting $\tilde{y}=y+z\tau_1(y)$, $\tilde{z}=z\sqrt[3]{\tau_2(\eta,z)}$ the map takes the form

$$t=\tilde{y}^2+\tilde{z}^3.$$

Secondly, suppose $\omega \not\equiv 0$. Let $\omega = d \cdot \log(1 + t'^{2l+1})$. To simplify notation, replace t by t' and choose new y, z, τ so that $t = y^2 + z^2 \tau$ still. Then on Σ_0, $y = \sqrt{t}$, so $\eta = d \cdot \log(1 + y^{2l+1})$. It follows that

$$d \log \left(\frac{\tau \left(\frac{\partial \tau}{\partial y}\right)^2 + \left(\frac{\partial \tau}{\partial z}\right)^2}{1 + y^{2l+1}} \right) \Bigg|_{z=0} \equiv 0,$$

hence

$$\frac{\tau \left(\frac{\partial \tau}{\partial y}\right)^2 + \left(\frac{\partial \tau}{\partial z}\right)^2}{1 + y^{2l+1}} = \phi(y)^2 + z\,\psi(y, z).$$

But now expand τ:

$$\tau(y, z) = \tau_0(y)^2 + y \cdot \tau_1(y)^2 + z \cdot \tau_2(y) + z^2 \cdot \tau_3(y, z).$$

Note that $\tau_2(0) \neq 0$. Substituting, we find:

$$(\tau_0^2 + y\,\tau_1^2)(\tau_1^4) + \tau_2^2 = (1 + y^{2l+1})\,\phi^2$$

hence

$$\begin{aligned} &\tau_0\,\tau_1^2 + \tau_2 = \phi \\ &\qquad \tau_1^3 = y^l \cdot \phi. \end{aligned}$$

Here, if $l > 0$, then $y \mid \tau_1$, hence $\tau_1(0) = 0$, hence from the first equation $\phi(0) \neq 0$. Thus l is the exact power of y dividing τ_1^3, and so $l = 3k$. Write $\tau_1 = y^k \cdot \tilde{\tau}_1$, so that $\tilde{\tau}_1^3 = \phi = \tau_2 + \tau_0\,\tau_1^2$. Substituting into $t = y^2 + z^2\,\tau$, we get:

$$t = (y + \tau_0\,z)^2 + z^2\,\tilde{\tau}_1^2 (y^{2k+1} + z\,y^{2k}\,\tau_0 + z\,\tilde{\tau}_1) + z^4 \cdot t_3(y, z).$$

Letting $\tilde{y} = y + \tau_0\,z$, $\tilde{z} = z\,\tilde{\tau}_1$, this becomes

$$t = \tilde{y}^2 + \tilde{z}^2 (\tilde{y}^{2k+1} + \tilde{z}) + \tilde{z}^4 \cdot \tilde{\tau}_3(\tilde{y}, \tilde{z}),$$

which is exactly what we want except for the $\tilde{z}^4$-term. It is easy to check that a suitable substitution $\tilde{y} \mapsto \tilde{y}$, $\tilde{z} \mapsto \tilde{z} + \tilde{z}^2 f(\tilde{y} \cdot \tilde{z})$ will get rid of this $\tilde{z}^4$-term. Q.E.D.

There is a rather beautiful geometric interpretation of the invariant η. First of all, in char. 2, the tangent line to the curve of cusps Σ_0 is always *transversal* to the tangent line to each curve $f^{-1}(P)$ at its cusp, rather than being equal as in char. 3. In fact, if we write f by

$$t = y^2 + \sigma^2 \cdot x,$$

then $y = 0$ is the tangent line to the curve $f^{-1}(0)$ at the cusp $(0, 0)$ and $\sigma = 0$ is the local equation of Σ_0; and in the cotangent space to X at $(0, 0)$, dy and $d\sigma$ are always independent. Secondly, suppose we consider the tangent line to the fibre $f^{-1}(0)$ at points Q besides $(0, 0)$:

$$T_{Q, f^{-1}(0)} \subset T_{Q, X}$$

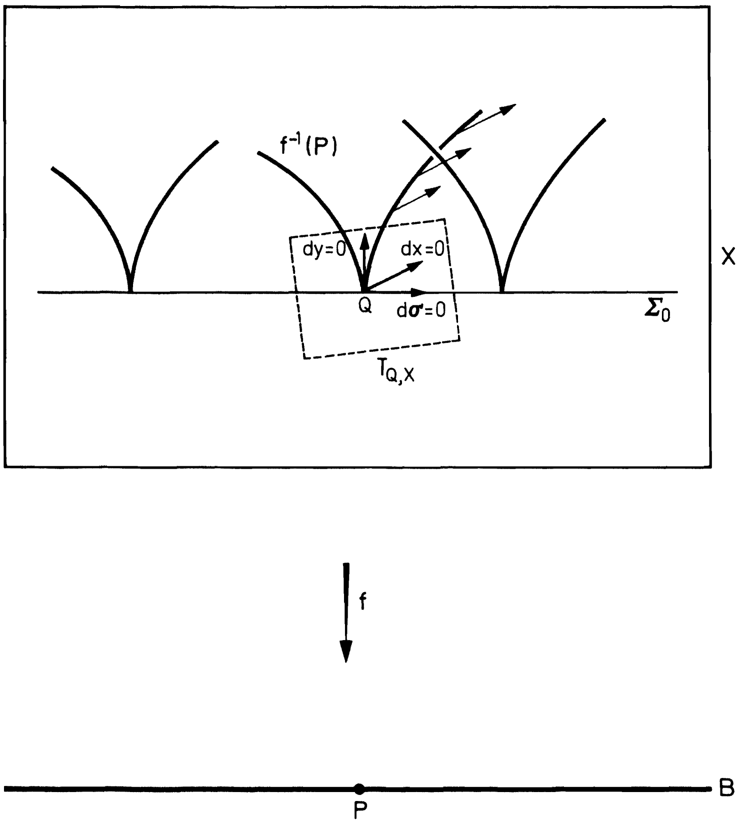

Fig. 2

and then take the limit of $T_{Q, f^{-1}(0)}$ as $Q \to (0,0)$. $T_{Q, f^{-1}(0)}$ is the subspace of $T_{Q,X}$ defined by $dt=0$. Since $dt=\sigma^2 \cdot dx$, this is the same as the subspace defined by $dx=0$ and as $dx|_{(0,0)}$ is still non-zero, this limit is the subspace $dx|_{(0,0)}=0$. In general, this is a 3rd one-dimensional subspace of $T_{(0,0),X}$!

Thus at each point if Σ_0, the cuspidal fibration $f: X \to B$ determines canonically the 3 tangent directions $dy=0$, $d\sigma=0$ and $dx=0$. Here $dy=0$ is always distinct from $d\sigma=0$ and $dx=0$ (because we have an ordinary cusp) but $d\sigma=0$ and $dx=0$ may be equal. We get the following curious picture: Here is the geometric interpretation of η: look at the differential $f^*\left(\frac{dt}{t}\right)$ on X. It has a simple pole on the fibre $f^{-1}(0)$ and a double zero on the curve Σ_0. Thus at $(0,0)$ it is indeterminate. However, take any curve $C \subset X$ on which $(0,0)$ is a smooth point with tangent line $dx=0$ at $(0,0)$. Then we claim

$$\lim_{\substack{Q\to(0,0)\\ Q\in C}} \left(f^*\frac{dt}{t}\right)\bigg|_Q = \eta|_{(0,0)}.$$

To see this, let $x=g(y)$ be the equation of C where $g'(0)=0$. Then

$$f^*\left(\frac{dt}{t}\right)\bigg|_{(g(y),\,y)}=\frac{\sigma(x,y)^2\,dx}{y^2+\sigma(x,y)^2\,x}\bigg|_{(g(y),\,y)}=\frac{\sigma(g(y),y)^2}{y^2+\sigma(g(y),y)^2\,g(y)}\,dx,$$

hence

$$\lim_{y\to 0} f^*\left(\frac{dt}{t}\right)\bigg|_{(g(y),\,y)}=\frac{\partial\sigma}{\partial y}(0)^2\cdot dx=\eta|_{(0,0)}.$$

Moreover, in this picture, note that the zeroes of the differential form η are precisely those points where the tangent lines $d\sigma=0$ and $dx=0$ are equal $\left(\text{in fact, } \frac{\partial\sigma}{\partial y}(0)=0 \text{ if and only if } d\sigma \text{ is a multiple of } dx\right)$.

As in characteristic 3, we may also reduce the study of cuspidal fibrations to that of smooth fibrations plus a "rank one distribution". In fact, as above, define

$X^*=$ *normalization of* $X\times_B B^{(1/2)}$.

Choosing t, x, y as in Proposition 4, then $s=t^{1/2}$ is a local coordinate on $B^{(1/2)}$ and $X\times_B B^{(1/2)}$ is given locally by $s^2=y^2+x^3+x^2y^{2k+1}$ (if $k=\infty$, the last term is omitted). If we put $r=\frac{s+y}{x}$, then $r^2=x+y^{2k+1}$, so r is integral over $k[[x. y, s]]$. As $x=r^2+y^{2k+1}$, $s=rx+y=r^3+ry^{2k+1}+y$, it follows that X^* is smooth with local coordinates (r, y), and the map $X^*\to B^{(1/2)}$ is smooth. Thus we have a diagram

$$\begin{array}{ccc} X^* & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle f^*} & & \downarrow \\ B^{(1/2)} & \xrightarrow{F} & B \end{array}$$

as before.

However, unlike char. 3, Σ_0 does not lift to X^* unless $k=\infty$: for finite k, the inverse image Σ_1 of Σ_0 in X^* is the curve $r^2=y^{2k+1}$ which is purely inseparable of degree 2 over $B^{(1/2)}$, i.e., is generically $B^{(1/4)}$, and which even has singularities if $k\geqq 1$, i.e., over the zeroes of η.

Since the map π is given by

$$x=r^2+y^{2k+1},$$
$$y=y$$

we have that if L is the line sub-bundle of T_{X^*} generated locally by $\partial/\partial r$, then interpreting the sections of L as derivations of $\mathcal{O}_{X^*}$, we have:

$$\mathcal{O}_X=\ker(L\colon \mathcal{O}_{X^*}\to\mathcal{O}_{X^*}).$$

This means that X is the quotient of X^* by the rank 1 distribution L.

One final remark: we can get still another definition of our invariant η from this construction. In fact, in char. 2, suppose Y is any smooth surface and

$$L_i\subset T_Y,\qquad i=1,2$$

are 2 rank 1 integrable distributions (i.e., $D \in \Gamma(U, L_i) \Rightarrow D^2 \in \Gamma(U, L_i)$). Let C be the curve of points P where $L_{1,p} = L_{2,p}$. Then we get an invariant differential ζ on C as follows:

i) Since the L_i are integrable, there are locally functions $x_i \in \mathcal{O}_Y$ such that $dx_i \neq 0$ and $D(x_i) = 0$, all $D \in L_i$. If x_i' is another such function, $dx_i' = u_i^2\, dx_i$, u_i a unit.

ii) Along C, dx_1 is a multiple of dx_2. Define

$$\zeta = d \log (dx_1/dx_2\,|_C).$$

Then we claim:

Proposition 5. *Let $f\colon X \to B$ be a cuspidal fibration, and let $X^* =$(normalization of $X \times_B B^{(1/2)}$) as above. Let $L_1, L_2 \subset T_{X^*}$ be the distributions which are tangent to the fibres of $f^*\colon X^* \to B^{(1/2)}$ and to the fibres of $\pi\colon X^* \to X$ respectively. Let $\Sigma_1 \subset X^*$ be the curve where L_1, L_2 are tangent. Assume the invariant $\omega \in \Gamma(\Sigma_0, \Omega^1_{\Sigma_0})$ is not identically zero. Then $\Sigma_1 = \pi_*^{-1}(\Sigma_0)$ and we get a diagram:*

$$\begin{tikzcd} \Sigma_0^{(1/2)} \arrow[rr, "g"] \arrow[dr, "F"'] & & \Sigma_1 \arrow[dl, "\pi_*"] \\ & \Sigma_0 & \end{tikzcd}$$

where g is birational. Then ζ and η are related by:

$$g^* \zeta = \eta^{(1/2)}.$$

Proof. This is a simple calculation in the normal forms of Proposition 4. The details are left to the reader.

§ 2. Analysis of Quasi-Hyperelliptic Surfaces in Char. 2, 3

In this section we study quasi-hyperelliptic surfaces X. By definition, these are surfaces satisfying:

a) $K_X \sim 0$ ($\sim$ being numerical equivalence),

b) the Albanese mapping is $\pi\colon X \to E$, E elliptic,

c) almost all fibres C_x of π are rational with a cusp. By Table 1 of the Introduction to [2], it follows also that

d) $B_2 = 2$, $c_2 = 0$, $\chi(\mathcal{O}_X) = 0$. Moreover, by Proposition 5 [2], it follows that

c′) *all* fibres C_x are rational with a cusp. In Theorem 3, [2], we saw that:

e) There is a second pencil $\pi'\colon X \to \mathbb{P}^1$ on X, this time with elliptic fibres.

We can follow now the same argument used in § 3, [2] for hyperelliptic surfaces to construct an action of E on X. Now we denote by C_0 a fixed rational curve with one ordinary cusp (all such are isomorphic) and deduce, exactly as in § 3, [2]:

Theorem 1. *Every quasi-hyperelliptic surface X is of the form:*

$$X = E_1 \times C_0/K,$$

E_1 an elliptic curve, where K is a finite subgroupscheme of E_1 and K acts by

$$k(u, v) = (u + k, \alpha(k)(v))$$

for some injective homomorphism

$$\alpha\colon K \to \mathbf{Aut}(C_0).$$

Moreover the 2 fibrations on X are given by:

$$\begin{array}{ccc} E_1 \times C_0/K & \qquad & E_1 \times C_0/K \\ \Big\downarrow{\scriptstyle \text{fibres} \approx C_0} & & \Big\downarrow{\scriptstyle \text{fibres elliptic}} \\ E_1/K, \text{ elliptic} & & C_0/\alpha(K) \cong \mathbb{P}^1. \end{array}$$

To use the Theorem, we must calculate $\mathbf{Aut}(C_0)$:

Proposition 6. *Let ∞ be the cusp on C_0 and let t be an affine coordinate on $C_0 - (\infty)$. Then:*

i) *if* $\mathrm{char} \neq 2, 3$, $\mathbf{Aut}(C_0) =$ *reduced group of automorphisms* $t \to at + b \cong$ *semi-direct product $\mathbb{G}_m \cdot \mathbb{G}_a$, ($\mathbb{G}_a$ the normal factor).*

ii) *if* $\mathrm{char} = 3$, $\mathbf{Aut}(C_0) =$ *group scheme of automorphisms* $t \to at + b + ct^3$, *where* $c^3 = 0 \cong$ *"semi-direct" product of 3 factors $\mathbb{G}_m \cdot A \cdot \mathbb{G}_a$, where $\mathbb{G}_m$ normalizes A and $\mathbb{G}_a$, and A normalizes $\mathbb{G}_a$ and $A \cong \alpha_3$.*

iii) *if* $\mathrm{char} = 2$, $\mathbf{Aut}(C_0) =$ *group scheme of automorphisms* $t \to at + b + ct^2 + dt^4$, *where* $c^4 = d^2 = 0 \cong$ *"semi-direct" product of 3-factors $\mathbb{G}_m \cdot A \cdot \mathbb{G}_a$, where $\mathbb{G}_m$ normalizes A and $\mathbb{G}_a$ and A normalize $\mathbb{G}_a$ and A is an infinitesimal group scheme of order 8.*

Proof. In fact, $\mathbb{G}_m \cdot \mathbb{G}_a$ obviously acts on C_0, and an S-valued automorphism of C_0 is a product of an S-valued point of $\mathbb{G}_m \cdot \mathbb{G}_a$ times another S-valued automorphism which a) fixes the point $t = 0$, b) acts as the identity on $\mathfrak{m}_{0, C_0}/\mathfrak{m}_{0, C_0}^2$. It suffices to show that the group of automorphisms with properties a) and b) is (e), α_3, or the above A of order 8, depending on the characteristic. Now if $L = \mathcal{O}_{C_0}((0))$, then L^3 is very ample and any automorphism is determined by its action on $\Gamma(L^3)$. But it must preserve the filtration:

$$\underset{\substack{\text{basis} \\ 1}}{\Gamma(L)} \subset \underset{\substack{\text{basis} \\ x = 1/t^2}}{\Gamma(L^2)} \subset \underset{\substack{\text{basis} \\ x = 1/t^2 \\ y = 1/t^3}}{\Gamma(L^3)},$$

and act identically on $\Gamma(L)$, $\Gamma(L^2)/\Gamma(L)$, $\Gamma(L^3)/\Gamma(L^2)$. Let it therefore act via

$$x \mapsto x + a,$$

$$y \mapsto y + bx + c.$$

But $x^3 = y^2$, hence we must have

$$x^3 + 3x^2 a + 3xa^2 + a^3 = y^2 + 2byx + b^2 x^2 + 2cy + 2bcx + c^2.$$

If $p \neq 2, 3$, this shows $a = b = c = 0$; if $p = 3$, this shows $b = c = 0$ and $a^3 = 0$; if $p = 2$, this shows $a = b^2$ and $b^4 = c^2 = 0$. The Proposition now follows by examining the effect of these substitutions on $t = x/y$. Q.E.D.

The remarkable feature of the situation of Theorem 1 is the fact that $E_1 \times C_0$ has a singular line on it, while its quotient by K is non-singular. We can give the following criterion, starting with any action of such a group K, for $E_1 \times C_0/K$ to be non-singular:

Proposition 7. *Let* $\mathrm{Stab}(\infty) \subset \mathbf{Aut}(C_0)$ *be the stabilizer of* ∞ *(as a subgroup-scheme). Then for any action of* $K \subset E_1$ *on* $E_1 \times C_0$ *as in Theorem 5,*

$$E_1 \times C_0/K \text{ non-singular} \Leftrightarrow \alpha(K) \not\subset \mathrm{Stab}(\infty).$$

Proof. Let $E = E_1/K$, let $X = E_1 \times C_0/K$ and let $\pi: x \to E$ be the natural map. Since E_1 acts on $E_1 \times C_0/K$ by translation and permutes the fibres of X over E transitively, it follows

$$\begin{aligned} X \text{ singular anywhere} &\Leftrightarrow \text{the whole curve } E_1 \times (\infty)/K \text{ is singular on } X \\ &\Leftrightarrow \text{the generic fibre } \pi^{-1}(\eta_E) \text{ of } \pi \text{ is not regular} \\ &\quad\ (\eta_E = \text{generic point of } E). \end{aligned}$$

Now $\pi^{-1}(\eta_E)$ is a curve of arithmetic genus 1 over the field $k(\eta_E)$ which becomes isomorphic to C_0 over $\overline{k(\eta_E)}$. Let ∞ be the image of the cusp on it. Since ∞ is singular on $\pi^{-1}(\eta_E) \times \mathrm{Spec}\, \overline{k(\eta_E)}$, ∞ is a non-smooth point of $\pi^{-1}(\eta_E)$ over $k(\eta_E)$. Therefore

$$\infty \text{ a } k(\eta_E)\text{-rational point} \Rightarrow \infty \text{ not regular on } \pi^{-1}(\eta_E).$$

And if ∞ is *not* $k(\eta_E)$-rational, then ∞ is regular, or else comparing the genus of $\pi^{-1}(\eta_E)$ and its normalization it would force $p_a(\pi^{-1}(\eta_E))$ to be bigger than 1: Thus

$$\begin{aligned} \pi^{-1}(\eta_E) \text{ not regular} &\Leftrightarrow \infty \text{ is } k(\eta_E)\text{-rational} \\ &\Leftrightarrow \text{the curve } E_1 \times (\infty)/K \text{ maps by } \pi \text{ isomorphically to } E \\ &\Leftrightarrow \text{the singular line } E_1 \times (\infty)/K \text{ defines a section} \\ &\quad\ \text{of } \pi: X \to E. \end{aligned}$$

But now $E_1 \times C_0$ is recovered by fibre product:

$$\begin{array}{ccc} X & \longleftarrow & E_1 \times C_0 \\ \downarrow & & \downarrow \\ E & \longleftarrow & E_1 \end{array}$$

hence the above is equivalent to $E_1 \times (\infty)$ defining a K-equivariant section of $p_1: E_1 \times C_0 \to E_1$, or to ∞ being a fixed point by $\alpha(K)$. Q.E.D.

Since if $\mathrm{char} \neq 2, 3$, the whole of $\mathbf{Aut}(C_0)$ stabilizes ∞, this shows again Tate's result that smooth surfaces like X can exist only if $\mathrm{char} = 2$ or 3. Now:

$$\mathrm{char} = 2 \Rightarrow \mathrm{Stab}(\infty) = \mathbb{G}_a \cdot A_0 \cdot \mathbb{G}_m, \qquad A_0 = \left\{ \begin{matrix} \text{gp. of automorphisms} \\ t \mapsto t + ct^2,\ c^2 = 0 \end{matrix} \right\}$$

$$\mathrm{char} = 3 \mapsto \mathrm{Stab}(\infty) = \mathbb{G}_a \cdot \mathbb{G}_m\text{: set } A_0 = (e) \text{ here for consistency.}$$

The infinitesimal orbit of ∞ can be pictured like this:

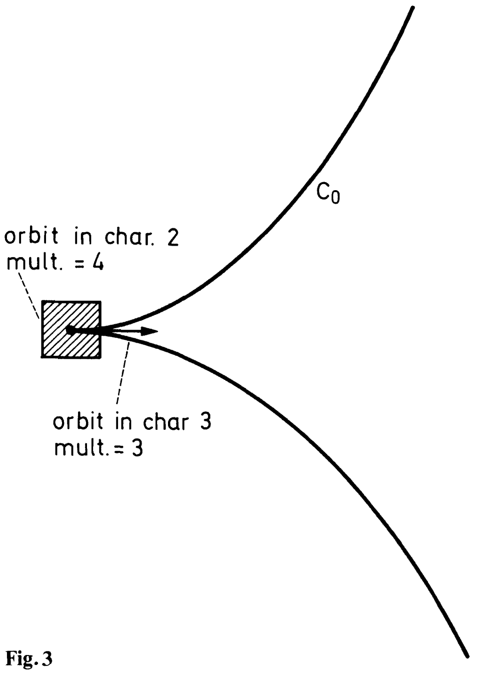

Fig. 3

Moreover, one checks that Stab(∞) is also the subgroup-*scheme* of $\mathbf{Aut}(C_0)$ of all automorphisms that lift to the normalization $\mathbb{P}^1$ of C_0. We don't know whether this can be deduced from a general principle but in our case, the automorphisms ϕ of $\mathbb{P}^1$ such that $\phi(0) \neq \infty$ are given by

$$\begin{aligned}\phi(t) &= \frac{at+b}{1-ct}, \qquad a+bc \neq 0\\ &= b+(a+bc)\,t+(ac+bc^2)\,t^2+(ac^2+bc^3)\,t^3+\cdots.\end{aligned}$$

If char $=3$, and ϕ is also of the form $\phi(t)=\alpha t+\beta+\gamma t^3$, with α unit, $\gamma^3=0$, it follows that $c=0$, i.e., $\phi \in \mathrm{Stab}(\infty)$. If char $=2$ and ϕ is also of the form $\phi(t)=\alpha t+\beta+\gamma t^2+\delta t^4$ with α unit, $\gamma^4=0$, $\delta^2=0$, it follows that $c^2=0$, hence $\gamma^2=\delta=0$, i.e., $\phi \in \mathrm{Stab}(\infty)$. Thus by explicit calculation we see that an S-valued point of $\mathbf{Aut}(C_0)$ lifts to $\mathbb{P}^1$ if and only if it fixes ∞.

To classify quasihyperelliptic X's, the next step is to enumerate modulo conjugacy all subgroups:

$$K \subset \mathbb{G}_m \cdot A \cdot \mathbb{G}_a$$

such that:

1) $K \not\subset \mathbb{G}_m \cdot A_0 \cdot \mathbb{G}_a$,
2) K commutative,
3) Lie K and Lie K^D at most one-dimensional (K^D = Cartier dual).

This is a tedious problem but, as near as we can tell, the following is the list of all such subgroups K:

Char. 3

a) the μ_3 of maps $t \mapsto at + (1-a)t^3$, $a^3 = 1$,
b) this μ_3, plus $t \mapsto \pm t$,
c) this μ_3, plus $t \mapsto t + i$, $i \in \mathbb{Z}/3\mathbb{Z}$,
d) the α_3 of maps $t \mapsto t + at^3$, $a^3 = 0$,
e) this α_3, plus $t \mapsto \pm t$,
f) the group scheme of order 9 of maps

$$t \mapsto t + a + a^3 t^3, \ a^9 = 0.$$

Char. 2 – for all $\lambda \in k$:

a) the μ_2 of maps $t \mapsto$ at $+ \lambda(a+1)t^2 + (a+1)t^4$, $a^2 = 1$,
b) if $\lambda = 0$, this μ_2, plus $t \mapsto \omega t$, $\omega^3 = 1$,
c) this μ_2, plus $t \mapsto t + \xi$ where ξ is a root of

$$x^4 + \lambda x^2 + x = 0,$$

d) the μ_4 of maps $t \mapsto at + (a + a^2)t^2 + (1 + a^2)t^4$, $a^4 = 1$,
e) this μ_4, plus $t \mapsto t + 1$,
f) the α_2 of maps $t \mapsto t + \lambda a t^2 + a t^4$, $a^2 = 0$,
g) if $\lambda = 0$, this α_2, plus $t \mapsto \omega t$, $\omega^3 = 1$,
h) the group scheme of order 4 of maps

$$t \mapsto t + a + \lambda a^2 t^2 + a^2 t^4, \qquad a^4 = 0, \ \lambda \neq 0.$$

Combined with Theorem 1, this leads immediately to a complete list of quasi-hyperelliptic surfaces. Moreover, it has the following Corollary:

Proposition 8. *If X is quasihyperelliptic, then:*

i) *if* char. $= 3$, *then* $6K_X \equiv 0$,
ii) *if* char. $= 2$, *then* $6K_X \equiv 0$ *or* $4K_X \equiv 0$.

Proof. In the notation of Theorem 1, the inverse image on $E_1 \times C_0$ of Ω_X^2 is the invertible sheaf $\omega_{E_1 \times C_0}$. But $\omega_{E_1 \times C_0} \cong \Omega_{E_1}^1 \otimes \omega_{C_0}$ and K acts trivially on $\Omega_{E_0}^1$. Therefore

$$(\text{order } K_X) = (\text{least } n \text{ such that } \alpha(K) \text{ acts trivially on } \omega_{C_0})$$

But dt spans ω_{C_0} so $\mathbf{Aut}(C_0)$ acts on ω_{C_0} through the character:

$$\mathbf{Aut}(C_0) \to \mathbf{Aut}(C_0)/\mathbb{G}_a \cdot A \cong \mathbb{G}_m.$$

Now the Proposition follows by examining all the cases in the above list and noting that the subgroups are all contained in $\mu_6 \cdot A \cdot \mathbb{G}_a$ (char. 2 or 3) or in $\mu_4 \cdot A \cdot \mathbb{G}_a$ (char. 2). Q.E.D.

Another way to describe quasihyperelliptic surfaces X is to reduce their study to that of ruled surfaces X^* with rank one distributions $L \subset T_{X^*}$, as in § 1. To do this, define:

$$X^* = \textit{normalization of } X \times_E E^{(1/p)}$$
$$\pi: X^* \to X, \quad \text{the natural map}$$
$$L = \operatorname{Ker}[d\pi: T_{X^*} \to \pi^*(T_X)].$$

The fibres of $p: X^* \to E^{(1/p)}$ are smooth and rational, so X^* is an elliptic ruled surface. We may also described X^* in terms of the homomorphism $\alpha: K \to \mathbf{Aut}(C_0)$ as follows. Let

$$q: \mathbb{P}^1 \to C_0$$

be the normalization map, and let

$$K_0 = \alpha^{-1}(\operatorname{Stab}(\infty)).$$

Note that $K_0 \subsetneq K$; examining the various cases in our list above, we see that we have an exact sequence

$$0 \to K_0 \to K \to (\alpha_p \text{ or } \mu_p) \to 0,$$

hence E_1/K_0 must be the elliptic curve $E^{(1/p)}$. Moreover, we have seen that the action of K_0 on C_0 lifts to an action α_0 of K_0 on $\mathbb{P}^1$. Therefore we get a diagram:

$$\begin{tikzcd}
E_1 \times \mathbb{P}^1 \arrow[r] \arrow[d] & E_1 \times \mathbb{P}^1/K_0 \arrow[d] & \\
E_1 \times C_0 \arrow[r] \arrow[d] & E_1 \times C_0/K_0 \arrow[r] \arrow[d] & X \arrow[d] \\
E_1 \arrow[r] & E^{(1/p)} \arrow[r] & E
\end{tikzcd}$$

hence:

$$E_1 \times C_0/K_0 \approx X \times_E E^{(1/p)}$$

and

$$E_1 \times \mathbb{P}^1/K_0 \cong X^*.$$

Thus X^* is the ruled surface obtained by "twisting" $E^{(1/p)} \times \mathbb{P}^1$ by the cocycle

$$\alpha_0: K_0 \to \mathbf{Aut}(\mathbb{P}^1).$$

Note that in char 3, $\alpha_0(K_0) \subset \mathbb{G}_m \cdot \mathbb{G}_a$, hence K_0 stabilizes $E_1 \times (\infty)$, hence the line of cusps $E_1 \times (\infty)/K_0$ is a section of $X^* \to E^{(1/3)}$, as we saw had to happen in § 1. On the other hand, in char 2, $\alpha_0(K_0) \subset \mathbb{G}_m \cdot A_0 \cdot \mathbb{G}_m$, so K_0 need not stabilize $E_1 \times (\infty)$ and the line of cusps need not be a section of $X^* \to E^{(1/2)}$. In fact, if ω is the invariant, defined in § 1, in $\Gamma(E, \Omega_E^1)$ defined by the cuspidal fibration $X \to E$, then we have:

$\omega \equiv 0 \Leftrightarrow$ the line of cusps is a section of $X^* \to E^{(1/2)}$

$\Leftrightarrow \alpha_0(K_0) \subset \mathbb{G}_m \cdot \mathbb{G}_a$.

$\Leftrightarrow \alpha(K)$ belongs to one of the types a, b, c, f, g, h.

However, if $\alpha(K)$ is of type d or e, then $\omega \not\equiv 0$. Since $K \subset E_1$, E_1 is an ordinary elliptic curve, hence so is E, and ω must be the unique non-zero global 1-form such that $C\omega = \omega$ (C = Cartier's operator).

§ 3. Preliminary Analysis of Enriques' Surfaces

We have defined Enriques' surfaces by the conditions:

a) $K_X \sim 0$ ($\sim$ is numerical equivalence),

b) $B_2 = 10$.

By the Basic Table in Part II, we see that such surfaces also have the properties:

c) $B_1 = 0$, $c_2 = 12$, $\chi(\mathcal{O}_X) = 1$.

Moreover either $\dim H^1(\mathcal{O}_X) = \dim H^2(\mathcal{O}_X) = 0$ hence $K_X \not\equiv 0$; or $\dim H^1(\mathcal{O}_X) = \dim H^2(\mathcal{O}_X) = 1$ hence $K_X \equiv 0$. In the first case, by Castelnuovo's Theorem, since X is not rational we find $|2K_X| \neq \emptyset$, hence in fact $2K_X \equiv 0$. We have seen in Part II, Theorem 5, that the case $K_X \equiv 0$ only occurs in char. 2. We divide Enriques' surfaces into three types according to the action of the Frobenius cohomology operation F on $H^1(\mathcal{O}_X)$:

Definition. Enriques' surfaces are called:

i) classical if $\dim H^1(\mathcal{O}_X) = 0$, hence $K_X \not\equiv 0$, $2K_X \equiv 0$;

ii) singular if $\dim H^1(\mathcal{O}_X) = 1$, hence $K_X \equiv 0$, and F is bijective on $H^1(\mathcal{O}_X)$;

iii) supersingular if $\dim H^1(\mathcal{O}_X) = 1$, hence $K_X = 0$, and F is zero on $H^1(\mathcal{O}_X)$.

The essential similarity of these three classes is brought out in the following Theorem.

Theorem 2. *Let X be an Enriques' surface. Then*[1] $\mathbf{Pic}^\tau_X$ *is a group scheme of order* 2. *Moreover:*

$$\mathbf{Pic}^\tau_X \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } X \text{ is classical} \\ \mu_2 & \text{if } X \text{ is singular} \\ \alpha_2 & \text{if } X \text{ is supersingular.} \end{cases}$$

We give a short proof which handles the classical and non-classical cases separately and then we will sketch a uniform proof which is, perhaps, more natural.

First Proof. If X is classical, $H^1(\mathcal{O}_X) = (0)$ implies $\mathbf{Pic}^\tau_X$ has trivial tangent spaces hence is a finite discrete group. If D is a divisor numerically equivalent to 0 then

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) = 1,$$

[1] Note. $\mathbf{Pic}^\tau_X$ denotes the open subgroup of $\mathbf{Pic}_X$ parametrizing divisor classes numerically equivalent to 0.

hence $\dim H^0(\mathcal{O}_X(D))>0$ or $\dim H^2(\mathcal{O}_X(D))>0$. Therefore $|D|\neq\emptyset$ or $|K_X-D|\neq\emptyset$, hence in fact $D\equiv 0$ or $D\equiv K_X$. Therefore $\mathbf{Pic}^\tau_X\cong \mathbb{Z}/2\mathbb{Z}$. If X is not classical then the same argument shows that any such D is linearly equivalent to 0, hence $\mathbf{Pic}^\tau_X$ consists of one point. Since its tangent space is 1-dimensional, we must show that there is a non-trivial obstruction to extending a non-zero morphism $\operatorname{Spec} k[t]/(t^2) \to \mathbf{Pic}_X$ to a morphism on $\operatorname{Spec} k[t]/(t^3)$. By the theory of [5], last lecture, this follows from:

Lemma 1. *If X is a non-classical Enriques' surface, the first Bockstein operation*

$$\beta_1\colon H^1(\mathcal{O}_X)\to H^2(\mathcal{O}_X)$$

is an isomorphism.

Proof of Lemma 1. It is well-known that in char. 2, $\beta_1(x)$ is the cup-product $\beta_1(x)=x\cup x$. By Serre's duality

$$H^1(\mathcal{O}_X)\otimes H^1(\Omega^2_X)\to H^2(\Omega^2_X)$$

is an isomorphism. Since $K_X\equiv 0$, it follows that

$$H^1(\mathcal{O}_X)\otimes H^1(\mathcal{O}_X)\to H^2(\mathcal{O}_X)$$

is an isomorphism. Then β_1 is this map composed with:

$$H^1(\mathcal{O}_X)\to H^1(\mathcal{O}_X)\otimes H^1(\mathcal{O}_X)$$

where $x\mapsto x\otimes x$. Q.E.D.

This proves that $\mathbf{Pic}^\tau_X=\mu_2$ or α_2 when X is non-classical. Now the isomorphism $H^1(\mathcal{O}_X)\cong \operatorname{Lie}(\mathbf{Pic}^\tau_X)$ carries the Frobenius F on $H^1(\mathcal{O}_X)$ to p-th power on $\operatorname{Lie}(\mathbf{Pic}^\tau_X)$, so $F\neq 0$ is equivalent to p-th power $\neq 0$ on $\operatorname{Lie}(\mathbf{Pic}^\tau_X)$, hence is equivalent to $\mathbf{Pic}^\tau_X=\mu_2$. Q.E.D.

Second Proof. Let us first sketch some quite general facts about the Picard scheme of any projective variety Y. Let P be the universal invertible sheaf on $Y\times\mathbf{Pic}_Y$. Then:

(a) consider the functor which, for all morphisms $f\colon S\to\mathbf{Pic}^\tau_Y$ assigns $p_{2,*}((1\times f)^*P)$:

$$\begin{tikzcd} Y\times S \arrow[r, "1\times f"] \arrow[d, "p_2"'] & Y\times \mathbf{Pic}^\tau_Y \arrow[d] \\ S \arrow[r, "f"] & \mathbf{Pic}^\tau_Y. \end{tikzcd}$$

By Grothendieck's theory of cohomology and base change (cf. EGA, Ch. 3, 2nd part; or Mumford, Abelian Varieties, § 1) there is a coherent sheaf $\mathscr{F}$ on $\mathbf{Pic}^\tau_Y$ such that

$$p_{2,*}((1\times f)^*P)=\operatorname{Hom}_{\mathcal{O}_S}(f^*\mathscr{F},\mathcal{O}_S).$$

Applying this first with $S=\mathrm{Spec}(k)$, Image(f) = point corresponding to the sheaf L on Y, it follows that Supp$(\mathscr{F})=(0)$, since $H^0(Y,L)=(0)$ if $L\not\cong\mathcal{O}_Y$. Choosing next $S=\mathrm{Spec}(k[\varepsilon])$, Image$(f)=(0)$, and using the fact that if L on $Y\times\mathrm{Spec}(k[\varepsilon])$ is a non-trivial deformation of $\mathcal{O}_Y$ (that is, fits into a non-split sequence

$$0\to\mathcal{O}_Y\xrightarrow{\times\varepsilon}L\to\mathcal{O}_Y\to 0)$$

then $H^0(Y\times\mathrm{Spec}(k[\varepsilon]),L)\cong k$, it follows that for all surjective homorphisms $\mathcal{O}_{0,\mathbf{Pic}_Y}\twoheadrightarrow k[\varepsilon]$ we have

$$\mathrm{Hom}(\mathscr{F}_0\otimes_{\mathcal{O}_{0,\mathbf{Pic}_Y}}k[\varepsilon],k[\varepsilon])\cong k$$

and we deduce from this that $\mathscr{F}\cong k_0$, a sheaf with support (0) and stalk k there. We now apply this to the inclusion $f\colon H\hookrightarrow\mathbf{Pic}_Y^\tau$ of a finite group scheme into $\mathbf{Pic}_Y^\tau$. In this case

$$\mathcal{O}_{0,H}\cong k[t_1,\dots,t_n]/(t_1^{p^r},\dots,t_n^{p^{r_n}})$$

hence

$$\mathrm{Hom}_{\mathcal{O}}(k,\mathcal{O}_{0,H})\cong k.$$

This proves:

Lemma 2. *If* $\mathrm{Spec}(A)\hookrightarrow\mathbf{Pic}_Y^\tau$ *is a finite subgroup scheme, then*

$$\dim H^0(Y\times\mathrm{Spec}(A),P\otimes_{\mathcal{O}_{\mathbf{Pic}}}A)=1.$$

(b) for all finite subschemes $\mathrm{Spec}(A)\hookrightarrow\mathbf{Pic}_Y^\tau$, we may consider $P\otimes_{\mathcal{O}_{\mathbf{Pic}}}A$ as an invertible $\mathcal{O}_Y\otimes_k A$-module, hence if $n=\dim_k A$ it is also a locally free $\mathcal{O}_Y$-module of rank n. Now if $\mathrm{Spec}(A)$ is a subgroup, with points $a_1,\dots,a_m$ let translation by a_i define an automorphism $T_i\colon A\to A$, define also

$$\pi\colon A\to\mathcal{O}_{0,A}\cong k[t_1,\dots,t_n]/(t_1^{p^{r_1}},\dots,t_n^{p^{r_n}})$$

by restriction to (0), and then define $t\colon A\to k$ by

$$t(f)=\sum_{i=1}^m\left[\text{coeff. of}\prod_{i=1}^n r_i^{p^{r_i}-1}\ \text{ in }\ \pi(T_i(f))\right].$$

Then $(x,y)\mapsto t(x.y)$ is a non-degenerate quadratic form on the n-dimensional k-vector space A. Now we can pair the two locally free rank n $\mathcal{O}_Y$-sheaves $P\otimes_{\mathcal{O}_{\mathbf{Pic}}}A$ and $P^{-1}\otimes_{\mathcal{O}_{\mathbf{Pic}}}A$, via

$$(P\otimes_{\mathcal{O}_{\mathbf{Pic}}}A)\times(P^{-1}\otimes_{\mathcal{O}_{\mathbf{Pic}}}A)\xrightarrow{\text{mult.}}\mathcal{O}_{Y\times\mathbf{Pic}}\otimes_{\mathcal{O}_{\mathbf{Pic}}}A=\mathcal{O}_Y\otimes_k A\xrightarrow{1\otimes t}\mathcal{O}_Y.$$

This is similarly non-degenerate, which proves:

Lemma 3. *If* $\mathrm{Spec}(A)$ *is a finite subgroup schme of* $\mathbf{Pic}_Y^\tau$, *then*

$$P^{-1}\otimes_{\mathcal{O}_{\mathbf{Pic}}}A\cong\mathrm{Hom}_{\mathcal{O}_Y}(P\otimes_{\mathcal{O}_{\mathbf{Pic}}}A,\mathcal{O}_Y).$$

Apply this to the case in which $Y=X$, an Enriques' surface. If $\mathbf{Pic}_X$ contains a subgroup scheme $\mathrm{Spec}(A)$ of order n, then $E_A=P\otimes_{\mathcal{O}_{\mathbf{Pic}}}A$ is a locally free $\mathcal{O}_X$-

module of rank n. By Lemma 2, we have

$$\dim H^0(X, E_A) = 1,$$

and by Lemma 3 and Serre duality

$$\begin{aligned}\dim H^2(X, E_A) &= \dim H^0(X, \Omega_X^2 \otimes \mathrm{Hom}(E_A, \mathcal{O}_X)) \\ &= \dim H^0(X \times \mathrm{Spec}(A), [(\Omega_X^2 \otimes \mathcal{O}_{\mathbf{Pic}}) \otimes P^{-1}] \otimes_{\mathcal{O}_{\mathbf{Pic}}} A).\end{aligned}$$

Assume $\mathrm{Spec}(A)$ contains the point a of order 1 or 2 of $\mathbf{Pic}_X$ representing the sheaf Ω_X^2. Now under the automorphism $(x, p) \mapsto (x, -p)$ of $X \times \mathbf{Pic}_X$, the pull-back of P is P^{-1}; and under the automorphism $(x, p) \mapsto (x, p+a)$ the pull-back of P is $(\Omega_X^2 \otimes \mathcal{O}_{\mathbf{Pic}}) \otimes P$. Restricting the automorphism $(x, p) \mapsto (x, a-p)$ to $X \times \mathrm{Spec}(A)$, it follows that

$$\begin{aligned}&\dim H^0(X \times \mathrm{Spec}(A), [(\Omega_X^2 \otimes \mathcal{O}_{\mathbf{Pic}}) \otimes P^{-1}] \otimes_{\mathcal{O}_{\mathbf{Pic}}} A) \\ &\quad = \dim H^0(X \times \mathrm{Spec}(A), P \otimes_{\mathcal{O}_{\mathbf{Pic}}} A) = 1.\end{aligned}$$

Thus $\chi(E_A) \leqq 2$.

On the other hand, if $a_1, \ldots, a_m$ are the points of $\mathrm{Spec}(A)$ and if a_i corresponds to a sheaf L_i on X, then E_A is a direct sum of m sheaves, the i-th of these being a successive extension of copies of L_i. To see this, note that

$$A = \bigoplus_{i=1}^{m} \mathcal{O}_{a_i, \mathrm{Spec}(A)}$$

and choose filtrations

$$\mathcal{O}_{a_i, \mathrm{Spec}(A)} = I_{i,0} \supset I_{i,1} \supset \cdots \supset I_{i,s} = (0)$$

of each factor by ideals with $\dim I_{i,j}/I_{i,j+1} = 1$. Then

$$E_A = \bigoplus_{i=1}^{m} (E_A \otimes_A \mathcal{O}_{a_i, \mathrm{Spec}(A)})$$

and each factor is a successive extension of the sheaves

$$E_{i,j} = (E_A I_{i,j})/(E_A I_{i,j+1}).$$

If $t_{i,j} \in I_{i,j} - I_{i,j+1}$, then multiplication by $t_{i,j}$ defines

$$L_i = E_A I_{i,0}/E_A I_{i,1} \xrightarrow{\sim} E_A I_{i,j}/E_A I_{i,j+1}.$$

Since $\chi(L_i) = \chi(\mathcal{O}_X) = 1$, this proves: $\chi(E_A) = n$. Therefore $n \leqq 2$, and $\mathbf{Pic}_X^\tau$ has order 2. The rest of the argument now follows the first proof.

Combining the Theorem with the fact that for all finite commutative group schemes H, principal covering spaces

$$\pi: \tilde{X} \to X$$

with structure group H are classified by:

$$\mathrm{Hom}(H^D, \mathbf{Pic}_X)$$

(H^D = Cartier dual of H), cf. [8], p. 50, it follows that:

Corollary. *Let X be an Enriques surface. Then X has a canonical principal covering space*

$$\pi\colon \tilde{X} \to X$$

of degree 2, whose structure group is:

μ_2 *if X is classical*

$\mathbb{Z}/2\mathbb{Z}$ *if X is singular*

α_2 *if X is supersingular.*

(Recall that if char $\neq 2$, then $\mu_2 \cong \mathbb{Z}/2\mathbb{Z}$.)

We may construct this covering explicitly as follows:

Case i: X classical. Let $\{f_{ij}\} \in Z^1(\{U_i\}, \mathcal{O}_X^*)$ be a 1-cocycle representing the invertible sheaf Ω_X^2. Since $2K_X \equiv 0$, $\{f_{ij}^2\}$ is a 1-coboundary and we can write

$$f_{ij}^2 = g_i/g_j \quad \text{on } U_i \cap U_j$$

where $g_i \in \Gamma(U_i, \mathcal{O}_X^*)$. Now we define $\pi\colon \tilde{X} \to X$ locally by the covering

$$z_i^2 = g_i \quad \text{on } U_i \times \mathbb{A}^1$$

the glueing being given by

$$z_i/z_j = f_{ij} \quad \text{on } (U_i \times \mathbb{A}^1) \cap (U_j \times \mathbb{A}^1).$$

Case ii: X singular. Let $\{a_{ij}\} \in Z^1(\{U_i\}, \mathcal{O}_X)$ be a 1-cocycle representing an element $\eta \in H^1(\mathcal{O}_X)$ such that $F\eta = \eta$. Following Serre [11], this means that $\{a_{ij}^2 - a_{ij}\}$ is a 1-coboundary, and we can write

$$a_{ij}^2 - a_{ij} = b_i - b_j \quad \text{on } U_i \cap U_j$$

where $b_i \in \Gamma(U_i, \mathcal{O}_X)$. Now we define $\pi\colon \tilde{X} \to X$ locally by the Artin-Schreier covering:

$$z_i^2 - z_i = b_i \quad \text{on } U_i \times \mathbb{A}^1$$

the glueing being given by

$$z_i - z_j = a_{ij} \quad \text{on } (U_i \times \mathbb{A}^1) \cap (U_j \times \mathbb{A}^1).$$

Case iii: X supersingular. Let $\{a_{ij}\} \in Z^1(\{U_i\}, \mathcal{O}_X)$ be any 1-cocycle which is not a coboundary. Since F is zero on $H^1(\mathcal{O}_X)$, $\{a_{ij}^2\}$ is a 1-coboundary and we can write

$$a_{ij}^2 = b_i - b_j \quad \text{on } U_i \cap U_j$$

where $b_i \in \Gamma(U_i, \mathcal{O}_X)$. Now we define $\pi\colon \tilde{X} \to X$ locally by the inseparable covering:

$$z_i^2 = b_i \quad \text{on } U_i \times \mathbb{A}^1$$

the glueing being given by

$$z_i - z_j = a_{ij} \quad \text{on } (U_i \times \mathbb{A}^1) \cap (U_j \times \mathbb{A}^1).$$

What sort of scheme is $\tilde{X}$? If X is classical and char $\neq 2$ or if X is singular and char $= 2$, π is étale of degree 2, hence $\tilde{X}$ is a smooth surface. On the other hand, if

X is classical or supersingular and char $=2$, then π is purely inseparable. Still $\tilde{X}$ is a reduced Gorenstein surface because: a) it is codimension 1 in a smooth threefold, and b) in the notation above, g_i (resp. b_i) are not squares because $\{f_{ij}\}$ (resp. $\{a_{ij}\}$) is not itself a 1-coboundary. Moreover, in the first set of cases, $\tilde{X}$ is certainly a $K3$-surface because $K_{\tilde{X}} \sim 0$ and $\chi(\mathcal{O}_{\tilde{X}}) = 2\chi(\mathcal{O}_X) = 2$, and X is quotient of a $K3$-surface by means of a fixed point free involution. In the second set of cases, $\tilde{X}$ must be singular. To see this, note that

$$\begin{aligned} &\eta = dg_i/g_i, && \text{case i, char. 2} \\ &\eta = db_i, && \text{case iii} \end{aligned}$$

detines a global 1-form on X with no poles, and that for all $P \in \tilde{X}$:

P is singular on $\tilde{X}$ if and only if all derivatives of $z_i^2 - g_i$ (resp. $z_i^2 - b_i$) vanish at P, hence if and only if η is zero at $\pi(P)$. Since $c_2(X) = 12$, a 1-form like η will generically have 12 zeros, and it must always vanish somewhere, hence $\tilde{X}$ must be singular.

However, $\tilde{X}$ is always "$K3$-like":

Proposition 9. *The double covering $\pi\colon \tilde{X} \to X$ satisfies:*

$$\dim H^i(\mathcal{O}_{\tilde{X}}) = \begin{cases} 1 & i=0 \\ 0 & i=1 \\ 1 & i=2 \end{cases}$$

and $\omega_{\tilde{X}}$ (the dualizing sheaf on $\tilde{X}$) is isomorphic to $\mathcal{O}_{\tilde{X}}$.

Proof. By Grothendieck's duality theory (cf. [4]), for any finite flat morphism $g\colon \tilde{Y} \to Y$, write $\mathscr{A} = g_*(\mathcal{O}_{\tilde{Y}})$, so that $\tilde{Y} \cong \operatorname{Spec}(\mathscr{A})$. Then

$$\omega_{\tilde{Y}} \cong [\operatorname{Hom}_{\mathcal{O}_Y}(\mathscr{A}, \omega_Y)]^{\sim}$$

(i.e. regard $\operatorname{Hom}_{\mathcal{O}_Y}(\mathscr{A}, \omega_Y)$ as a sheaf of $\mathscr{A}$-modules and take the associated sheaf on $\tilde{Y}$). So to show that $\omega_{\tilde{Y}} \cong \mathcal{O}_{\tilde{Y}}$ we must show

$$\phi\colon \mathscr{A} \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_Y}(\mathscr{A}, \omega_Y) \quad \text{as } \mathscr{A}\text{-modules},$$

or equivalently we must construct an $\mathcal{O}_Y$-linear map

$$t\colon \mathscr{A} \to \omega_Y$$

such that $t(x \cdot y)$ is a non-degenerate quadratic form on the locally free sheaf $\mathscr{A}$ with values in the invertible sheaf ω_Y, and then define ϕ by setting $\phi(x)$ equal to the homomorphism $y \mapsto t(x \cdot y)$. Returning to Enriques' surfaces, we have:

Case i: $\pi_*(\mathcal{O}_{\tilde{X}}) \cong \mathcal{O}_X \oplus L$

where $L = \mathcal{O}_X \cdot z_i$ locally.

Since $z_i/z_j = f_{ij}$, we have $L \cong \Omega_X^2$. Now define t to be 0 on $\mathcal{O}_X$ and this isomorphism on L.

Cases ii, iii: we get an exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\alpha} \pi_*(\mathcal{O}_{\tilde{X}}) \xrightarrow{\beta} \mathcal{O}_X \to 0$$

where $\alpha(a) = a$, $\beta(a + b z_i) = b$.

Since $\Omega_X^2 \cong \mathcal{O}_X$, define t to be the map β.

This proves statement 2). Moreover, it also shows that $\chi(\mathcal{O}_{\tilde{X}}) = \chi(\pi_* \mathcal{O}_{\tilde{X}}) = 2\chi(\mathcal{O}_X) = 2$. Now $\tilde{X}$ connected and reduced implies $\dim H^0(\mathcal{O}_{\tilde{X}}) = 1$, so by 2) we have also

$$\dim H^2(\mathcal{O}_{\tilde{X}}) = \dim H^0(\omega_{\tilde{X}}) = \dim H^0(\mathcal{O}_{\tilde{X}}) = 1. \quad \text{Q.E.D.}$$

The same type of argument can be used to prove more generally that if $\pi\colon \tilde{Y} \to Y$ is any principal covering space with finite structure group scheme G, then, in Grothendieck's notation, $\pi^! \mathcal{O}_Y \cong \mathcal{O}_{\tilde{Y}}$, hence $\omega_{\tilde{Y}} \cong \pi^* \omega_Y$.

Finally, we would like to give some examples to show that all these types of Enriques' surfaces do exist and may even be part of the same connected family of surfaces. Enriques' original construction of his surfaces as the normalization of sextic surfaces in $\mathbb{P}^3$ passing doubly through the edges of a tetrahedron gives classical Enriques surfaces in all characteristics (cf. [6], where it was remarked that in char. 2 these surfaces carried regular 1-forms). To get all the types at once, we follow an idea of Miles Reid which adapts Serre's construction [10] to the case of Enriques' surfaces: we construct $\tilde{X}$ in $\mathbb{P}^5$ as the intersection of three quadrics and an action of μ_2, $\mathbb{Z}/2\mathbb{Z}$ or α_2 on $\mathbb{P}^5$ which restricts to a free action of this group on $\tilde{X}$. Then define X to be the orbit space.

Let $x_1, x_2, x_3, y_1, y_2, y_3$ be homogeneous coordinates on $\mathbb{P}^5$. Consider the action of

$$G = \text{group of matrices } \left\{ \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix},\ b \in k^*,\ a \in k \right\}$$

on $\mathbb{P}^5$ given by

$$(x_i, y_i) \mapsto (x_i, a x_i + b y_i).$$

Inside G, we consider the following subgroup schemes of order 2:

a) for all $\sigma \in k^*$, $H_{(\sigma,0)} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \sigma & 1 \end{pmatrix} \right\} \cong \mathbb{Z}/2\mathbb{Z}$,

b) $H_{(0,0)} = \left\{ \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}, \varepsilon^2 = 0 \right\} \cong \alpha_2$,

c) for all $\tau \in k^*$, $H_{(0,\tau)} = \left\{ \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 + \tau\varepsilon \end{pmatrix}, \varepsilon^2 = 0 \right\} \cong \mu_2$.

Note that as either σ or τ goes to 0, $H_{(\sigma,0)}$ or $H_{(0,\tau)}$ approaches $H_{(0,0)}$. In fact, altogether they form a finite and flat group scheme over $\operatorname{Spec} k[\sigma,\tau]/(\sigma\tau)$. These group schemes act on $\mathbb{P}^5$ and the subrings of $k[x_1, \ldots, y_3]$ of H-invariants of even degree are readily computed to be:

a) $(k[x_1, \ldots, y_3]^{H_{(\sigma,0)}})_{\substack{\text{even}\\\text{degree}}} = k[x_k^2, x_i x_j, y_k^2 + \sigma x_k y_k, y_i x_j + y_j x_i]$
where $1 \leqq i < j \leqq 3$, $1 \leqq k \leqq 3$;

b) $(k[x_1, \ldots, y_3]^{H_{(0,0)}})_{\substack{\text{even}\\\text{degree}}} = k[x_k^2, x_i x_j, y_k^2, y_i x_j + y_j x_i]$
where $1 \leqq i < j \leqq 3$, $1 \leqq k \leqq 3$;

c) $(k[x_1, \dots, y_3]^{H(0,\tau)})_{\substack{\text{even}\\ \text{degree}}} = k[x_k^2, x_i x_j, y_k^2, y_i x_j y_j x_i + \tau y_i y_j]$
where $1 \leqq i < j \leqq 3$, $1 \leqq k \leqq 3$.

In all three cases we take the twelve invariant quadratic forms on the right and use these to define a morphism:

$$\Phi_{(\sigma,\tau)}\colon \mathbb{P}^5 \to \mathbb{P}^{11}$$

for $\sigma\tau = 0$. Then $\Phi_{(\sigma,\tau)} \mathbb{P}^5 \cong \mathbb{P}^5 / H_{(\sigma,\tau)}$. Define $X_{(\sigma,\tau)}$ to be a generic 2-dimensional section of this quotient:

$$X_{(\sigma,\tau)} = \Phi_{(\sigma,\tau)}(\mathbb{P}^5) \cap (L_1 = L_2 = L_3 = 0)$$

where the L_i are generic linear forms, and define $\tilde{X}_{(\sigma,\tau)}$ to be the inverse image of $X_{(\sigma,\tau)}$ in $\mathbb{P}^5$, hence

$$\tilde{X}_{(\sigma,\tau)} = \text{locus}\,(f_1 = f_2 = f_3 = 0)$$

where the f_i are the quadratic forms obtained by pull-back of the linear forms L_i. Dropping for an easier notation the subfix (σ, τ), let π be the restriction of Φ to $\tilde{X}$. Since in all cases the fixed point locus F of H is one or two planes, $\tilde{X}$ does not meet F and $\pi\colon \tilde{X} \to X$ is a principal H-bundle. Since $\Phi(\mathbb{P}^5)$ is smooth outside $\Phi(F)$, this shows that X is smooth too. On the other hand, $\tilde{X}$, being the intersection of three quadratic forms in $\mathbb{P}^5$, is Gorenstein with dualizing sheaf given by:

$$\omega_{\tilde{X}} \cong \Omega_{\mathbb{P}^5}(-\textstyle\sum \deg f_i)|_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}.$$

As we have seen, $\omega_{\tilde{X}} \cong \pi^* \omega_X$, so

$$\omega_X^{\otimes 2} \cong \mathrm{Norm}_{\tilde{X}/X}(\omega_{\tilde{X}}) \cong \mathcal{O}_X.$$

Moreover, again because $\tilde{X}$ is a complete intersection, $H^1(\mathcal{O}_{\tilde{X}}) = (0)$; since $\dim H^2(\mathcal{O}_{\tilde{X}}) = \dim H^0(\omega_{\tilde{X}}) = \dim H^0(\mathcal{O}_{\tilde{X}}) = 1$, we find that $\chi(\mathcal{O}_{\tilde{X}}) = 2$, hence $\chi(\mathcal{O}_X) = \frac{1}{2}\chi(\mathcal{O}_{\tilde{X}}) = 1$. By the Basic Table in Part II, X must be an Enriques surface.

§ 4. Linear Systems on Enriques' Surfaces

In this section we shall continue our investigation of Enriques' surfaces and prove that they are all elliptic or quasi-elliptic. This result was proved for classical surfaces by Enriques himself; modern proofs can be found in [1, 12]. The basic argument in his proof was to show that every linear system $|D| \neq \emptyset$ with $D^2 > 0$ contains reducible curves, hence with components of lower arithmetic genus; this implies easily the existence on X of a curve of canonical type. His method of proof however breaks down in the non-classical case and we have to exploit the special features of char. 2 in order to obtain the same result.

Theorem 3. *Every Enriques' surface is elliptic or quasi-elliptic.*

Proof. In order not to repeat well-known arguments, we shall consider only the non-classical cases in char. 2. By the results in Part I, [7], it is sufficient to show the existence of a curve of canonical type on X.

Let L be an invertible sheaf on X, $L \not\cong \mathcal{O}_X$, and $|L| \neq \emptyset$. If $s \in \Gamma(L)$ and $C = \operatorname{div}(s)$ we have the exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{s} L \to \mathcal{O}_C \otimes L \to 0$$

and we denote by $\Gamma(L)_0$ the vector subspace of $\Gamma(L)$ consisting of those sections s for which

$$H^1(\mathcal{O}_X) \xrightarrow{s} H^1(L) \quad \text{is the zero map,}$$

and $|L|_0$ will be the associated linear system.

Lemma 4. *Assume that there exists* $C \in |L|$ *with* $\dim H^0(\mathcal{O}_C) = 1$. *Then we have: either*

a) $\dim H^1(L) = 0$ *and* $\dim |L| = \dim |L|_0 = \frac{1}{2}(L^2)$

or

b) $\dim H^1(L) = 1$ *and* $\dim |L|_0 = \frac{1}{2}(L^2)$.

Moreover, in case b) *we have that* $D \in |L|_0$, $D > 0$ *if and only if* $\dim H^0(\mathcal{O}_D) = 2$, *hence every element of* $|L|_0$ *is reducible.*

Proof of Lemma. Since $K_X \equiv 0$, the Riemann-Roch Theorem yields

$$\dim |L| = \tfrac{1}{2}(L^2) + \dim H^1(L).$$

The cohomology sequence of

$$0 \to \mathcal{O}_X \xrightarrow{s} L \to \omega_D \to 0$$

where $D = \operatorname{div}(s)$, gives

$$H^1(\mathcal{O}_X) \xrightarrow{s} H^1(L) \to H^1(\omega_D) \to H^2(\mathcal{O}_X) \to 0.$$

Now if $\dim H^1(\omega_D) = \dim H^0(\mathcal{O}_D) = 1$, we see that $\dim H^1(L) \leqq 1$, hence we obtain either a) or b). We also get, for any D,

$$\dim H^1(L) = -1 + \dim H^0(\mathcal{O}_D) + \dim \operatorname{Im}\{H^1(\mathcal{O}_X) \xrightarrow{s} H^1(L)\}$$

and we obtain the last clause of Lemma 4. Q.E.D.

Lemma 5. *If* $L^2 > 0$, $\dim H^1(L) = 0$ *and* $|L|$ *contains an irreducible curve then*

$$\dim H^1(L^{\otimes 2}) = 0 \quad \textit{and} \quad \dim |L^{\otimes 2}| = 2(L^2).$$

Proof of Lemma. Let $C = \operatorname{div}(s) \in |L|$ be irreducible. Clearly $H^1(\mathcal{O}_X) \xrightarrow{s} H^1(L^{\otimes 2})$ is the zero map, since it factors through multiplication by s alone, hence $s^2 \in \Gamma(L^{\otimes 2})_0$. On the other hand, C irreducible and $C^2 = L^2 > 0$ imply $\dim H^0(\mathcal{O}_{2C}) = 1$, and the result follows from the previous Lemma. Q.E.D.

Proposition 10. *Let C be an irreducible curve on X with* $C^2 > 0$. *Then the linear system* $|C|$ *contains a reducible divisor D which is not the sum of two non-singular rational curves.*

Proof of Proposition. We choose a non-trivial 1-cocycle $\{a_{ij}\} \in Z^1(\{U_i\}, \mathcal{O}_X)$ such that, as remarked in the proof of Corollary to Theorem 2,

$$a_{ij}^2 - \varepsilon_X a_{ij} = b_i - b_j \quad \text{on } U_i \cap U_j$$

with $b_i \in \Gamma(U_i, \mathcal{O}_X)$, where

$$\varepsilon_X = \begin{cases} 1 & \text{if } X \text{ is singular} \\ 0 & \text{if } X \text{ is supersingular.} \end{cases}$$

We also fix once for all such a datum $b = \{b_i\}$. Now let $\{f_{ij}\} \in Z^1(\{U_i\}, \mathcal{O}_X^*)$ be a 1-cocycle representing the class of L in $\operatorname{Pic}(X)$. If s is a section of L, the image of the class of $\{a_{ij}\}$ by $H^1(\mathcal{O}_X) \xrightarrow{s} H^1(L)$ is represented by the 1-cocycle $\{s_i a_{ij}\} \in Z^1(\{U_i\}, L)$, hence if $s \in \Gamma(L)_0$ then this is a 1-coboundary:

$$s_i a_{ij} = \sigma_i - f_{ij} \sigma_j \quad \text{on } U_i \cap U_j$$

with $\sigma_i \in \Gamma(U_i, \mathcal{O}_X)$. The corresponding datum $\sigma = \{\sigma_i\}$ is determined uniquely modulo sections of L. We use b and σ to construct sections of $L^{\otimes 2}$ as follows: if s, $t \in \Gamma(L)_0$ and σ, τ are associated data then $s\tau + \sigma t$ and $\sigma^2 + \varepsilon_X \sigma s + b s^2$ are sections of $L^{\otimes 2}$, as a straightforward calculation in char. 2 shows.

Now take $L = \mathcal{O}_X(C)$. By Lemma 4, if $\dim H^1(L) = 1$ every element of $|C|_0$ is reducible and is not the sum of two non-singular rational curves, since by hypothesis $C^2 > 0$, and $\dim H^0(\mathcal{O}_C) = 2$.

Therefore we have to consider only the case in which $\dim H^1(L) = 0$. We fix a basis $s_0, \ldots, s_n$ of $\Gamma(L)$, where $n = \frac{1}{2}(L^2)$ and we fix associated data $\bar{\sigma}_0, \ldots, \bar{\sigma}_n$ which we extend by linearity to a well-defined datum $\bar{\sigma}$ associated to $s \in \Gamma(L)$. Any other datum σ associated to s is of type $\sigma = \bar{\sigma} + s'$, $s' \in \Gamma(L)$. Now consider in $\Gamma(L^{\otimes 2})$ sections of the form

$$\sigma^2 + \varepsilon_X \sigma s + b s^2 + s\tau + \sigma t \tag{A'}$$

and

$$s\tau + \sigma t \tag{A''}$$

There are some obvious cases in which they vanish identically: in case (A'), if $s = \sigma = 0$, or if $s = 0$, $\sigma = t$; in case (A''), if $s = \mu t$ and $\sigma = \mu\tau$ with $\mu \in k^*$, or if $s = \sigma = 0$, or $s = t = 0$, or $t = \tau = 0$. We call these the trivial relations. Writing

$$\begin{aligned} s &= \sum x_i s_i, \\ \sigma &= \sum x_i \bar{\sigma}_i + \sum u_i s_i, \\ t &= \sum y_i s_i, \\ \tau &= \sum y_i \bar{\sigma}_i + \sum (v_i + \varepsilon_X u_i) s_i \end{aligned}$$

we obtain

$$\begin{aligned} &\sigma^2 + \varepsilon_X \sigma s + b s^2 + s\tau + \sigma t \\ &\quad = \sum x_i^2 (\bar{\sigma}_i^2 + \varepsilon_X \bar{\sigma}_i s_i + b s_i^2) \\ &\qquad + \sum_{i<j} (\varepsilon_X x_i x_j + y_i x_j + y_j x_i)(\bar{\sigma}_i s_j + \bar{\sigma}_j s_i) \\ &\qquad + \sum (u_i^2 + v_i x_i + u_i y_i) s_i^2 \\ &\qquad + \sum_{i<j} (v_i x_j + v_j x_i + u_i y_j + u_j y_i) s_i s_j. \end{aligned}$$

In the same way, setting $v_i' = v_i + \varepsilon_X u_i$ we find

$$\begin{aligned} s\tau + \sigma t = & \sum_{i<j} (y_i x_j + y_j x_i)(\bar{\sigma}_i s_j + \bar{\sigma}_j s_i) \\ & + \sum (v_i' x_i + u_i y_i) s_i^2 \\ & + \sum_{i<j} (v_i' x_j + v_j' x_i + u_i y_j + u_j y_i) s_i s_j. \end{aligned}$$

Now let V be a vector space over k, with basis denoted by $e_i, e_{ij}, e_i', e_{ij}', 0 \leqq i < j \leqq n$ and let M' be the algebraic cone in V consisting of points

$$\begin{aligned} & \sum x_i^2 e_i + \sum_{i<j} (\varepsilon_X x_i x_j + y_i x_j + y_j x_i) e_{ij} \\ & \quad + \sum (u_i^2 + v_i x_i + u_i y_i) e_i' + \sum_{i<j} (v_i x_j + v_j x_i + u_i y_j + u_j y_i) e_{ij}' \end{aligned}$$

and in the same way let M'' be the algebraic cone in V consisting of points

$$\begin{aligned} & \sum_{i<j} (y_i x_j + y_j x_i) e_{ij} + \sum (v_i x_i + u_i y_i) e_i' \\ & \quad + \sum_{i<j} (v_i x_j + v_j x_i + u_i y_j + u_j y_i) e_{ij}'. \end{aligned}$$

The main fact is:

(i) $M = M' \cup M''$ is closed in V,

(ii) $\dim M = 4n + 2$.

If $\psi: V \to \Gamma(L^{\otimes 2})$ is the homomorphism

$$\begin{aligned} \psi(e_i) &= \bar{\sigma}_i^2 + \varepsilon_X \bar{\sigma}_i s_i + b s_i^2, \\ \psi(e_{ij}) &= \bar{\sigma}_i s_j + \bar{\sigma}_j s_i, \\ \psi(e_i') &= s_i^2, \\ \psi(e_{ij}') &= s_i s_j \end{aligned}$$

then its kernel is a linear subspace of V of codimension

$$\operatorname{codim} \ker(\psi) \leqq \dim \Gamma(L^{\otimes 2}) = 4n + 1$$

by Lemma 5, hence by (i) and (ii) there is a point $m \in M$, $m \neq 0$, with $\psi(m) = 0$. Thus we obtain a relation of the form $(A') = 0$ or $(A'') = 0$ and this relation is non-trivial since trivial relations correspond to the origin of the cone, while $m \neq 0$.

We claim that $\operatorname{div}(s)$ is reducible and is not the sum of two non-singular rational curves. Suppose for example that the relation is

$$\sigma^2 + \varepsilon_X \sigma s + b s^2 + s\tau + \sigma t = 0$$

and that $\operatorname{div}(s)$ is irreducible; note that s is not 0, because the relation is non-trivial. Hence

$$s(\tau + \varepsilon_X \sigma + b s) = \sigma(t + \sigma)$$

and, denoting restriction to U_i by a subscript, we find either

$$\sigma_i = s_i g_i, \quad g_i \in \Gamma(U_i, \mathcal{O}_X)$$

or

$$t_i + \sigma_i = s_i g_i, \quad g_i \in \Gamma(U_i, \mathcal{O}_X)$$

because we assume that div(s) is irreducible. This implies easily, using $\sigma_i - f_{ij}\sigma_j = s_i a_{ij}$, that either

$$\sigma_i = s_i g_i \quad \text{for every } U_i,$$

or

$$t_i + \sigma_i = s_i g_i \quad \text{for every } U_i$$

holds. Replacing the representative σ by $t+\sigma$ if needed, we obtain $\sigma_i = s_i g_i$ for every U_i. This implies that

$$a_{ij} = g_i - g_j \quad \text{on } U_i \cap U_j$$

with $g_i \in \Gamma(U_i, \mathcal{O}_X)$, hence $\{a_{ij}\}$ would be a 1-coboundary, which is a contradiction.

Now assume that div$(s) = E' + E''$ where E', E'' are non-singular rational curves on X. Let $\{f'_{ij}\}$, $\{f''_{ij}\}$ be 1-cocycles in $Z^1(\{U_i\}, \mathcal{O}_X^*)$, with $f_{ij} = f'_{ij} f''_{ij}$, representing the classes of $\mathcal{O}_X(E')$ and $\mathcal{O}_X(E'')$ in Pic(X) and let s', s'' be corresponding sections with $s = s' s''$. Reasoning as before, we can assume that

$$\sigma_i = s'_i g_i \quad \text{on } U_i$$

with $g_i \in \Gamma(U_i, \mathcal{O}_X)$, and we get that

$$s''_i a_{ij} = g_i - f''_{ij} g_j \quad \text{on } U_i \cap U_j,$$

hence

$$H^1(\mathcal{O}_X) \xrightarrow{s''} H^1(\mathcal{O}_X(E'')) \quad \text{is the zero map.}$$

This contradicts the exact sequence

$$H^0(\mathcal{O}_{E''}(E'')) \to H^1(\mathcal{O}_X) \xrightarrow{s''} H^1(\mathcal{O}_X(E'')),$$

because $E''.E'' = -2$, hence $H^0(\mathcal{O}_{E''}(E'')) = (0)$.

The same argument applies in case of a relation of type $(A'') = 0$. Q.E.D.

We now return to the proof of Theorem 3. Let $p = \frac{1}{2}(C^2) + 1$ be the smallest genus of irreducible curves C with $C^2 > 0$, and let $D \in |C|$ be a reducible element as in the previous Proposition. We conclude our proof in the following steps:

Step 1. If E is an irreducible component of D, then $p(E) < p(C)$ hence $p(E) = 0$ or 1. In fact, let $D = E + D'$. We have, since C is irreducible and $C^2 > 0$, $DD' = CD' \geqq 0$ hence $DE \leqq C^2$. Now $DE = E^2 + ED'$ and D_{red} is connected by the degeneration principle of Enriques-Zariski, hence either $ED' > 0$ or $E^2 < 0$, and in both cases $E^2 < C^2$. Q.E.D.

Step 2. We may assume $D = \sum m_i E_i$ where $\sum m_i \geqq 3$ and the E_i are non-singular rational curves with $E_i^2 = -2$ and $E_i E_j \leqq 2$. In fact, $p(E_i) = 1$ implies that E_i is of canonical type, hence X would be elliptic or quasi-elliptic. If instead $p(E_i)=0$ for all i, then $\sum m_i \geqq 3$ by the previous Proposition. Finally, $E_i E_j \leqq 2$, otherwise $(E_i + E_j)^2 \geqq 2$, and $\dim |E_i + E_j| \geqq 1$ by the Riemann-Roch Theorem. Now $|E_i + E_j| + (D - E_i - E_j) \subseteq |C|$, and a general element of $|E_i + E_j|$ is irreducible; if $F \in |E_i + E_j|$, then $F^2 < C^2$ by the argument in Step 1, which contradicts $p(C) = \text{minimum}$.

Q.E.D.

Final Step. X contains a curve of canonical type. If $E_i E_j = 2$, then $E_i + E_j$ is of canonical type, and we are done. So assume $E_i E_j \leqq 1$ for all i, j. Consider the connected graph with vertices E_i and edges connecting E_i, E_j if $E_i E_j = 1$. If the graph contains a subgraph which is a complete Dynkin diagram $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ (e.g. $\tilde{A}_n$ is a loop) then we find a curve of canonical type: e.g., for $\tilde{E}_8$ the curve

$$\begin{tikzcd}[column sep=small] & & 3 \arrow[d, no head] & & & & & \\ 2 \arrow[r, no head] & 4 \arrow[r, no head] & 6 \arrow[r, no head] & 5 \arrow[r, no head] & 4 \arrow[r, no head] & 3 \arrow[r, no head] & 2 \arrow[r, no head] & 1 \end{tikzcd}$$

where the numbers denote the multiplicities of the components. Otherwise, the graph must itself be a Dynkin diagram, and since the associated self-intersection quadratic form is negative definite we would obtain $D^2 < 0$, a contradiction.

Q.E.D.

From the computation of the canonical bundle of an elliptic or quasi-elliptic surface in Part II, we also get:

Proposition 11. *Let $f\colon X \to \mathbb{P}^1$ be an elliptic or quasi-elliptic fibration on an Enriques surface. Then we have:*

(i) *if X is classical, f has 2 ordinary double fibres and*

$R^1 f_*(\mathcal{O}_X) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$;

(ii) *if X is non-classical, f has exactly one wild double fibre $2P = f^{-1}(x)$ and*

$R^1 f_*(\mathcal{O}_X) \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus k_x$

where k_x is a sheaf with support x and stalk k there.

Theorem 4. *If X is an Enriques' surface then its Picard number is $\rho = B_2 = 10$.*

Proof. If X is quasi-elliptic, this comes from the following more general result:

Proposition 12. *For all quasi-elliptic surfaces we have $\rho = B_2$.*

Proof of Proposition. By the results of Section 1, there is a smooth ruled surface Y and a proper map $\pi\colon Y \to X$ of degree $p = \operatorname{char}(k)$. Now use

$$\begin{tikzcd} \operatorname{Pic}(Y) \otimes \mathbb{Z}_l \arrow[r, twoheadrightarrow] \arrow[d, shift left, "\pi_*"] & H^2_{\acute{e}t}(Y, \mathbb{Z}_l) \arrow[d, shift left, "\pi_*"] \\ \operatorname{Pic}(X) \otimes \mathbb{Z}_l \arrow[r] \arrow[u, shift left, "\pi^*"] & H^2_{\acute{e}t}(X, \mathbb{Z}_l) \arrow[u, shift left, "\pi^*"] \end{tikzcd}$$

where $l \neq p$. Since $\pi_* \pi^* = \text{mult. by } p$, we obtain that $\operatorname{Pic}(X) \otimes \mathbb{Z}_l \to H^2_{\acute{e}t}(X, \mathbb{Z}_l)$ is onto. Q.E.D.

Now suppose that X is elliptic. We have:

Proposition 13. *Let $X \xrightarrow{f} C$ be an elliptic surface and let $J(X)$ be the associated Jacobian surface. Then if $\rho = B_2$ for $J(X)$ we have also $\rho = B_2$ for X.*

Proof of Proposition. We use an idea of M. Artin. From the Kummer sequence

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 1$$

we get the exact sequence

$$0 \to \operatorname{Pic}(X)^{(n)} \to H^2(X, \mu_n) \to H^2(X, \mathbb{G}_m)_n \to 0$$

where $H^2(X, \mathbb{G}_m)_n$ is the subgroup of $H^2(X, \mathbb{G}_m)$ killed by multiplication by n, and where $\operatorname{Pic}(X)^{(n)} = \operatorname{Pic}(X)/n \operatorname{Pic}(X)$. This gives, for each prime $l \neq p$, that

$$B_2(X) - \rho(X) = \operatorname{corank} H^2(X, \mathbb{G}_m)(l)$$

where the symbol (l) denotes the l-torsion part.

Now the Leray spectral sequence for $f\colon X \to C$ shows that, denoting by $\approx$ an homomorphism with finite kernel and cokernel, we have:

$$H^2(X, \mathbb{G}_m) \approx H^1(C, R^1 f_* \mathbb{G}_m);$$

also, if η denotes the generic point of C and if $i\colon \eta \to C$ is the inclusion, then if J is the Jacobian of the generic fibre we have:

$$H^1(C, R^1 f_* \mathbb{G}_m) \approx H^1(C, i_* J)$$

(cfr. [3], §4 and [9]). Since the last group is the same for both fibrations $X \to C$ and $J(X) \to C$ we obtain

$$B_2(X) - \rho(X) = B_2(J(X)) - \rho(J(X))$$

and Proposition 13 is proven. Q.E.D.

Proof of Theorem 4. We have to show that $\rho(J(X)) = B_2(J(X))$ where $J(X)$ is the Jacobian fibering of the elliptic Enriques' surface X. Now the elliptic surface $J(X)$ has the same Betti numbers as X and has a section, hence no multiple fibres. The canonical bundle formula proved in Part II of this work now shows that a canonical divisor on $J(X)$ is given by the opposite of a fibre, hence all plurigenera of $J(X)$ vanish and $J(X)$ is rational by Castelnuovo's criterion. Hence $B_2(J(X)) - \rho(J(X)) = 0$. Q.E.D.

§ 5. Two Examples

Let X be a supersingular Enriques surface. We have seen in §3 of this work that X has a regular 1-form η which is locally of the form $\eta = db_i$, and also X has a nowhere vanishing regular 2-form ω. Now the formula

$$df \wedge \eta = (\vartheta f)\,\omega$$

defines a regular vector field ϑ on X. Since we are in char. 2, we also have

$$\vartheta^2 = \delta\vartheta$$

for some constant δ; replacing η by a multiple, we may assume that

$$\vartheta^2 = \delta_X \vartheta,$$

with $\delta_X = 1$ or 0.

Proposition 14. *If $\delta_X = 1$ then η has exactly 12 isolated simple zeros. If instead $\delta_X = 0$ then all zeros of η have even multiplicity.*

Proof. Let u, v be local formal parameters at a point P of X. Now write $\omega = \varphi\, du \wedge dv$, $\eta = db = b_u\, du + b_v\, dv$. A simple computation now shows that

$$\delta_X = (\varphi^{-1})_u b_v + \varphi^{-1} b_{uv} + (\varphi^{-1})_v b_u$$

where the subscripts denote partial derivatives. Taking the partial derivatives with respect to u, v we see that $(\varphi^{-1})_{uv} b_u = (\varphi^{-1})_{uv} b_v = 0$, hence φ_{uv} is identically 0. Thus we must have

$$\varphi = A^2 + uB^2 + vC^2$$

for some power series A, B, C with $A(0,0) \neq 0$. If we introduce the new local parameters

$$\tilde{u} = u(A^2 + C^2 v),$$

$$\tilde{v} = v(1 + (B/A)^2 u)$$

we easily check that

$$\omega = (A^2 + uB^2 + vC^2)\, du \wedge dv = d\tilde{u} \wedge d\tilde{v},$$

whence we have shown that changing the local parameters u, v by multiplication with suitable units, we have the normal form $\omega = du \wedge dv$ for ω. It then follows that $\delta_X = b_{uv}$ and now we can write

$$b = A^2 + uB^2 + vC^2 + \delta_X uv$$

for suitable power series A, B, C. Hence

$$\eta = (\delta_X v + B^2)\, du + (\delta_X u + C^2)\, dv.$$

If η vanishes at P, then $B(0,0) = C(0,0) = 0$; if $\delta_X = 0$, then η vanishes at P to even order, while if $\delta_X = 1$ then η has a simple isolated zero at P, since $v + B^2$ and $u + C^2$ generate the maximal ideal of the formal local ring of X at P. Noting that $c_2(X) = 12$ we get the required result. Q.E.D.

Let $Y \to X$ be the α_2-covering associated to X defined in § 3; this is locally of type $z^2 = b$, hence of type

$$z^2 = \delta_X uv + uB^2 + vC^2.$$

It follows from this that if $\delta_X = 1$ then Y has exactly 12 ordinary rational double points, i.e. those over the zeros of the 1-form η.

Proposition 15. *Let X be a supersingular Enriques surface with $\delta_X = 1$, and let $\tilde{Y}$ be a non-singular minimal model of the associated surface Y. Then $\tilde{Y}$ is a $K3$-surface and its Picard number is $\rho(\tilde{Y}) = B_2(\tilde{Y}) = 22$, i.e. $\tilde{Y}$ is a supersingular $K3$-surface.*

Proof. We know already that Y is $K3$-like, that is $\chi(\mathcal{O}_Y)=2$ and $\omega_Y \cong \mathcal{O}_Y$. Since Y has only 12 rational double points, $\tilde{Y}$ is a $K3$-surface. Now Theorem 4, together with the method of proof of Proposition 12, shows that $\rho(\tilde{Y}) \geqq \rho(X)+12=22=B_2(\tilde{Y})$, since the resolution of the 12 singular points brings in 12 new independent curves on $\tilde{Y}$. Q.E.D.

We have computed two examples using the construction at the end of § 3. In the first example, X is obtained from the α_2-covering $Y \to X$, where Y is the complete intersection of the three quadrics

$$x_1^2+x_1x_2+x_2^2+x_1y_2+x_2y_1+y_3^2=0,$$
$$x_2^2+x_2x_3+x_3^2+x_2y_3+x_3y_2+y_1^2=0,$$
$$x_3^2+x_3x_1+x_1^2+x_3y_1+x_1y_3+y_2^2=0.$$

The surface Y has exactly 12 rational double points, namely: the point

$$(1, 1, 1, 1, 1, 1),$$

the 9 points

$$(1, t, t, t, t^2, t^2),$$
$$(t, 1, t, t^2, t, t^2),$$
$$(t, t, 1, t^2, t^2, t)$$

where $t^3+t^2+1=0$, and the 2 points

$$(1, t, t^2, 0, 0, 0)$$

where $t^2+t+1=0$.

The quotient surface X by the α_2-action $(x_i, y_i) \to (x_i, \varepsilon x_i+y_i)$ where $\varepsilon^2=0$, is a smooth supersingular Enriques surface with $\delta_X=1$.

In our second example, Y is the complete intersection of the three quadrics

$$x_1^2+x_1x_2+y_3^2+y_1x_2+x_1y_2=0,$$
$$x_2^2+x_2x_3+y_1^2+y_2x_3+x_2y_3=0,$$
$$x_3^2+x_1x_3+y_2^2+y_1x_3+x_1y_3=0.$$

The surface Y has exactly 6 isolated singular points, namely: the point

$$(1, 1, 1, 0, 0, 0),$$

the 3 points

$$(t^3, t, 1, t^3, t, 1)$$

where $t^3+t^2+1=0$, and the 2 points

$$(t^3, t^2, t, t^2, t, 1)$$

where $t^2+t+1=0$.

The quotient surface X by the α_2-action $(x_i, y_i) \to (x_i, \varepsilon x_i+y_i)$ where $\varepsilon^2=0$, is a smooth supersingular Enriques surface with $\delta_X=0$. The regular 1-form η on X has now exactly 6 double zeros (cf. Proposition 14).

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Received March 1, 1976

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